EXOTIC OPTIONS I

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Exotic options are a generic name given to derivative securities which have more complex cash-flow structures than standard puts and calls. The principal motivation for trading exotic options is that they permit a much more precise articulation of views on future market behavior than those offered by "vanilla" options. Like options, exotics can be used as part of a

risk-management strategy or for speculative purposes. From the investor's perspective, some exotics provide high leverage because they can focus the payoff structure very precisely (this is the case of barrier options discussed below). Exotics are usually traded over the counter and are marketed to sophisticated corporate investors or hedge funds. Exotic option **dealers** are generally banks or investment houses. They manage their risk-exposure by

- making two-way markets and attempting to be market-neutral as much as possible, and
- hedging with their "vanilla" option book and cash instruments.

The risk-management of exotics is more delicate than that of standard options because they are less liquid. This means that the seller of an exotic may not be able to buy it back if his theoretical hedging strategy failed without having to pay a large premium. Therefore, making a market in exotic options requires acute timing skills in hedging and the use of options to manage volatility risk. Roughly speaking, we can say that " exotic options are to standard options what options are to the cash market". By this we mean that exotic options are very sensitive to higher-order derivatives of option prices such as Gamma and Vega. Some exotics can be seen essentially as bets on the future behavior of higher order "Greeks" Gamma and Vega. Another important issue is the notion of **pin risk**: since some exotics have discontinuous payoffs, they can have huge Deltas and Gammas near expiration that make them very difficult, if not impossible, to Delta-hedge.

1. List of the most common exotics

- Digital options
- Barrier options
- Look-back options
- Average-rate options (Asian options)

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

- Options on baskets
- Forward-start options
- Compound options (options on options)

This lecture gives an introduction to these derivatives. We will discuss various aspects of exotics, namely (i) binomial tree pricing, (ii) closed-form solutions assuming the spot price follows a Geometric Brownian Motion, (iii) price sensitivity and hedging. The second point will require that we develop several mathematical results on the distribution of first-passage times and of the supremum of Brownian motion with drift over a given time-interval.

Aside from providing an introduction to these instruments, this study is interesting because it gives us a better perspective on the risks associated with hedging derivative products in general, including the risk-management of portfolios of standard options.[1]

2. Digital options

A **digital**, or **binary** option is a contingent claim on some underlying asset or commodity that has an "all-or-nothing" payoff. A digital call has a the payoff

$$F(S_T) \;=\; \begin{cases} 1 \quad \text{if} \;\; S_T \;\geq\; K \\ \\ 0 \quad \text{if} \;\; S_T \;\geq\; K \end{cases} \tag{1}$$

(A digital put has payoff $1 - F(S_T)$). Like standard options, digitals can be classified as **European** or **American** style. The European digital provides a payoff of \$1 if the asset end above the strike price at the option's maturity date and zero otherwise. The American digital has a payoff of \$1 if the underlying asset reaches the value K before or at the expiration date T.

2.1 European Digitals

The fair value of the digital call with payoff (1) can be derived easily under the assumptions of lognormal prices (the "Black-Scholes world'). In fact, the fair value of the digital is given by

$$V(S,T) \;=\; e^{-r\,T}\,\mathbf{E}\left\{\,H(S_T \,-\, K)\right\} \;=\; e^{-r\,T}\,\mathbf{P}\left\{S_T \,\geq\, K\right\}\,, \tag{2}$$

[1] The material for this lecture was taken from various research publications and my own notes. Recommended reading: (a) J.Hull: *An introduction to ...*, chapter on exotics (b) Mark Rubinstein: *Exotic options*, preprint Berkeley University, 1992, (a compilation of his articles in *RISK* Magazine.) and (c) *"From Black-Scholes to black holes"*, anther compilation of articles by several authors from *RISK* Magazine.

