CONTINUOUS-TIME FINANCE: AN INTRODUCTION

1. The basic model

We present a model for a securities market with intertemporal investment opportunities.[1] We assume an idealized world in which the prices of securities (stocks or bonds) and the dividends which they provide are functions of an M-dimensional random process

$$\mathbf{X}(t) \;=\; (\, X_1(t),\, X_2(t),\, X_3(t),\, ...\,, X_M(t)\,) \quad , \quad t \;>\; 0 \;. \tag{1}$$

The random variables $X_i(t)$ are **factors** or **state variables** which describe the economy at any given time. We do not assume necessarily that these variables represent security prices: for instance $X_1(t)$ could represent the value of a consumer price index, the number of laptop computers sold by IBM in 1995 or the inches of rainfall in Iowa in 1994. State variables can represent, more abstractly, *investors' beliefs* about future states of the market. Some state variables $X_i(\cdot)$ may not be tradeable or even observable.

We assume that the state variables follow a system of stochastic differential equations

$$dX_i(t) \;=\; \sum_0^{\nu} \alpha_{i,k}\;(\mathbf{X}(t),\, t\,)\; dZ_k(t) \;+\; \beta_i\;(\mathbf{X}(t),\, t\,)\; dt \quad , \tag{2}$$

for $i \;=\; 1 \ldots M$. Here, $Z_k(\cdot)$, $k \;=\; 1 \ldots \nu$ are independent Brownian motions. Notice that $\alpha_{i,k}$ and β_i are functions of $(\mathbf{X}(t),\, t\,)$ at time t. Thus, in particular, the current state of the system completely determines the local parameters (means and volatilities) and hence the statistics of future moves. The latter property is called the Markov property.[2] The Markovian assumption on the evolution of economic factors is not unreasonable, provided that all relevant past information is reflected in the current state variables.

[1] This lecture is based partly on Cox, Ingersoll and Ross, "An Intertemporal General Equilibrium Model of Asset Prices", *Econometrica*, 53 ,1985, and D. Duffie, *Dynamical Asset Pricing Theory*, Princeton Univ. Press, 1992. See also Chapter 12 in J. Hull, *Options, Futures and other Derivative Securities*, 2nd ed., Prentice-Hall, 1993.

[2] Ito processes which have the Markov property are called diffusion processes.

Another consequence of (2) is the continuity of the paths $X_i(\cdot)$ (recall that Ito processes have continuous paths). This assumption may not apply, in practice, if we expect sudden large-scale jumps in indices or prices (e.g., sudden transition from centralized to free-market economy, devaluation of a currency, sudden bankruptcy, etc.)

We assume that the dividends of traded securities and their market prices are functions the current state variables. Specifically, every security is characterized by its **dividend** process

$$D(t) \;=\; \widetilde{D}\;(\,X_1(t),\,X_2(t)\ldots X_M(t),\,t)$$

and by its **market price**,

$$\begin{aligned} P(t) \;&=\; \widetilde{P}\;(\,X_1(t),\,X_2(t)\ldots X_M(t),\,t) \\ &=\; \widetilde{P}\;(\mathbf{X}(t),\,t\,)\;\;. \end{aligned}$$

By definition, the dividend process represents the cashflow that an investor receives from holding the security. For mathematical simplicity, we think of this cashflow as an annualized yield. Thus, the holder of one unit of this security over a period dt will be able to purchase

$$1 \;+\; D(t)\,dt$$

units of the security after this period.[3]

We shall assume that there is short-term lending, i.e., that there exist short-term default-free notes (e.g. Treasury bills) with maturity dt which pay the holder

$$1 \;+\; r(t)\,dt \;=\; 1 \;+\; \widetilde{R}\;(\mathbf{X}(t),\,t\,)\;dt$$

times face value at maturity. Here, $r(t)$ is the annualized short-term interest rate, or **short rate**. "Rolling over" this short-term note and reinvesting the interest income produces a cumulative wealth of

$$B(t) \;=\; \exp.\left[\int_0^t r(s)\,ds\right] \qquad (3)$$

[3] This concept applies in a straightforward way to investment in foreign currencies or in coupon-bearing bonds. It can be generalized to stocks by using more complicated dividend functions.

times the original investment. (Thus, a money-market account can be viewed as a "security" with $P(t) = 1$ and $D(t) = r(t)$.)

