FIXED-INCOME SECURITIES AND THE TERM-STRUCTURE OF INTEREST RATES

This lecture is an introduction to the modeling of fixed-income securities. We introduce standard terminology and conventions for debt securities and concepts from bond mathematics, such as yield and duration. We also discuss the valuation of debt securities in terms of the *term-structure of interest rates*. The term-structure is usually represented as a curve that displays the value of an index (price or rate) as a function of the time-to-maturity. The value of "straight debt " (bonds, swaps) is determined entirely from the current term-structure. On the other hand, options and instruments with embedded options such as callable bonds derive their value from the volatility of interest rates, i.e. the evolution of the term-structure under a risk-neutral probability measure.

In the last section, we present several algorithms for constructing an *instantaneous forward-rate curve* from prices of bonds and/or swaps observed in the market. Procedures for curve construction are important in the practical implementation of term-structure models.

1. Bonds

Bonds are perhaps the simplest fixed-income securities. When a bond is issued, the investor (bond buyer) is is essentially lending money to the issuer in exchange of interest payments and the promise of repayment of the principal at a future date. Most bonds also trade in the secondary market, like stocks and other securities. Understanding the pricing of a bond and, in particular, the relative valuation of different bond issues is very important for investors and traders. The main considerations that enter the pricing of a bond are

- Principal (notional amount of the loan)
- Maturity
- Interest payments
- Call provisions and other features such as conversion to shares, etc.
- Credit quality of the issuer.

The first three points define the structure of cash flows that the investor expects to receive if the bond was held to maturity. **Call provisions** exist in many bond issues: this means that the issuer has the option to "call" (retire) the bond after a certain time, stipulated in the bond agreement. For the investor, this means that there is uncertainty as to the true maturity of the loan.

Putable bonds are bonds that can be "put back" to the issuer at a fixed notional value at a given date or dates. In this case, the bondholder essentially owns an option.

Convertible bonds (usually issued by corporations) can be converted into stock at a given price (the "conversion ratio"). Conversion events complicate the pricing because the investor is uncertain about the cash-flows that will be received.

Credit quality is a very important variable because it represents our beliefs about the issuer's capacity to repay the principal and interest. U.S. Government bonds are considered to be the most creditworthy, since they are backed by the "full faith and power" of the government.[1]

Bonds issued by corporations have a certain probability of defaulting in case the corporation cannot meets its obligations. Thus, corporate bonds have a lower credit quality. Low-credit bonds have higher coupons than high-grade bonds trading at the same price. Investors demand a premium for the default risk.

In this lecture, we shall ignore credit quality considerations altogether. The reason for this is that we wish to focus primarily on the interest rates and interest rate-risk. we will therefore assume issuers will not default on bond payments or, alternatively, that issuers have the highest credit quality.

Also, for simplicity, we will not discuss call provisions or convertible bonds right away, since call and conversion features are "embedded options", since these require option valuation methods. The question of pricing embedded options is postponed until we develop the mathematical machinery for modeling interest-rate volatility.

A **zero-coupon bond** (also called a "zero" or a "pure discount bond") is a bond that has a single payment of principal at maturity, without no intermediate interest payments. Let us denote the principal amount by $Pr.$ and the bond maturity date by T. Intuitively, the fair value of the bond at date $t < T$ should be less than $Pr.$ dollars. The difference is sometimes called the **liquidity premium** in finance theory. It comes from the fact that investor would rather have cash now, for investment or consumption purposes, as opposed to having the same amount in the future. In other words, interest-rates are positive.

Arbitrage Pricing Theory gives a way of expressing the value of a zero-coupon bond in terms of a **risk-neutral measure** on the paths of short-term interest rates.[2] We define

[1] More generally, the debt of sovereign issuers (countries) in the local currency usually has the highest rating among debt securities issued in the local currency.

[2] We are not making the claim that risk-neutral measure is unique, i.e. that we are considering a complete market. Arbitrage pricing theory postulates the existence of *at least* one probability measure on the interest rates paths that is such that interest-rate-contingent claims are priced consistently with the expected values of discounted cashflows. In the sequel, we will use the terminology "the risk-neutral measure" as referring to a particular such measure, without implying that such measure is unique.

the short-term interest rate r_t as the rate associated with a money-market account, in the framework of APT. For mathematical simplicity, we assume that r_t is a continuously-compounded interest rate. Since there is a single cashflow of $Pr.$ dollars at time T, according to APT, the value of this zero coupon bond at time t is

$$Z_t^T \;=\; Pr. \times \; \mathbf{E}_t^P \left\{ e^{-\int_t^T .r_s \; ds} \right\} \tag{1}$$

Here, $\mathbf{E}_t^P$ represents the expected value operator with respect to the risk-neutral measure; the subscript t reminds us that this expectation is taken conditionally on the state of the market at date t. The expression

$$P_t^T \;=\; \mathbf{E}_t^P \left\{ e^{-\int_t^T r_s \, ds} \right\} \tag{2}$$

is called the **discount factor** or the present value of one dollar (**PV01**) corresponding to date T. The function

$$T \;\longrightarrow\; P_t^T \quad , \quad T \;>\; t$$

edskipis the **discount curve** at time t. Notice that T represents "absolute time", *i.e.* a fixed date in the future of t. It is often customary to use "relative time" or "time-to-maturity" $\tau \;=\; T \;-\; t$ when dealing with the discount curve. In the latter case, the function

$$\tau \;\longrightarrow\; P_t^{t\,+\,\tau} \quad , \quad \tau \;>\; 0$$

is said to represent the discount curve at time t. We shall use whichever representation of the discount curve is more convenient according to the context.

Clearly, if the short-term interest rate is non-negative under the risk-neutral measure we have $P_t^T \;\leq\; 1$. The **yield** of the zero-coupon bond is , by definition, the constant interest rate that would make the bond price equal to the discounted value of the final cash-flow. The continuously compounded yield of the zero is therefore

$$Y_t^T \;=\; -\;\frac{ln\left(P_t^T\right)}{T-t} \;, \tag{3}$$

so that

$$P_t^T \;=\; e^{-\,(T-t)\,Y_t^T} \;.$$

Equivalently, the yield is the continuously compounded (constant) rate of return that the investor would receive if the zero-coupon bond was bought and held to maturity.[3]

There is an important practical consideration regarding the calculation of yields. In fact, expressing the time-to-maturity, $T - t$, as a decimal number requires using a *day-count count* (DCC), or convention to convert days and months into fractions of a year. A DCC determines unequivocally the decimal fraction of year that corresponds to the period between two calendar dates.[4] Notice that the price and the yield vary inversely to each other: an increase in price corresponds to a decrease in yield and vice-versa. Moreover, the price is a convex function of yield.