where r is the interest rate (assumed constant) and the expectation is taken with respect to a risk-neutral probability. Here,

$$H(X) \;=\; \begin{cases} 1 & \text{if} X \;\geq\; 0 \\ \\ 0 & \text{if} X \;<\; 0 \end{cases}$$

is the **Heaviside step function**. The calculation of the last probability in (2) is straightforward. Since the terminal price of the underlying asset satisfies

$$S_T \;=\; S\, e^{\sigma\, Z\, \sqrt{T}\; + (r - q - \frac{1}{2}\sigma^2)\, T} \;,$$

where Z is normal with mean zero and variance 1, we have

$$\mathbf{P}\left\{S_T \;\geq\; K\right\} \;=\; \int\limits_{Z_K}^{+\infty} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{z^2}{2}}\, dz$$

$$=\; N(-Z_K)\;.$$

Here $N(\cdot)$ is the cumulative distribution function of the standard normal and Z_K is defined by the equation

$$S\, e^{\sigma\, Z_K\, \sqrt{T}\; + (r - q - \frac{1}{2}\sigma^2)\, T} \;=\; K\;,$$

i.e.,

$$Z_K \;=\; \frac{1}{\sigma\,\sqrt{T}} \left\{ \ln\left(\frac{K}{S}\right) \;-\; (r - q - \frac{1}{2}\sigma^2)\, T \;\right\}\;.$$

Therefore, defining

$$d_2 \;\equiv\; -\, Z_K \;=\; \frac{1}{\sigma\,\sqrt{T}} \ln\left(\frac{S\, e^{(r-q)\, T}}{K}\right) -\; \frac{1}{2}\,\sigma\,\sqrt{T}\;, \tag{3}$$

we conclude that the fair value of the European digital call is given by

$$V(S, T) \;=\; e^{-r\, T}\, N\left(d_2\right)\;. \tag{4}$$

This formula resembles strongly the expression derived previously for for the cash-amount to be held in the equivalent hedging portfolio for a *vanilla call*, namely

$$- K\, e^{-r\,T}\, N\,(d_2)\ .$$

The resemblance is not accidental. The holder of a European call is – by equivalence of the final cash-flows – long a contingent claim that delivers one share if $S_T \geq K$ or nothing if $S_T < K$ and short K European digital options. In other words, the standard call can be viewed as a "portfolio" of two digital options (one digital with payoff consisting of one share and $-K$ digitals with payoff of \$ 1). On the other hand, from the Black-Scholes formula, the value of the call is

$$S \cdot e^{-q\,T}\, N(d_1)\ -\ K \cdot e^{-r\,T}\, N(d_2)\ , \tag{5}$$

where

$$d_1\ =\ \frac{1}{\sigma\sqrt{T}} \ln\left(\frac{S\, e^{(r-q)\,T}}{K}\right) +\ \frac{1}{2}\,\sigma\sqrt{T}\ .$$

We can interpret the two terms in this formula as the values of the two digital payoffs that "make up" the standard option.

Mathematically, European digital options are even simpler to price than standard options. On the other hand, from the point of view of risk-management, the difference between vanillas and digitals is substantial. There are two fundamental differences:

- Digital options have **mixed convexity**
- Digitals have **discontinuous payoffs**.

The issue of mixed convexity is important for the hedger because it means that the risk-exposure is complex as volatility changes. Recall that if the hedger is **short Gamma** he is vulnerable to large moves in the underlying asset whereas if he is **long Gamma** he is vulnerable to small moves, i.e. to time-decay. In contrast, if the hedger is short a standard option, the risk is just one-sided because the position is short Gamma at all levels of spot. In particular, when making a market in digital options, agents may not gain a market advantage by quoting a price with an implied volatility that is higher than the one of standard options.

Let us examine these issues by looking at the Greeks of the European digital. Differentiation with respect to S in (5) gives

$$\Delta_{digital}\ =\ \frac{e^{-r\,T}\, e^{-\frac{d_2^2}{2}}}{S\sqrt{2\pi\,\sigma^2\,T}} \tag{6}$$

and

$$\Gamma_{digital}\ =\ -\ \frac{e^{-r\,T}\, e^{-\frac{d_2^2}{2}}}{S^2\,\sigma^2\,T\sqrt{2\pi}} \cdot d_1\ . \tag{7}$$