A consequence of this model is that *security prices are Ito processes.* In fact, from the generalized Ito's Lemma, we have

$$dP(t) = \sum_{i=1}^{\nu} \frac{\partial \widetilde{P}}{\partial X_i} \cdot dX_i + \frac{\partial \widetilde{P}}{\partial t} \cdot dt + \frac{1}{2} \sum_{i=1\,,\,j=1}^{\nu} \frac{\partial^2 \widetilde{P}}{\partial X_i \, \partial X_j} \cdot (\, dX_i \, dX_j \,) \; . \tag{4}$$

Using Ito's multiplication rule,

$$dZ_k(t)\, dZ_l(t) = \begin{cases} dt & \text{if } \; k = l \\ \\ 0 & \text{if } \; k \neq l \; , \end{cases}$$

and (2), we find that

$$dX_i(t)\, dX_j(t) = \left(\sum_{k=1}^{\nu} \alpha_{i,k}\, \alpha_{j,k} \right) dt \; .$$

Therefore, the price $P(t)$ satisfies the stochastic differential equation

$$dP = P \cdot \sigma_P(t)\, dZ_P + P \cdot \mu_P \, dt \; , \tag{5}$$

with

$$\sigma_P = \frac{1}{P} \sqrt{ \sum_{k=1}^{\nu} \left[\sum_{i=1}^{M} \alpha_{i,k} \, \frac{\partial \widetilde{P}}{\partial X_i} \right]^2 } \; , \tag{6}$$

and

$$\mu_P = \frac{1}{P} \sum_{i=1}^{M} \beta_i \, \frac{\partial \widetilde{P}}{\partial X_i} + \frac{1}{2P} \sum_{i=1\, j=1}^{M} \sum_{k=1}^{\nu} \alpha_{i,k}\, \alpha_{j,k} \, \frac{\partial^2 \widetilde{P}}{\partial X_i \partial X_j} \; , \tag{7}$$

where Z_P is a Brownian motion defined by the stochastic differential equation

$$dZ_P = \frac{1}{\sigma_P} \sum_{k=1}^{\nu} \left[\sum_{I=1}^{M} \alpha_{i,k} \frac{\partial \widetilde{P}}{\partial X_i} \right] dZ_k$$

$$= \frac{1}{\sigma_P} \sum_{k=1}^{\nu} \sigma_{k,P} \, dZ_k \; .$$

Equations (6) and (7) show that the local parameters driving the price equation (5), (σ_P, μ_P), can be complicated functions of the state-variables. A major challenge in implementing this basic model is to decide which variables determine security prices. Another problem, of econometric nature, is the estimation of the local parameters $\alpha_{i,k}$ and β_i appearing in the dynamical equations (2). Even if it were possible to solve these problems (a big if), one must still find the prices of securities (the functions $\widetilde{P}$) from their cash-flows $\widetilde{D}$. The latter problem has been studied considerably in Mathematical Economics literature. Economists have found that price functions $\widetilde{P}$ can be derived under the assumption that investors behave rationally and maximize some aggregate utility function.[4]

In this lectures, we shall take the point of view that prices and financial indices follow Ito processes, making different assumptions about the local parameters σ and μ according to the problem of interest.

2. Trading Strategies

To fix ideas, we shall assume that there are N traded securities, with prices $P_1(t)$, $P_2(t)$, ... $P_N(t)$. A **trading strategy** consists of an N-tuple of non-anticipative processes (with respect to the economic factors X_i, the Brownian motions Z_k which drive the prices, etc.)

$$\Theta(t) = (\theta_1(t), \theta_2(t), ... \theta_N(t)) \; . \tag{8}$$

The entry $\theta_i(t)$ represents the number of units of the i^{th} security held in an investment portfolio at time t. The value of this portfolio at time t is

$$V(t) = \sum_{j=1}^{N} \theta_i(t) \, P_i(t) \; . \tag{9}$$

We will assume that investors can purchase and short-sell arbitrary numbers of securities.[5] We also neglect transaction costs and bid-offer spreads in this first analysis.

[4] See Cox, Ingersoll and Ross and/or D. Duffie.

[5] In particular, the variables θ_i can take arbitrary real values.