Most bonds have interest payments (coupons) as well as payment of the principal. The "generic" bond will therefore specify a maturity date, in which the principal payment is made, as well as a schedule of interest payments.

- Maturity date
- Principal ("face value")
- Coupon (interest)
- Frequency & payment dates

The coupon is the annualized interest rate of the bond. The frequency represents how many payments are made per year ($\omega = 1, 2, 4,$ or 12). Most bonds have annual or semi-annual interest payments ($\omega = 1$ or 2). Thus, a 10-year bond with face value of \$1,000 and a semiannual coupon of 6.25will pay the investor an interest of $0.5 \times 0.0625 \times 1{,}000 =$ \$31.25 every six months (20 payments) and the principal will be paid at the 20th payment date.

To derive a mathematical formula for the value of a coupon-bearing bond in terms of a risk-neutral probability measure on future interest rate scenarios, we set $Pr. =$ principal, $C =$ coupon, $\omega =$ frequency.[5] Let $t_n,\ n \le N$ represent the cash-flow dates, assuming that $t = 0$ now. Then, the value of the bond is given by

$$B = \sum_{n:t_n \ge t}^{N} \frac{C \cdot Pr.}{\omega} \mathbf{E}_t^P \left\{ e^{-\int_t^{t_n} r_s\, ds} \right\} + Pr. \cdot \mathbf{E}_t^P \left\{ e^{-\int_t^{t_N} r_s\, ds} \right\}, \tag{4a}$$

[3]In Chapter 5, we discussed the notion of forward rates in the context of option pricing. There, we converted term rates into forward rates. In practice, the term rates are often taken to be the yields on short-term debt securities such as Treasury bills.

[4]Day-count conventions are important when yields are used to quote bond prices. This is somewhat analogous to quoting option prices in terms of implied volatilities rather than in a currency. A day-count convention in fixed-income markets typically specifies (i) the number of days in a month and (ii) the number of days in a year. The reader interested in DCC's should consult, for example F. Fabozzi: *Fixed Income Markets*, Irwin (???) or ISDA (International Swap Dealers Association) publications.

[5]It is customary to use $Pr. = 100$ for quoting bond prices.

or, simply,

$$B \;=\; \sum_{n:\, t_n \;\geq\; t}^{N} \frac{C \cdot Pr.}{\omega} \cdot P_t^{t_n} \;+\; Pr.\; P_t^{t_N} \;, \tag{4b}$$

where $P_t^{t_n}$ are the discount factors. Notice that this formula is an immediate consequence that a bond is equivalent to a series of zero-coupon bonds maturing at th coupon-payment dates and at maturity. In particular, *the value of a bond on a particular date is completely determined by the discount curve at that date.*

Although equations (4a) and (4b) give the true *market value* of the security, notice that the function B has a jump of $\frac{C \cdot Pr.}{\omega}$ at each date $t = t_n$. Thus, the value of the bond changes discontinuously.

The bond "price" can be made continuous, however, if we subtract from expression (4b) the interest accrued to the bondholder between the last coupon date and the present date. The usual convention is that the interest on the next coupon accrues *linearly.* Thus, if the present date is t and the last coupon date was t_n, the accrued interest "earned" by the bondholder is

$$AI(t,\, t_n) \;=\; \frac{C \,\cdot\, Pr.}{\omega}\, \frac{t \;-\; t_n}{t_{n+1} \;-\; t_n} \;.$$

By definition, the **clean price** of a bond corresponds to the price at which the transaction takes place *without* including accrued interest. The **dirty price** is the price including accrued interest, i.e. how much money trades hands (so to speak). Hence,

$$\text{Dirty price} \;=\; \text{Clean price} \;+\; AI(t,\, t_n) \;.$$

In an arbitrage-free economy, the dirty price should be equal to the theoretical value (4a)-(4b). In particular, the theoretical clean price can be expressed in terms of the term-structure of interest rates as

$$\text{Clean price} \;=\; \sum_{n:\, t_n \;\geq\; t}^{N} \frac{C \cdot Pr.}{\omega} \cdot P_t^{t_n} \;+\; Pr.\; P_t^{t_N} \;-\; AI(t;\, t_n) \;.$$

The clean and dirty prices coincide on the coupon date after the coupon is paid (since $AI(t_n;\, t_n) \;=\; 0$).

Bond quotes in the US Treasury, international and corporate markets are usually in terms of clean prices.[6]

[6]Since the clean price varies continuously, it is not uncommon for practitioners to model the clean bond price as a diffusion process for the purposes of pricing options. We shall discuss this procedure in a future lecture.

The yield of a bond (or yield-to-maturity) is the effective constant interest rate which makes the bond price equal to the future cashflows discounted at this rate. The yield-to-maturity is usually computed using the same frequency as the bond's interest payments (*e.g.* semiannual), rather than the continuously compounded yield used for zeros.

Assume that the current date coincides with a coupon payment date, so that $t = t_m$. In this case, we define the yield-to-maturity of the bond (after the coupon was paid) to be the value of Y such that

$$B \;=\; \sum_{n=m+1}^{N} \frac{C \cdot Pr.}{\omega} \left(\frac{1}{1+Y/\omega}\right)^{n-m} \;+\; Pr. \left(\frac{1}{1+Y/\omega}\right)^{N-m} \quad . \tag{5a}$$

If the current date does not coincide with a coupon date, we should take into account the fraction of year corresponding to the period between now and the next coupon date. Accordingly, assume that $t_m < t < t_{m+1}$ and that f represents the ratio of the number of days in remaining until the next coupon date and the number of days in the coupon period, using the appropriate DCC. (Hence, $0 < f < 1$). The bond yield Y is defined by the relation

$$B \;=\; \sum_{n=m+1}^{N} \frac{C \cdot Pr.}{\omega} \left(\frac{1}{1+Y/\omega}\right)^{f\,+\,n\,-\,m-\,1} \quad Pr. \left(\frac{1}{1+Y/\omega}\right)^{f\,+\,N\,-\,m-\,1} \tag{5b}$$

Equations (5a) and (5b) define Y implicitly in terms of the bond value. It is easy to see that B is a decreasing function of Y. Moreover, B is convex in Y. To obtain the yield from the bond value, equations (5a), (5b) must be solved numerically. Nevertheless, (recalling the price relation (4b)) the yield of a bond is a well-defined function of its theoretical value B (dirty price) and thus of the discount factors $P_t^{t_1}$, ... $P_t^{t_N}$.