These sensitivities become large as $T \to 0$ for $S \approx e^{-(r-q)\,T}\,K$. As T converges to zero, the Delta of the Digital option approaches the Dirac delta-function. This has two consequences: first, far away from expiration the value of the digital option is small compared to \$1 and the Deltas and Gammas are small. As the expiration date approaches, hedging the digital becomes much more complicated due to the unbounded Deltas and Gammas. The second consequence is **pin risk**: if the price of the underlying asset oscillates around the strike price near expiration, the hedger will have to buy and sell large numbers of shares very quickly to replicate the option. At some point, the amount of shares bought or sold can be so large that the risk due to a small change of the stock price may exceed the maximum liability of the digital. At this point, Delta-hedging becomes extremely risky.[2]

The Gamma of the option vanishes for

$$S \;=\; S^*(T) \;=\; K \cdot e^{-(r-q)\,T} \cdot e^{\frac{1}{2}\sigma^2\,T}\;. \tag{8}$$

(This is the value of spot for which the Delta of the standard call is exactly 1/2). For $S < S^*(T)$, Gamma is large and negative. This means that the hedger is exposed to significant risk near expiration if there is a big move in the spot price. If $S > S^*(T)$, the hedger is subject to time-decay risk: he must rebalance his position frequently in order to offset time-decay. When $S = S^*(T)$ the hedger is subject to risk from both large and small moves!

The sensitivity of the price of the digital with respect to the volatility parameter is

$$\begin{aligned} Vega_{\,digital} \;&=\; \frac{\partial V(S,T)}{\partial \sigma} \\ &=\; -\;\frac{e^{-r\,T}\,e^{-\frac{d_2^2}{2}}\,\sigma}{\sqrt{2\pi}} \cdot d_1\;. \end{aligned} \tag{9}$$

Vega also changes sign at $S = S^*(T)$. In, particular, the seller of the option is vulnerable to an increase or a decrease in market volatility according to whether S is smaller or greater than $S^*(T)$.

One way to understand the European digital option in terms of standard options is in term of **call spreads**. Recall that a call spread is a position which consists of being long one call with a given strike and short another call with a different strike. Let ϵ denote a small number.Then, the position

[2] To understand better pin risk, we should recall that prices do not change continuously — the lognormal approximation is just a convenient device for generating simple pricing formulas. The *discrete nature* of price movements can make continuous-time hedging techniques very risky when Delta changes rapidly as the spot moves. This observation applies also to highly leveraged option portfolios.

- long $1/\epsilon$ calls with strike $K-\epsilon$ and
- short $1/\epsilon$ calls with strike K ,

where the calls have the same expiration date as the digital, has a final payoff

$$Min\ \{ Max\ [\, S_T - (K-\epsilon)\, ,\, 0\,]\ ,\ 1\ \}\ . \tag{10}$$

This function is greater than $H(S_T-K)$ for all S_T. This implies that the value of the digital is less than that of $1/\epsilon$ $(K-\epsilon, K)$ call-spreads. We say that the call spreads **dominate** the digital. This observation suggests that a good hedging strategy for digital options would be to use call-spreads instead of Delta-hedging, i.e. to adopt a **static** hedging strategy. If ϵ is large, this will require relatively few options, but the fair value of the spread may be much greater than that of the digital. On the other hand, diminishing ϵ will make the difference in prices arbitrarily small but the hedge requires many options. This may be difficult and costly to execute. Moreover, options with strikes very close to K may not exist in the market. Nevertheless, the idea of using call spreads is a useful one. In fact, hedging with an option spread which *approximates* the binary payoff (but not necessarily dominates it or replicates it exactly) can help offset pin risk by diminishing the magnitude of the jump. In other words, a portion of the risk can be diversified by hedging with options and the **residual**, i.e. the difference between the payoffs of the digital and the option spread, can be hedged in the cash market at a lesser risk.

Example: An important example where digitals appear in finance is in the pricing of **contingent premium options**. These are derivative securities which are structured as standard European options, except for the fact that the holder pays the premium *at maturity* and *only if the option is in the money*. A contingent premium option can be viewed as a portfolio consisting of

- Long one standard option with strike K and maturity T,
- Short V binary calls with strike K and maturity T,

where V represents the premium, to be paid if and only if $S_T > K$. The intrinsic value of a contingent-premium call option is therefore zero for $S_T \leq K$ and $S_T - K - V$ for $S_T > K$. Notice, in particular, that the investor makes money only if $S_T > K + V$ and will actually lose money if $K < S_T < K + V$. (The situation is analogous to that of a person which has "free" medical insurance but with with a large "deductible".) If the option is structured so that no down-payment is required, then, according to equations (4) and (5), V should satisfy

$$e^{-rT}\, \mathbf{E}\ \{ Max\, (\, S_T - K\, , 0\,)\, \}\ -\ e^{-r\,T}\, V\, \mathbf{P}\ \{ S_T > K\ \}\ =\ 0. \tag{11}$$