For simplicity, it is convenient to restrict our attention to trading strategies which are **self-financed**. These are strategies in which the investor reallocates his wealth among different investments without adding or withdrawing capital. After a small period of time dt, the value of the portfolio (8) would be

$$V(t+dt) \approx V(t) + \sum_{j=1}^{N} \theta_i(t)\, dP_i(t) + \sum_{j=1}^{N} \theta_i(t)\, P_i(t)\, D_i(t)\, dt ,$$

where $P_i(t)\, D_i(t)\, dt$ represents the dividend flow from the the i^{th} security. If the strategy $\Theta(\cdot)$ is self-financed, then the value of the portfolio at time $t+dt$ after trading should also satisfy

$$V(t+dt) = \sum_{j=1}^{N} P_i(t+dt)\, \theta_i(t+dt) .$$

Thus, a self-financed trading strategy in a non-anticipative process that satisfies the stochastic differential equation

$$\begin{aligned} dV &= d\left(\sum_{i=1}^{N} \theta_i\, P_i \right) \\ &= \sum_{j=1}^{N} \theta_i\, dP_i + \sum_{j=1}^{N} \theta_i\, P_i\, D_i\, dt . \end{aligned} \tag{10}$$

Often, it is convenient to analyze the profits/losses of trading strategies in constant dollars, i.e. discounting with respect to the short-term interest rate. The value of a portfolio worth $V(t)$ at time t in dollars-at-time-0 is

$$\hat{V}(t) \equiv \frac{V(t)}{B(t)} = e^{-\int_0^t r(s)\, ds}\, V(t) .$$

Using this equation and (10), we find that the self-financing equation in constant dollars is

$$\begin{aligned} d\left(\frac{V(t)}{B(t)}\right) &= \frac{1}{B(t)} \sum_{j=1}^{N} \theta_i(t)\, dP_i(t) + \frac{1}{B(t)} \sum_{j=1}^{N} \theta_i(t)\, P_i(t)\, (\, D_i(t) - r(t)\,)\, dt \\ &= \sum_{j=1}^{N} \theta_i(t)\, d\left(\frac{P_i(t)}{B(t)}\right) + \sum_{j=1}^{N} \theta_i(t) \left(\frac{P_i(t)}{B(t)}\right) D_i(t)\, dt , \end{aligned} \tag{11}$$

or, with $\hat{P}_i \;=\; P_i / B$,

$$d\hat{V} \;=\; \sum_{j=1}^{N} \theta_i \, d\hat{P}_i \;+\; \sum_{j=1}^{N} \theta_i \, \hat{P}_i \, D_i \, dt \; . \tag{12}$$

This equation is analogous to (10); it just expresses the self-financing relation in a different unit of account.

3. Arbitrage Pricing Theory.

Definition. *A self-financed trading strategy* $\{\Theta(t) \quad, 0 \,\leq\, t \,\leq\, T\,\}$ *is said to be an* arbitrage strategy *if the profit/loss that it generates,*

$$\hat{V}(T) \;-\; V(0) \;=\; \sum_{j=1}^{N} \int_0^T \theta_i(t) \, d\hat{P}_i(t) \;+\; \sum_{j=1}^{N} \int_0^T \theta_i(t) \, \hat{P}_i(t) \, D_i(t) \, dt \;\; ,$$

is (i) non-negative with probability 1, and (ii) positive with positive probability.

In previous lectures, we discussed arbitrage strategies from both the theoretical and practical points of view. The most common "arbitrages" are cash-and-carry trades, which try to exploit discrepancies between spot and forward prices. Other types of arbitrage opportunities may arise if, say, options can be replicated with market instruments (statically or dynamically) at a profit. Arbitrage opportunities may arise sporadically but they cannot subsist for long in efficient markets due to the forces of supply and demand.

The following result shows that the securities market model must satisfy certain constraints if there are no arbitrage opportunities. Roughly speaking, the constraints can be viewed as a relation between the volatility and the returns of traded securities.