Notice that if $t = t_m$ we can use the summation formula for a geometric series to obtain

$$B \;=\; \frac{C \cdot Pr.}{Y} \cdot \left(1 \;-\; \left(\frac{1}{1+Y/\omega}\right)^{N-m} \right) \;+\; Pr. \left(\frac{1}{1+Y/\omega}\right)^{N-m} \quad ,$$

This formula shows that if the yield is equal to the coupon rate, the value of the bond is equal to its face value. From this fact and the monotonicity of the price/yield relationship, we can derive some elementary relationships between price, yield and coupon.

If, immediately after a coupon payment, a bond trades at 100% of the principal, we say that the bond trades **at par**. In this case, its yield is exactly equal to the coupon rate. If the bond price is less than 100% of face value, we say that the bond trades **at a discount**. In this case, its yield is higher than the coupon rate. If the bond trades above 100% of face value, we say that bond trades *at a premium*. In this case, the yield is lower than the coupon rate.

In an arbitrage-free market, two bonds with same price and same cashflow dates cannot have different coupons (otherwise, we can short the one with the smaller coupon and buy the one with the larger one). Similarly two bonds with the same price and payment dates cannot have different yields. The notion of **par yield** – the yield of a par bond – is sometimes used to represent the term structure of interest rates implied by the bond market. In this case, one speaks of the **par yield curve**.

2. Duration

The price-yield relation gives rise to several quantities which are commonly used in bond risk-management. The first notion is that of **duration** (or **average duration**, or **McCauley duration**) which is defined as

$$D = \frac{1}{B} \cdot \left(\sum_{n:t_n \geq t}^{N} \frac{C \cdot Pr.}{\omega} \cdot (t_n - t) P_t^{t_n} + Pr.\,(T_n - t)\, P_t^{t_N} \right)$$

$$= \frac{\sum\limits_{n:t_n \geq t}^{N} \frac{C \cdot Pr.}{\omega} \cdot (t_n - t) P_t^{t_n} + Pr.\,(t_N - t)\, P_t^{t_N}}{\sum\limits_{n:t_n \geq t}^{N} \frac{C \cdot Pr.}{\omega} \cdot P_t^{t_n} + Pr.\, P_t^{t_N}} \tag{6}$$

Thus, duration represents a weighted average of the cashflow dates, weighted by the cash-flows measured in constant dollars. (Mathematically, it is the "barycenter" of the cashflow dates).

A closely related quantity is obtained by differentiating the bond price with respect to the yield-to-maturity. In fact, differentiating (5b) with respect to Y, we obtain

$$\frac{\partial B}{\partial Y} = - \sum_{n=m+1}^{N} \frac{C \cdot Pr.}{\omega} \left(\frac{f + n - m - 1}{\omega} \right) \left(\frac{1}{1 + Y\omega} \right)^{f + n - m} -$$

$$Pr. \left(\frac{f + N - m - 1}{\omega} \right) \left(\frac{1}{1 + Y/\omega} \right)^{f + N - m} .$$

It follows from the definition of f, that $f + n - m - 1$ represents the time between t and t_n measured in *coupon periods* ($\frac{1}{\omega}$ years). Therefore, the number

$$\frac{f + n - m - 1}{\omega}$$

represents the time interval between t and the n^{th} coupon date measured in years. We conclude that

$$\frac{1}{B}\frac{\partial B}{\partial Y} = -\frac{D}{1 + Y/\omega} .$$

Thus, the percent sensitivity of the bond (dirty) price with respect to yield is of opposite sign and proportional to the average duration. The quantity

$$D_{mod} = \frac{D}{1 + Y/\omega}$$

which represents the exact magnitude of the percentage change o is known as **modified duration**. These equations express the fact that the longer the duration, the greater the sensitivity of a bond to a change in yield, in percentage terms. These relations are sometimes expressed in the form

$$dB = -\frac{B \times D}{1 + Y/\omega} \cdot dY = -B\, D_{mod}\, dY . \tag{7}$$

Clearly, a zero-coupon bond has duration equal to the time to maturity. The duration of a coupon-bearing bond trading at par (face value) immediately after the coupon date is

$$D = \frac{1}{\omega} \sum_{n=0}^{N-1} \frac{1}{(1 + Y/\omega)^n}$$

$$= \frac{1}{Y} \cdot (1 + Y/\omega) \cdot \left(1 - \frac{1}{(1 + Y/\omega)^N}\right) . \tag{8}$$

(The derivation of this formula is left as an exercise to the reader).

Formula (8) shows that duration decreases with frequency. In fact, If the bond matures in T years and makes only a single payment, we have $N = 1$, $\omega = 1/T$. Sunbstituting these values into (8) we find $D = T$, the result for zeros. In the limit $\omega \gg 1$, setting $N = \omega T$, we have $D = \left(1 - e^{-YT}\right)/Y$.

The duration of a coupon-bearing bond is always smaller than the time-to-maturity, because far-away cashflow dates are "discounted" more than nearby dates. From formula (8) the modified duration of a par bond, which gives the price-yield sensitivity, is

$$D_{mod} = \frac{1}{Y} \cdot \left(1 - \frac{1}{(1 + Y/\omega)^N} \right) .$$

These formulas are useful for estimating the price-yield sensitivity of bonds.