Therefore, from (4) and (5), the "fair" deferred premium should be

$$V\ =\ S\, e^{(r-q)\,T}\, \frac{N(d_1)}{N(d_2)}\ -\ K \tag{12}$$

More generally, such options can be structured so that a portion of the premium is paid upfront and another is contingent on the option being in-the-money at maturity . All such options have "embedded" European binaries. The idea was implemented in recent years for designing debt securities known as **structured notes**. A simple example of a structured note would consist of a note with coupons indexed to LIBOR with the following characteristics:

- Coupon payment = $Max\,(\text{LIBOR}, 5\%)$
- Contingent premium = 0.25 % if LIBOR < 5% on the coupon date.

This note guarantees the holder a "floor" of 5% on the interest rate income. The structure resembles that of a floating-rate note with an interest-rate floor (series of puts on interest rates.) However, the investor does not pay for the floor when he buys the structured note. Instead, he can (and also must) take advantage of the interest-rate floor by paying 0.25% (25 basis points) if LIBOR goes below 5% on any given coupon date. This derivative security could be desirable to investors that believe that if interest rates go below 5% then they will be significantly *below* 5% for some period of time. Also, the structure may be desirable to investors seeking protection if interest rates drop but who are unwilling to finance the interest-rate insurance upfront.

2.1 American digitals

The payoff of the American digital option is similar, but now the holder receives \$1 *if and when the the underlying asset trades above \$K for the first time.* This introduces additional time-optionality to the problem. The "fair value" of the American digital is, according to the general principles,

$$V(S,T) \;=\; \mathbf{E}\{\, e^{-r\,\tau}\,;\, \tau \leq\; T\}\;, \tag{13}$$

where expectation is taken with respect to a risk-neutral probability measure and τ represents the **first hitting time** of the strike price $S = K$.

It is worthwhile to consider the two valuation methods that we are familar with: the binomial pricing model and the lognormal approximation. As we shall see, the former is relatively easy to program. The lognormal approximation gives insight into the sensitivities of the price with respect to the model parameters, aside from being useful to "benchmark" the binomial tree calculation. However, ir requires introducing new tools from Probability Theory.

To implement the binomial approach, we proceed as follows: by definition, the value of the American binary is \$1 if the option is in the money. Therefore, with the usual notation, we have

$$V_n^j \;=\; 1 \quad \text{if} \quad S_n^j \;\geq\; K \tag{14a}$$

and

$$V_N^j \;=\; 0 \quad \text{if} \quad S_N^j \;<\; K \; . \tag{14b}$$

The portion of the binomial tree which needs to be determined by roll-back corresponds to the nodes (n, j) such that

$$S_n^j \;<\; K \quad \text{with} \;\; n \;<\; N \; . \tag{15}$$

The value of the binary option at these nodes is calculated using the familiar recursive relation

$$V_n^j \;=\; e^{-r_n\, dt} \cdot \left\{ P_U^{(n)}\, V_{n+1}^{j+1} \;+\; P_D^{(n)}\, V_{n+1}^{j} \right\} \; . \tag{16}$$

The only difference, from the numerical point of view, between American and European binaries resides in the fact that the value at the nodes which are one step away from the boundary $\{S = K\}$ are computed using equation (14a) for the value of V_{n+1}^{j+1} (i.e. setting $V_{n+1}^{j+1} \;=\; 1$). This type of problem, that involves a lateral boundary condition, is known as a **Dirichlet problem** or **boundary-value problem** in the theory of Partial Differential Equations.

Notice that the formulation (16) allows for term-structures of volatility and interest rates. This point is significant because the hitting time of the barrier is unknown. Therefore, the effect on pricing due to a time-varying volatility is non-trivial.[3] From the point of view of interest-rate and volatility term structures, the main difference between European and American binaries is due to the fact that the relevant volatility parameter for European binaries is the annualized standard deviation of the change of price between now and the maturity date. In contrast, American binaries are sensitive to the entire volatility path.