Proposition 1. *Under the assumptions of the basic model, assume that security prices satisfy*

$$dP_i(t) \;=\; P_i(t) \left\{ \sum_{k=1}^{\nu} \sigma_{k,i}(t) \, dZ_k(t) \;+\; \mu_i(t) \, dt \right\} \;, \quad i = 1, \, ... \, N \;, \tag{13}$$

where $Z_k(\cdot)$ are independent Brownian motions. Let $D_i(t)$ denote the corresponding dividend processes and let $r(t)$ be the short-term interest rate. Then, if there are no arbitrage opportunities, there exist non-anticipative processes $\lambda_1(t)$, $\lambda_2(t)$, ... $\lambda_\nu(t)$ such that

$$\mu_i(t) \ + \ D_i(t) \ - \ r(t) \ = \ \sum_{k=1}^{\nu} \sigma_{k,i}(t) \ \lambda_k(t) \ . \qquad (14)$$

This proposition, which we will prove shortly, has an important corollary (sometimes referred to as the Fundamental Theorem of Arbitrage Pricing Theory):

Proposition 2. *Under the assumptions of the basic model, let* **P** *represent the probability measure on path-space associated with the security prices (13). A necessary and sufficient condition for the existence of no arbitrage opportunities over the time-interval $(0,T)$ is that there exists an equivalent probability measure* **Q** *such that all security prices satisfy*

$$P_i(t) \ = \ \mathbf{E}_t^Q \left\{ e^{-\int_t^T (\, r(s) - D_i(s)\,)\, ds} \ P_i(T) \right\} \quad , \quad i = 1, \, 2, \, \ldots N. \qquad (15)$$

Proof of Proposition 1: First, we will show that if there are no arbitrage opportunities equation (14) holds. For this, we consider the possibility of a specific "single-period" arbitrage at some point in time t. More precisely, assume that there exist quantities $\theta_i(t)$ ($i = 1, \ldots N$) such that the profit/loss over a small time-interval dt,

$$dV(t) \ = \ d\left(\sum_{i=1}^{N} \theta_i(t) \, dP_i(t) \right) \ ,$$

has *zero variance*, given the past up to time t. Consider a strategy that corresponds to holding the portfolio $\Theta(t) \ = \ (\, \theta_i(t)\,)_{i=1}^N$, over the period $(t, t+dt)$ and subsequently closing the position after dividends are paid out. The return on this investment should equal the return of a riskless money-market account over the same period. Indeed, the existence of two different riskless rates of return in the market would give rise to an obvious arbitrage opportunity.[6] Therefore, if $dV(t)$ has variance zero we must have

$$dV(t) \ = \ r(t) \, V(t) \, dt \ . \qquad (16)$$

[6]If the return is higher than $r(t)$, borrow money and purchase the portfolio. If the return is lower, short-sell the portfolio and invest the proceeds in short-term funds.

Now, using equation (13), we find that

$$dV(t) \;=\; \sum_{i=1}^{N} \theta_i(t)\,P_i(t)\,\left(\sum_{k=1}^{\nu} \sigma_{k,i}(t)\;dZ_k(t)\right) \;+\; \sum_{i=1}^{N} \theta_i(t)\,P_i(t)\;(\,\mu_i(t)\;+\;D_i(t)\,)\,dt$$

$$=\; \sum_{k=1}^{\nu} \left(\sum_{i=1}^{N} \theta_i(t)\,P_i(t)\,\sigma_{k,i}(t)\right)\;dZ_k(t)\;+\;\sum_{i=1}^{N} \theta_i(t)\,P_i(t)\;(\,\mu_i(t)\;+\;D_i(t)\,)\;dt \qquad (17)$$

It follows that $dV(t)$ has zero variance if and only if

$$\sum_{i=1}^{N} \theta_i(t)\,P_i(t)\,\sigma_{k,i}(t) \;=\; 0 \quad , \quad k = 1, \ldots \nu\;, \qquad (18)$$

and, moreover, that in the latter case equation (16) can be written as

$$\sum_{i=1}^{N} \theta_i(t)\,P_i(t)\;(\,\mu_i(t)\;+\;D_i(t)\,)\;dt \;=\; \sum_{i=1}^{N} \theta_i(t)\,P_i(t)\,r(t)\,dt\;,$$

or

$$\sum_{i=1}^{N} \theta_i(t)\,P_i(t)\;(\,\mu_i(t)\;+\;D_i(t)\;-\;r(t))\;=\;0\;. \qquad (19)$$

The conclusion is that if there are no arbitrage opportunities then *whenever the ν equations in (18) hold, equation (19) must hold as well.*