For example, if $N >> 1$ we can make the approximation $D_{mod} \approx 1/Y$. This approximation is exact for **perpetual** or **consol** bonds, which are fixed-income securites that pay a fixed coupon and have no redemption date. Due to the fact that the maturity is infinite, the above formulas apply even if the consol bond is not trading at par, by simply scaling the coupon. The modified duration of a consol is exactly equal to $1/Y$. Moreover, it is easy to see that $Y = C \times (Pr./B)$, where C B is the price and $Pr.$ is the face value. For example, a 9.25% consol bond trading at \$102 has a yield of $0.0925/1.02 = 9.07\%$. The modified duration is $\frac{1}{0.0907} = 11.02$ years. Therefore, according to (7), if the yield varies by 1 basis point (1bp = .01%), the price will vary by $102 \times 11.02 \times 10^{-4}$ dollars, or 11.24 cents.[7]

Treasury bond prices are usually quoted in yield and bonds usually trade close to par (this is true for recently-issued bonds). For example, using formulas (7) and (8), we find that a 30-year U.S. Treasury bond yielding 5.90% and trading at par has a modified duration of 13.98 years. The $1/Y$ approximation would give instead 16.94 years. This means that a 1 basis point variation in yield gives rise to a variation in price of 13.98 cents per 100 dollar face value.

Historically, duration was introduced as a measure of the risk-exposure of a bond portfolio and hence as a hedging tool. The rationale for this is that if we assume that bond yields vary in the same direction and by the same amount, i.e. if the yield curve shifts in parallell, we can use equation (7) to measure the total exposure of a portfolio to a shift in the yield curve. In fact, a portfolio consisting of M bonds with n_1 dollars invested in bond 1, n_2 dollars invested in bond 2, etc., has, under the parallel shift assumption, a first-order variation with respect to yield of

$$\sum_j n_j \frac{dB_j}{B_j} = \left(\sum_j n_j \, D_{mod\ j} \right) dY .$$

Thus, the sensitivity to a parallell shift in yields is equal to the *dollar-weighted modified duration* of the portfolio. A portfolio with vanishing dollar-weighted modified duration has no exposure to parallel shifts in the yield curve.

[7]Corporations sometimes issue very long term bonds, with maturities as long as 100 years, but this is not a common ocurrence. In such cases, the $1/Y$ approximation and the computation of the yield as coupon/price apply. Notice that such "century bonds" are closer in spirit to company preferred stock than to debt, since bondholders receive income but will will not redeem the principal from the issuer.

It has been recognized now for quite some time that duration-based hedging (under the tacit or explicit assumption of parallell shifts of the yield curve) is not precise enough to immunize a fixed income portfolio against interest rate risk. The reason is that yields of different maturities generally do not move together and by the same amount. Appropriate modeling of yield correlations is needed to produce efficient portfolio hedges and to correctly price fixed-income derivatives that are contingent on more than one yield. The modeling of yield correlations is an interesting subject that we will consider in depth in the next chapter.

3. Term rates and forward rates

This section discusses loans and forward-rate agreements between default-free counterparties. The objective is to relate the value of these loans and agrements to a risk-neutral probability (denoted by P).

Term rates are simply-compounded interest rates corresponding to loans of different maturities starting at the present date. For instance, a loan of \$1 for a period of time ΔT years starting today can be expressed in terms of a simply compounded rate $R(\Delta T)$. The relation between this rate and the discount factor of the same maturity is (assuming that today corresponds to $t = 0$),

$$\frac{1}{1 + R(\Delta T) \cdot \Delta T} = \mathbf{E}_0^P \left\{ e^{-\int_0^{\Delta T} r_s \, ds} \right\} = P_0^{\Delta T} .$$

Solving for $R(\Delta T)$, we obtain

$$R(\Delta T) = \frac{1}{\Delta T} \left(\frac{1}{P_0^{\Delta T}} - 1 \right) . \tag{9}$$

The same computation applies to term rates at *future dates*, which are random variables. Let us denote by $R(t, t + \Delta T)$ the term rate at a future date t for a loan of \$1 over the period $(t, t + \Delta T)$. This rate is not known today, since, by definition, it depends on the cost of money at the future t. Nevertheless, generalizing (9), we can express the stochastic rate in terms of conditional expectations and discount factors, *viz.*,

$$\frac{1}{1 + R(t, t + \Delta T) \cdot \Delta T} = \mathbf{E}_t^P \left\{ e^{-\int_t^{t+\Delta T} r_s \, ds} \right\} = P_t^{t+\Delta T} ,$$

where $\mathbf{E}_t^P$ represents the conditional expectation operator given the market information up to time t. Solving for $R(t, t + \Delta T)$, we obtain, as before,

$$R(t,\, t + \Delta T) \;=\; \frac{1}{\Delta T} \;\left(\frac{1}{P_t^{t+\,\Delta T}} \;-\; 1 \right) \;. \tag{10}$$

Notice that this equation gives a relation between two random variables, , the unknown future term rate and the conditional expectation at time t of a functional of the short-term rate over the loan period $(t,\, t + \Delta T)$.[8]

A **forward rate agreement** (FRA) is a contract between two counterparties to enter into a loan at a future date for a specified period of time. The rate at which the counterparties agree to this loan in the future is called a **forward rate**. Forward rate agreements are used by corporations and financial institutions to hedge interest rate risk in their portfolios.

In general, a forward rate depends on two variables: the starting date of the loan and the period of the loan in the future. What is the arbitrage-free value of a forward rate? Let us consider the situation from the lender's perspective. Assuming a notional amount of \$1, and denoting the forward rate by $F(t,\, t + \Delta T)$, the lender agrees to pay one dollar to the borrower at time t and will receive

$$1 \;+\; \Delta T \;F(t,\, t + \Delta T)$$

dollars from the borrower at date $t + \Delta T$. The net present value for the lender is therefore

$$-\; P_0^{t+\Delta T} \;+\; (1 \;+\; \Delta T \;F(t,\, t + \Delta T \;) \;\cdot P_0^{t+\Delta T} \;.$$

Since, by definition, a forward-rate agreement is entered at zero cost, the "fair" value for the contracted interest rate for the period $(t, t + \Delta T)$ should be

$$F(t,\, t + \Delta T) \;=\; \frac{1}{\Delta T} \;\left(\frac{P_0^t}{P_0^{t+\,\Delta T}} \;-\; 1 \right) \;. \tag{12}$$

Another way of phrasing this arbitrage relationship is that *in an FRA, the lender is long a zero with maturity* $t \;+\; \Delta T$ *and short a zero with maturity* t.