Closed-from expressions for American binary options can be obtained under the assumption that the volatility and interest rates remain constant. This requires introducing new mathematical tools. Let $f(\theta)$ represent the probability density function of the random variable τ, i.e.

$$\mathbf{P}\,\{\tau \;<\; T\,\} \;=\; \int_0^T f(\theta)\, d\theta \; . \tag{17}$$

Then, the value of the American digital in equation (13) can be written as

[3] In contrast, European binary options can be priced in a time-dependent volatility environment like standard options, using an "effective" mean-square volatility $\overline{\sigma}_T$ such that $\overline{\sigma}_T^2 \;=\; T^{-1} \int_0^T \sigma_t^2\, dt$.

$$V(S,T) \;=\; \int_0^T e^{-r\,\theta}\, f(\theta)\, d\theta \;. \tag{18}$$

An explicit expression for the probability distribution of the first-exit time can be derived from

Lemma 1. *Let Z_t^μ represent a Brownian motion with drift μ, i.e.*

$$Z_t^\mu \;=\; Z_t \;+\; \mu\, t$$

where Z_t is a Brownian motion and μ is a constant. Let τ_A represent the first time the path $Z_\cdot^\mu$ hits A, where $A \;>\; 0$, Then,

$$\mathbf{P}\,\{\tau_A \;<\; T\} \;=\; N\left(\frac{-A \,+\, \mu\, T}{\sqrt{T}}\right) \;+\; e^{2\,A\,\mu}\, N\left(\frac{-A \,-\, \mu\, T}{\sqrt{T}}\right) \;. \tag{19}$$

We defer the proof of this Lemma fro the moment. To apply this result to the case of a lognormal random walk of the form

$$S_t \;=\; S\, e^{\,\sigma\, Z_t \;+\; \left(r - q - \frac{1}{2}\sigma^2\right)\, t} \tag{20}$$

we set

$$\mu \;\equiv\; \frac{r - q}{\sigma} \;-\; \frac{1}{2}\sigma$$

and

$$A \;\equiv\; \frac{1}{\sigma}\ln\left(\frac{K}{S}\right) \;.$$

The reader will verify easily that if τ represents the first hitting time of $S \,=\, K$, then

$$\mathbf{P}\;\{\;\tau \;<\; T\} \;=\; \mathbf{P}\;\{\tau_A \;<\; T\} \;. \tag{21}$$

Using equation (19), we conclude that

$$\mathbf{P}\;\{\;\tau \;<\; T\} \;=\; N\,(d_2) \;+\; \left(\frac{K}{S}\right)^{\left(\frac{2(r-q)}{\sigma^2} - 1\right)} \cdot\; N\,(d_3) \tag{22}$$

where

$$d_3 \;\equiv\; \frac{1}{\sigma\sqrt{T}} \left[\ln\left(\frac{S}{K}\right) \;-\; \left(r - q - \frac{1}{2}\sigma^2\right) T \right] . \tag{23}$$

Equation (22) gives a closed-from expression for the probability distribution of the first-exit probability of the set $\{S \;<\; K\}$. The probability density of τ can then be computed by differentiating (22) with respect to T. More precisely, we have

$$\begin{aligned}
V(S,T) \;&=\; \int_0^T e^{-r\,\theta}\, f(\theta)\, d\theta \\
&=\; \int_0^T e^{-r\,\theta}\, \frac{d}{d\theta}\, \mathbf{P}\{\,\tau \;<\; \theta\,\}\, d\theta \\
&=\; \left[e^{-r\,\theta}\, \mathbf{P}\{\,\tau \;<\; \theta\,\} \right]_{\theta=0}^{\theta=T} \;+\; r \int_0^T e^{-r\,\theta}\, \mathbf{P}\{\,\tau \;<\; \theta\,\}\, d\theta \\
&=\; e^{-r\,T}\, \mathbf{P}\{\,\tau \;<\; T\} \;+\; r \int_0^T e^{-r\,\theta}\, \mathbf{P}\{\,\tau \;<\; \theta\,\}\, d\theta\,.
\end{aligned} \tag{24}$$