It is useful to re-interpret this in "geometric" terms. Define $\nu + 1$ vectors in N-dimensional Euclidean space by

$$\mathbf{s}_k \;=\; (\,\sigma_{1,k},\,\ldots\,\sigma_{N,k}\,) \qquad , k = 1, \ldots \nu\;,$$

and

$$\mathbf{m} \;=\; (\,\mu_1(t)\;+\;D_1(t)\;-\;r(t)\,,\;\ldots,\;\mu_N(t)\;+\;D_N(t)\;-\;r(t)\,)\;.$$

The statement "(18) implies (19)" is equivalent to saying that "whenever a vector is orthogonal to $\mathbf{s}_k$, $k \;=\; 1, \ldots \nu$, then it is also orthogonal to $\mathbf{m}$". It follows from linear algebra that this condition holds if and only if $\mathbf{m}$ is contained in the linear subspace generated by the vectors $\mathbf{s}_k$.[7] We conclude that there must exist functions $\lambda_k(t)$, $k = 1, .., \nu$, such that

$$\mathbf{m} \; = \; \sum_{k=1}^{\nu} \lambda_k(t)\, \mathbf{s}_k \;,$$

or, for all i,

$$\mu_i(t) \; + \; D_i(t) \; - \; r(t) \; = \; \sum_{k=1}^{\nu} \lambda_k(t)\, \sigma_{i,k} \; . \tag{20}$$

We have thus shown that (20) must hold in the absence of arbitrage. (Note that the scalars λ_k are generally functions of t, because the argument leading to (20) is "local in time".)

Conversely, let us establish that (20) is a sufficient condition for the absence of arbitrage.In fact, substituting the values for μ_i derived from this condition into the equation for security prices (13), we have

$$dP_i(t) \; = \; P_i(t) \cdot \left\{ \sum_{k=1}^{\nu} \sigma_{i,k}(t) \; (\, dZ_k(t) \; + \; \lambda_k(t)\, dt \;) \; + \; (\, r(t) \; - \; D_i(t)\,) \; dt \right\}$$

$$= \; P_i(t) \cdot \left\{ \sum_{k=1}^{\nu} \sigma_{i,k}(t)\, dW_k(t) \; + \; (\, r(t) \; - \; D_i(t)\,) \; dt \right\} \;, \tag{21}$$

where we set

$$W_k(t) \; \equiv \; Z_k(t) \; + \; \int_0^t \lambda_k(s)\, ds \; . \tag{22}$$

By Girsanov's Theorem, the processes $W_k(\cdot)$ are distributed like independent Brownian motions under the modified probability

$$\mathbf{Q}\,\{\,\mathcal{S}\,\} \; = \; \mathbf{E}\left\{ \mathcal{S}\; ; \; \exp.\left[-\sum_{k=1}^{\nu} \int_0^T \lambda_k(s)\, dZ_k(s) \; - \; \frac{1}{2} \int_0^T \sum_{k=1}^{\nu} \int_0^T \lambda_k^2(s)\, ds \right] \right\} \; . \tag{23}$$

[7] A proof of this result is included as an Appendix.

As shown in the previous lecture, equation (21) implies that $P_i(\cdot)$ satisfies the equation

$$P_i(t) \; = \; P_i(0) \cdot \, M_i(t) \cdot e^{\int_0^t (\, r(s) \, - \, D_i(s) \,) \, ds} \tag{24}$$

where

$$M_i(t) \; \equiv \; \exp. \left[\int_t^T \sum_{k=1}^{\nu} \sigma_{i,k}(s) \, dW_k(s) \; - \; \frac{1}{2} \int_t^T \sum_{k=1}^{\nu} \left(\sigma_{i,k}(s) \, \right)^2 \, ds \right] \tag{25}$$

is an exponential martingale under **Q**. Hence, the process

$$e^{-\int_0^t (\, r(s) \, - \, D_i(s) \,) \, ds} \, P_i(t) \; = \; e^{\int_0^t D_i(s) \, ds} \cdot \hat{P}_i(t)$$

is also a martingale under **Q**.