Equation (12) can be rewritten in a different form, which is useful to compare FRAs with futures contracts. In fact, we have, from (12),

[8]Note: it is easy to check from this equation that $\lim_{\Delta T \,\to 0} R(t,\, t \,+\, \Delta T) \;=\; r(t)$, a relation linking "real" (*i.e.* discretely compounded) term rates with the mathematical, continuously-compounded, short-term rate r_t.

$$F(t,\,t+\Delta T) \;=\; \frac{1}{\Delta T}\,\frac{1}{P_0^{t+\Delta T}}\cdot\left(P_0^t \;-\; P^{t+\Delta T}\;\right)$$

$$=\; \frac{1}{\Delta T}\,\frac{1}{P_0^{t+\Delta T}}\cdot\mathbf{E}\left\{e^{-\int\limits_0^t r_s\,ds}\;\left(1\;-\;P_t^{t+\Delta T}\;\right)\right\}$$

$$=\; \frac{1}{P_0^{t+\Delta T}}\cdot\mathbf{E}\left\{e^{-\int\limits_0^t r_s\,ds}\;P_t^{t+\Delta T}\,R(t,\,t+\Delta T)\right\}\;,$$

where we used equation (10). Therefore, since $R(t,\,t+\Delta T)$ is known at time t, we have

$$F(t,\,t+\;\Delta T) \;=\; \frac{1}{P_0^{t+\Delta T}}\cdot\mathbf{E}\left\{e^{-\int\limits_0^{t+\Delta T} r_s\,ds}\;R(t,\,t+\Delta T)\;\right\}$$

$$=\; \frac{\mathbf{E}^P\left\{R(t,\,t+\Delta T)\;e^{-\int_0^{t+\Delta T}}\right\}}{\mathbf{E}^P\left\{e^{-\int_0^{t+\Delta T}}\right\}}\;. \tag{13}$$

Thus, the forward rate can be viewed as an expectation of the corresponding term rate in the future, with respect to a probability measure given by expression (13). This probability is not the risk-netural measure, due to the presence of the discount term $e^{-\int\limits_0^t r_s\,ds}$. This fact proves useful in comparing interest rate forwards with interest rate futures.

Interest rate futures are exchange-traded contracts which are closely related to FRAs. The most popular contract is the CME 3-month Eurodollar futures contract, In this case, the underlying rate at the settlement date, $R(t,\,T+0.25)$, represents the average 3-month LIBOR rate on that date.[9]. The contract has a settlement price of $100*(1-R(t,\,t+0.25))$.

With this example in mind, consider a stylized futures contract with settlement date t, settling into the rate $R(t,\,t+\Delta T)$. The futures price today ($t=0$) implies therefore a rate for the loan over the period $(t,\,t+\Delta T)$ equal to

$$F_{fut.}(t,\,t+\Delta T) \;=\; \mathbf{E}_0^P\left\{R(t,\,t+\Delta T)\;\right\}. \tag{14}$$

[9]See Chapter 5 and also Siegel and Siegel, *The Futures Markets*, Probus.

This follows from the fact that, in a risk-neutral world, futures prices and the corresponding implied rates should be martingales.[10]

In particular the expressions for the interest rates implied by futures contracts and the forward rates for the same period are not equal (compare (13) and (14)). From these equations, it follows that

$$F(t,\, t+\Delta T) \;-\; F_{fut.}(t,\, t+\Delta T) \;=\; \frac{\mathbf{E}(R\,D) \;-\; \mathbf{E}(R)\,\mathbf{E}(D)}{\mathbf{E}(D)}$$

where $R \;=\; R(t,\, t+\, \Delta T)\;,\quad D \;=\; e^{-\int_0^t r_s\, ds}$, or

$$F(t,\, t+\Delta T) \;-\; F_{fut.}(t,\, t+\Delta T) \;=\; \mathbf{Cov}\left(R(t,\, t+\, \Delta T)\;,\;\; e^{-\int_0^t r_s\, ds} \right) \tag{15}$$

If we assume a positive correlation between the short rate $r_s,\;\, s\, <\, t$ and the term rate $R(t,\, t+\Delta T)$, which is reasonable, we conclude that R and D are negatively correlated. (For example, this assumption is consistent with the "parallell movements" approximation alluded to earlier.) Negative correlation implies that

$$F(t, t+\Delta T) \;<\; F_{fut.}(t,\, t+\Delta T)\;. \tag{16}$$

This result is consistent with empirical evidence of the spread between ED futures and forwards n the LIBOR market. The difference between futures and forwards is caused by the correlation between short-term financing and the rate under consideration.

3. Interest-rate swaps

A swap is a contractual agreement between two parties in which they agree to make periodic payments to each other according to two different indices.[11]

A "plain vanilla" interest-rate swap specifies the **notional** amount, or face value of the swap, the payment frequency (quartery, semi-annual, etc), the **tenor**, or maturity, the

[10] For example, in the 3-month Eurodollar contract, the implied interest rate is equal to (100-(futures price))%. The martingale property is due to the fact that futures are entered at no cost and are "marked-to-market" daily. Thus, the expected futures price tomorrow must be today's futures price under any risk-neutral probability.

[11] For an in-depth study of swaps, see J.F. Marshall and K. R. Kapner, *Understanding swaps*, Wiley Finance, 1993.

coupon, or fixed rate, and the floating rate. One party (Counterparty A) makes *fixed rate* payments on the stipulated notional. The other party (Counterparty B) makes *floating rate* payments to Counterparty A based on the same notional. Most swaps are arranged so that their value is zero at the starting date.

For US dollar swaps, floating rates are typically the 3-month or 6-month LIBOR rates prevailing over the period before the interest payment is made. (The interest rates are determined *in advance*, or equivalently, the payments are made *in arrears*).

In practice, there are many variants of this basic structure. For instance, swaps can be such that the notionals are different for the two counterparties, or the notional(s) amortize, or where the floating leg is LIBOR plus or minus a fixed coupon, etc. For simplicity, we shall concentrate primarily on standard swaps.

Swaps are the most popular fixed-income derivatives at this writing. They are used for many purposes by many different market participants. One of the most common applications is for hedging interest-rate risk by financial institutions, corporations and large institutional investors. An often-cited motivation for using swaps is the theory of *comparative advantages*: counterparties often have different abilities for borrowing in capital markets. This could be due to differences in credit rating or tax treatment, or for accounting reasons. Swaps can serve, for instance, to change the fixed- coupon debt into floating rate debt and vice-versa. Due to the liquidity of the swap market, they can be used to hedge the interest rate risk of fixed-income positions.[12].

In this section, we derive the basic pricing formulas for swaps. We assume that there are N cash-flow dates in the swap, t_1, t_2, ..., t_N. At each date, the floating rate for the following period set equal to the simply compounded term rate for that period. Accordingly, we model the floating rate for the period $(t_{n-1},\, t_n)$ as

$$R(t_{n-1},\, t_n) \ , \quad n = 1,\, 2,\, ...N \ .$$

If we denote the coupon by F and the notional $Not.$, the stream of cashflows for a swap at inception $(t = t_0 = 0)$

Fixed leg:

$$\Delta_n \cdot F \cdot Not. \quad \text{at dates } t = t_1, \ldots, t_n \quad \text{where} \quad \Delta_n \ = \ t_n \ - \ t_{n-1} \ ,$$

Floating leg:

$$\Delta_n \cdot R(t_{n-1},\, t_n) \cdot Not. \quad 1 \ \leq \ n \ \leq \ N \ .$$

The value at time $t_0 = 0$ of a swap from the point of view of the payer of the fixed leg is

[12] *ibid.*

$$Swap \;=\; \sum_{n=1}^{N} \Delta_n \; Not. \; \mathbf{E}^P \left\{ \left(\; R(t_{n-1},\, t_n) \;-\; F \; \right) \; e^{-\int_0^{t_n} \; r_s \, ds} \; \right\} . \tag{15}$$

This expression can be simplified considerably. Indeed, consider the series of floating-rate payments alone. If we add to this series of payments a fictitious payment of principal ($Not.$) at time t_N, this series becomes equivalent to a as a **floating-rate bond**. Therefore, the cash-flows of a swap are equivalent to those generated by being *long a coupon-bearing bond* and *short a floating rate bond* with the same principal.

As it turns out, the value of a floating rate bond in this case is equal to par. To see this, notice that the value of such floating-rate bond is

$$B \;=\; \sum_{n=1}^{N} \Delta_n \, Not. \; \mathbf{E}^P \left\{ \; R(t_{n-1},\, t_n) \;\; e^{-\int_0^{t_n} \; r_s \, ds} \; \right\}$$

$$+ \; Not. \; \mathbf{E}^P \left\{ \; R(t_{N-1},\, t_N) \;\; e^{-\int_0^{t_N} \; r_s \, ds} \; \right\} . \tag{16}$$

Each term in the series of interest payments can be re-written using (10) and properties of the conditional expectation, as

$$\Delta_n \, Not. \; \mathbf{E}^P \left\{ \; R(t_{n-1},\, t_n) \;\; e^{-\int_0^{t_n} \; r_s \, ds} \; \right\} \;= Not. \; \mathbf{E}^P \left\{ \left(\frac{1}{P_{t_{n-1}}^{t_n}} \;-\; 1 \right) \; e^{-\int_0^{t_n} \; r_s \, ds} \right\}$$

$$= \;\; Not. \; \mathbf{E}^P \left\{ \left(\frac{1}{P_{t_{n-1}}^{t_n}} \;-\; 1 \right) \; P_{t_{n-1}}^{t_n} \;\; e^{-\int_0^{t_{n-1}} \; r_s \, ds} \right\}$$

$$= \;\; Not. \; \left(\; P_0^{t_{n-1}} \;-\; P_0^{t_n} \; \right)$$

Therefore, that

$$B \;=\; \sum_{n=1}^{N} Not. \; \left(\; P_0^{t_{n-1}} \;-\; P_0^{t_n} \; \right) \;+\; Not. \; P_0^{t_N} \;=\; Not.$$

Returning to the valuation of the swap, we have

$$Swap \;=\; \text{Floating rate bond} \;-\; \text{Fixed Coupon bond} \, (Coupon = F)$$

$$= Not. - \left(\Delta_n \cdot F \cdot Not. \sum_{n=1}^{N} P_0^{t_n} + Not. P_0^{t_N} \right) \qquad (17)$$

We can therefore view a swap as *a synthetic position in which the investor buys a coupon-bearing bond, financing the purchase at the floating interest rate* . The crucial difference between the swap and the above transaction is that in a swap there is no exchange of principal; only interest payments are made. This means that

- There is less credit risk – a default will affect only future interest payments, but not the principal.
- The investor does not actually buy the bond - he does not need to utilize capital or credit lines to own or short the bond.

By definition, the **swap rate**(corresponding to a particular maturity and payment schedule) is the fixed rate F that makes the swap value equal to zero. In other words, it is the rate at which ("top credit") counterparties would agree to enter the swap without paying or receiving any premium. The swap rate is computed by setting Swap = 0 in in equation (17) and solving for F. The result is

$$F_{\text{swap}} = \frac{1 - P_0^{t_N}}{\sum_{n=1}^{N} \Delta_n \cdot P_0^{t_N}}. \qquad (18)$$

If this equality does not hold and a par bond with the same maturity and frequency trades in the market, investors can make a profit at no risk. For example, if the swap rate F is *less* than the theoretical F_{swap}. The arbitrage strategy consists in entering into the swap paying fixed (and receiving floating). Such position is equivalent to being long a floating-rate bond and short a coupon-bearing bond with coupon rate F. It can therefore be offset by buying the fixed-rate par bond and financing the purchase at the floating rate. This trade has zero cost and produces a profit due to the fact that the coupon of the par bond (F_{swap} is greater than F. A mirror argument applies if $F > F_{swap}$.

The equivalence between swaps and "long-short" combinations of a fixed and a floating rate bonds implies that we know the "duration" of a swap. In fact, notice that a floating rate bond has very little interest rate risk, since the interest rate resets after each period. Therefore, the "duration" of a floater is essentially $\frac{1}{\omega}$ (the fraction of year between resets). We conclude that the sensitivity of a swap to yield curve movements of a swap is roughly equivalent to the modified duration of the fictitious coupon bond.