From the self-financing equation (12), we conclude that the discounted value of any self-financed trading strategy satisfies

$$d\hat{V}(t) \; = \sum_{i=1}^{N} \theta_i(t) \left[d\hat{P}_i(t) \; + \; D_i(t) \, \hat{P}_i(t) \, dt \, \right]$$

$$= \; \sum_{i=1}^{N} \theta_i(t) \, e^{-\int_0^t D_i(s) \, ds} \, d \left(e^{-\int_0^t (\, r(s) \, - \, D_i(s) \,) \, ds} \, P_i(t) \right)$$

$$= \; \sum_{i=1}^{N} \theta_i(t) \, e^{-\int_0^t D_i(s) \, ds} \, P_i(0) \, dM_i(t) \; .$$

In particular, $\hat{V}(t)$ is a martingale under **Q** and

$$\mathbf{E}^Q \left\{ \hat{V}(t) \; - \; \hat{V}(0) \right\} \; = \; 0. \tag{26}$$

The fact that self-financed strategies are martingales under **Q** implies that arbitrage strategies cannot exits. In fact, since **P** and **Q** are equivalent, if the profit/loss generated by a strategy, $\hat{V}(T) - \hat{V}(0)$, is non-negative with **P**-probability 1, then it must be non-negative with **Q**-probability 1. But then, by (26), it must vanish with **Q**-probability 1. Using again the equivalence of the two measures, we conclude that $\hat{V}(T) - \hat{V}(0)$ must vanish with **P**-probability 1.

The proof of Proposition 1 is complete.

Proof of Proposition 2. In the proof of Proposition 1, we established that a necessary condition for the absence of arbitrage is that

$$e^{-\int_0^t (\, r(s) - D_i(s) \,)\, ds} \; P_i(t) \quad , \quad 0 \le t \le T \, , \tag{27}$$

is a martingale under the measure **Q** defined in (23). Therefore,

$$\mathbf{E}_t^Q \left\{ e^{-\int_0^T (\, r(s) - D_i(s) \,)\, ds} \; P_i(T) \right\} = e^{-\int_0^t (\, r(s) - D_i(s) \,)\, ds} \; P_i(t) \; .$$

Multiplying both sides of this equation by $e^{\int_0^t (\, r(s) - D_i(s) \,)\, ds}$ (which is measurable with respect to the past up to time t), we find that

$$\mathbf{E}_t^Q \left\{ e^{-\int_t^T (\, r(s) - D_i(s) \,)\, ds} \; P_i(T) \right\} = P_i(t) \quad , \quad t \le T \; . \tag{28}$$

This establishes (28) as a necessary condition for no-arbitrage.

Conversely, if equation (28) holds for some probability **Q** which is equivalent to **P**, then the process (27) is a martingale under **Q**. But then, we can follow verbatim the argument presented in the proof of Proposition 1[8] to conclude that there are no arbitrage strategies. Q.E.D.

[8] See p. 10 after eq. (25).

Another useful result which we have established along the way is

Proposition 3. *Under the assumptions of the basic model, a necessary and sufficient condition for the absence of arbitrage over the time interval* $[0,\, T\,]$ *is that there exists an equivalent probability* $\mathbf{Q}$ *such that the value of any self-financed trading strategy in constant dollars* $\hat{V}(t)$ *is a martingale under* $\mathbf{Q}$*. In particular,*

$$V(t) \;=\; \mathbf{E}_t^Q \left\{ e^{-\int_t^T r(s)\,ds}\; V(T) \right\} . \tag{29}$$

In most applications of Arbitrage Pricing Theory to pricing derivative securities, we shall focus our attention primarily on the measure $\mathbf{Q}$. The reason for this is that Proposition 2 relates the values of traded securities with their expected future cash-flows **under Q**. Since derivatives are securities with cash-flows which depend on the values of other securities, equation (15) provides a framework for pricing them in terms of their "payoffs" at a future date. In a similar vein, the knowledgeable reader will interpret (29) as a pricing formula for contingent claims based upon the notion of *replicating portfolio*: if a derivative security delivers a (single) payoff of $V(T)$ dollars at a date T, and if $V(T)$ is also the value at time T of a self-financed strategy $\Theta(s)\,, t \,\le\, s \,\le\, T$, then the value of the derivative security at time t rpresents the cost of entering into this dynamic strategy at time t, i.e. $V(t)$.

Some "concrete" applications of this theory are presented in the following lecture.