Another useful way to view a swap is as *a series of forward rate agreements.* This can be used to derive no-arbitrage relations that must hold between swaps and FRAs. In practice, traders "arbitrage away" any mispricing between swaps of different tenors using swaps, futures and FRAs.

3. Caps and Floors

Caps and floors are a basic type of option associated with swaps and FRAs. Parameters of a cap are

- Notional
- Cashflow dates
- Floating rate
- Strike rate

Denoting the cashflow dates by t_1, t_2, ... t_N, the floating rate by $R(t_{n-1},\, t_n)$, and the strike rate by F, the cashflow at the date t_n is

$$Not. \times (\; t_n \;-\; t_{n-1}\,)\; max\; (R(t_{n-1},\, t_n) \;-\; F,\; 0\;)$$

Notice that payments are made in arrears, to mimic the usual payment of interest rates. A cap is analogous to a series of calls on the floating rate. Similarly, a floor corresponds to a series of puts and has cashflows

$$Not. \times (\; t_n \;-\; t_{n-1}\,)\; max\; (F \;-\; R(t_{n-1},\, t_n)\;,\; 0\;)$$

Caps and floors are options and hence trade at a premium. The premiums of caps and floors are trivially related by the parity relation

$$Swap \;=\; Cap \;-\; Floor$$

Notice that the fixed rate of the swap is the strike so the swap is not "at market". In this case,

$$Swap \;=\; Not. \sum_{n=1}^{N} \Delta_n\,(F_{swap} \;-\; F)\, P_{t_0}^{t_n} \;.$$

Thus, we have

$$Floor \;=\; Not. \sum_{n=1}^{N} \Delta_n\,(F_{swap} \;-\; F)\, P_{t_0}^{t_n} \;+\; Cap\;.$$

seems to have disappeared in the above formula?

The ingredients needed to price caps and floors are (i) the current level of interest rates, (ii) the strike level and (iii) the volatility of future interest rates. In fact, the theoretical value of a cap is

$$Cap \;=\; Not. \sum_{n=1}^{N} (\, t_n \;-\; t_{n-1} \,)\, \mathbf{E}_t^P \left\{ e^{-\int_t^{t_n} r_s\, ds}\, max\, (F \;-\; R(t_{n-1}, t_n)\;,\; 0\;) \right\} \;,$$

where P is the risk-neutral measure. Therefore, a model for interest rate dynamics is needed to compute the premium. This matter will be the object of the next two chapters.

4. Swaptions and bond options

As their name indicates, swaptions are options to enter into swaps. In a **payer swaption**, the investor has the option to enter into a swap with given date, paying the fixed rate (strike) and receiving floating. A **receiver swaption** gives the right to enter into a swap receiving fixed and paying floating. Swaptions can be both European and American. In the latter case, the option can be exercised usually on cashflow dates, rather than "continuously" like American stock-options. American swaptions that can be exercised only at reset dates are often called "mid-Atlantic" or "Bermudan" because they are somewhere between American and European. The parameters that define a swaption are therefore

- Notional
- Maturity of the option
- Payer or receiver
- Type: American or European
- Maturity of the swap
- Cashflow dates of the swap
- Floating rate

For instance, a European **in-5-for-10** LIBOR payer swaption with strike 6.50% is an option to enter into a 10-year swap paying a fixed rate of 6.50% 5 years from now.

Let us denote by $T = t_0$ the maturity date of the option and by t=0 the present date. The payoff of the payer swaption can be expressed in the form

$$max\ (Swap,\ 0\ )\ =\ Not. \times\ max\left[1\ -\ \left(\sum_{n=1}^{N}\Delta_n\, F\, P_{t_0}^{t_n}\ +\ P_{t_0}^{t_N}\right)\ ,\ \ 0\right]$$

The theoretical value of a European payer swaption is

$$Payer\ =\ Not.\, \mathbf{E}_0^P\left\{\ e^{-\int_0^{t_0}\ r_s\, ds}\ max\left[1\ -\ \left(\sum_{n=1}^{N}\Delta_n\, F\, P_{t_0}^{t_n}\ +\ P_{t_0}^{t_N}\right)\ ,\ \ 0\right]\ \right\}\ .$$

An equivalent expression, which follows from equation ( ), is

$$Payer\ =\ Not.\, \mathbf{E}_0^P\left\{\ e^{-\int_0^{t_0}\ r_s\, ds}\ max\left[\ \sum_{n=1}^{N}\Delta_n\ \mathbf{E}_{t_0}^P\left\{e^{-\ \int_{t_0}^{t_n}\ r_s\, ds}\ \ (\ R(t_{n-1},\, t_n)\ \ -\ \ F\ )\ \right\}\ \ ,0\right]\ \right\}.$$

This formula can be used to compare the value of a forward starting cap with that of a European swaption. Since $max(X,\, 0)$ is an increasing function of X, we note that the right-hand side of the last equation can only increase if we replace $(R(t_{n-1},\, t_n)\ -\ F)$ by its positive part $max\,(R(t_{n-1},\, t_n)\ -\ F,\ 0)$. This change increases the value of the conditional expectations in the sum and hence of the sum. However, the value of the sum after the substitution is non-negative. We conclude that that

$$Payer\ \le\ \ Not.\, \mathbf{E}_0^P\left\{\ e^{-\int_0^{t_0}\ r_s\, ds}\ \sum_{n=1}^{N}\Delta_n\ \mathbf{E}_{t_0}^P\left\{e^{-\ \int_{t_0}^{t_n}\ r_s\, ds}\ \ max\,(\ R(t_{n-1},\, t_n)\ \ -\ \ F\ ,\ \ \ 0)\ \right\}\ \ \right\}$$

$$=\ \ Not.\sum_{n=1}^{N}\Delta_n\ \mathbf{E}_0^P\left\{e^{-\ \int_0^{t_n}\ r_s\, ds}\ max\,(\ R(t_{n-1},\, t_n)\ \ -\ \ F\ ,\ \ \ 0)\ \right\}\ \ \ =\ \ Cap.$$

We have just verified what is obvious financially: the theoretical value of a European swaption cannot exceed the value of a cap on the same underlying swap with same rate. Essentially, the cap is a series of calls on interest rates, whereas a payer swaption is a single option on a "basket" of interest rates. The cap guarantees protection against the rise of interest rates above the level F at each period; the swaption provides protection "on average' and therefore should be cheaper.

In the “ideal world” that we are considering here, credit risk is neglected. Furthermore, recall that APT assumes traders can borrow and lend at the same rates in arbitrary amounts. The question of liquidity (i.e. availability of a particular security in the market) is not directly addressed by APT. Because of this, our model cannot address two important aspects of bond pricing: credit quality, on the one hand, and the existence of bonds that are “on special”, i.e. that are in short supply. Credit quality was discussed previously. The second consideration is also very important. Bonds that are in high demand (“on special”) trade at a premium over other bonds of similar maturities. The rate at which these bonds can be financed via repurchase agreements are lower than the standard bond financing rate (repo rate).

Once these issues are separated from the modeling of interest rate risk, we are left with a simplified “world” where bond options can be priced using the same risk-neutral measure as swaptions.

We can write down the a formula for pricing a bond option, using the notation of §1. The intrinsic value of a European-style call option with expiration date T and strike K on a coupon-bearing bond with coupon rate C, payment frequency ω, cashflow dates t_n and principal $Pr.$ is (with the notation of §1)

$$Bondoption \;=\; \mathbf{E}^P \left\{ e^{-\int_0^T r_s\,ds} \; max \left(\sum_{n:\,t_n \,\geq\, T}^{N} \frac{C\cdot\; Pr.}{\omega} P_T^{t_n} \;+\; Pr.\, P_T^{t_N} \;-\; K \;,\; 0 \right) \right\} ,$$

where P is the risk-neutral measure (compare this with equation ()). Notice that payer and receiver swaptions are equivalent, respectively, to calls and puts on a bond with coupon rate F, with strike equal to the principal of the bond (the par amount).

5. The term-structure of interest rates

The examples of the previous paragraphs show that, in general, the cashflows of an interest-rate sensitive security at time t can be expressed in terms of the collection of discount factors

$$P_t^T \;=\; \mathbf{E}_t^P \left\{ e^{-\int_t^T r_s\,ds} \right\} \quad , \; T \;\geq\; t \;.$$

The variable t is the **calendar date** – it corresponds to the date at which interest rates are observed. The variable QT o often called the **forward date**. It labels the maturity date of a loan. The discount factors at time t represent the totality of term rates and forward rates prevailing at time t. This means that all rates corresponding to swaps and FRAs as well as bond yields are functions of the discount fact. Because of this fact, any model of the economy suitable for pricing simultaneously all fixed-income securities must describe the the evolution of the discount factors $\{ P_t^T,\ T > t \}$. Such models are called **term-structure models** because they describe the complete evolution of the term structure of interest rates across time. The reader can think of a term structure model as describing the dynamics of a *curve* (a function of T) across time.

An alternative description of the term-structure of interest rates often favored by theoreticians and practitioners is the so-called **term-structure of instantaneous forward rates**, or term structure of forward rates for short. An instantaneous forward rate is , by definition, the forward rate that would prevail for an infinitesimal time-interval.

Consider first, for simplicity, the current ($t = 0$) term-structure of interest rates. We have seen in §2 that the forward rate for a loan over the period $(T, T + \Delta T)$ is

$$F(T, T + \Delta T) \;=\; \frac{1}{\Delta T} \left(\frac{P_0^T}{P^{T+\Delta T}} \;-\; 1 \right)$$

If we consider a very short time-internal in the future of T, then the corresponding instantaneous forward rate should be defined by the limit

$$f(0; T) \;=\; \lim_{T \to 0} \frac{1}{\Delta T} \left(\frac{P_0^T}{P^{T+\Delta T}} \;-\; 1 \right) \;=\; -\, \frac{\partial}{\partial T} \log \left(P_0^T \right)$$

Here, the first variable in $f(0; T)$ is written to emphasize that these are *current* forward rates, corresponding to $t = 0$. There is a one-to-one correspondence between discount factors and instantaneous forward rates. Indeed, we can recover the discount factors from instantaneous forward rates by the formula

$$P_0^T \;=\; e^{-\int_0^T f(0;s)\, ds}$$

This formalism applies to instantaneous forward at a *future* date t (which are now random variables). We define the instantaneous forward rate $f(t; T)$ at time t for a loan over an infinitesimal period after date T by

$$f(t; T) \;=\; -\, \frac{\partial}{\partial T} \log \left(P_t^T \right) \;,$$

which is equivalent to setting

$$P_t^T \;=\; e^{-\int_t^T f(t;s)\, ds} \;.$$

Notice that, by definition, we have

$$r_t \;=\; f(t;\, t)\;.$$

In some sense, the description of the evolution of interest rates in terms of forward rates rather than discount factors seems more appealing, since the latter contains the short rate as a special case, whereas the description in terms of discount factors requires adding the extra equation defining the spot rate as

$$r_t \;=\; \left[\, -\,\frac{\partial}{\partial T}\, log(P_t^T)\,\right]_{T=t}\;.$$

(This distinction is nevertheless purely formal, since the information contained in $\{P_t^T\}$ and $\{f(t;\, T)\}$ is the same.)

In any case, a model for the *evolution* of the curve of instantaneous forward rates, $\{\; f(t;\, \bullet),\; t \;>\; 0\}$ under a suitable risk-neutral measure would allow us to price the entire universe of interest-rate sensitive securities,in the absence of credit risk and assuming infinite liquidity. More importantly, it would allow us to compare the theoretical values of different securities and to study how their values would change across time. It is important to note that such dynamic model is not needed if we were to limit ourselves to the universe of straight bonds (without call provisions), swaps and FRAs; we showed that the prices of the latter are determined by the *current* term-structure of interest rates. On the other hand, the relative values of bond options, caps, swaptions and other, more exotic, interest rate derivatives can only be assessed by means of a model for the dynamics of the term structure. As we shall see, there exist economic constraints, based on no-arbitrage considerations (APT), that must be taken into account when specifying the evolution of interest rates. In the next lecture, we shall discuss the fundamentals of term-structure modeling from this perspective.