The final expression for $V(S,T)$ is, from (22),

$$V(S,T) \;=\; e^{-r\,T}\cdot N(d_2) + e^{-r\,T}\cdot \left(\frac{K}{S}\right)^{\left(\frac{2(r-q)}{\sigma^2} - 1\right)} N(d_3) + r \int_0^T e^{-r\,\theta}\, \mathbf{P}\{\,\tau \;<\; \theta\,\}\, d\theta\,, \tag{25}$$

where the probability inside the integral is given by (22) (with T replaced by θ). This last integral can be computed numerically by quadrature. Notice that in the special case $r = 0$, the formula simplifies further. The same is true if the option is modified so that the holder collects \$1 *at time* T if the price ever touches K (and not at time τ). In both cases, the value of the American digital is

$$V(S,T) \;=\; e^{-r\,T}\cdot N(d_2) \;+\; e^{-r\,T}\cdot \left(\frac{K}{S}\right)^{\left(\frac{2(r-q)}{\sigma^2} - 1\right)} N(d_3)\,. \tag{26}$$

Next, we discuss the option's sensitivities to changes in spot price or market volatility.

Both the Delta and Gamma of the American digital are monotone increasing for $0 < S < K$ and become unbounded as $T \to 0$ in a neighborhood of $S = K$. Therefore, the option has significant pin risk. The "worst-case scenario" for the hedger would be a market rallying slowly towards the strike level which collapses immediately before the option's maturity. In this event, Delta-hedging builds up a large spot position in the rally (long the market). If the market falls suddenly, the hedger may incur a significant loss, defeating the purpose of Delta-hedging. There is, however, an important difference with respect to European binaries: the hedger does not have to worry about market "whipping" around the strike price, since the option expires after K is hit for the first time. The exposure to Gamma and the pin risk are simpler than for the European counterpart.

The Vega of the American digital option is positive at all values of spot. (This follows from the convexity with respect to the spot price.) Thus, the risk-exposure due to an incorrect estimate of the volatility is only one sided.[4] Just like with standard options, the seller fears an increase in volatility and the buyer a decrease in volatility.

Near the "barrier" $S = K$, we can make a straightforward analysis of the sensitivity of the Delta of the option to volatility (what some traders call colloquially "D-Delta-D-Vol"). In fact, we know that $V(S,T)$ is non-decreasing as a function of the volatility and that $V(K,T) = 1$. This implies that the difference quotient

$$\frac{V(K,T) \; - \; V(K-\epsilon,T)}{\epsilon} \;\approx\; \Delta(K,T)$$

decreases as σ increases. Hence, Δ varies inversely to σ in a neighborhood of K. Therefore, increasing the volatility parameter (with respect to, say, the implied volatility of vanilla options traded in the market) will provide protection against slippage when the spot is away from K improve the exposure to pin risk by decreasing the Delta at the barrier.[5]

3. Barrier options

Barrier options are a generic name given to derivative securities with payoffs which are contingent on the spot price reaching a given level, or barrier, over the lifetime of the option. The most common types or barrier options are

- **Knock-out options**. These are contingent claims that *expire* automatically when the spot price touches one or more predetermined barriers.

[4] Recall that the European digital has two-sided volatility risk.

[5] Of course, since increasing the volatility increases the premium, the seller will have to charge more if he wishes to follow this augmented- volatility strategy. He must therefore charge above the market volatility or else "set aside" some of his other funds to finance the strategy.

- **Knock-in options**. These contingent claims are *activated* when the spot price touches one or more predetermined barriers.

The most common barrier options are structured as standard European puts and calls with one knock-in or knock-out barrier. For instance,

- a **down-and-out call** with strike K, barrier H and maturity T is an option to buy the underlying asset for $\$K$ at time T, provided that the spot price never goes below $\$H$ between now and the maturity date.
- An **up-and-out call** with strike K, barrier H and maturity T is an option to buy the underlying asset for $\$K$ at time T, provided that the spot price never goes above $\$H$ between now and the maturity date.
- A **down-and-in call** with strike K, barrier H and maturity T is an option to buy the underlying asset for $\$K$ at time T, provided that the spot price goes below $\$H$ between now and the maturity date.
- An **up-and-in call** with strike K, barrier H and maturity T is an option to buy the underlying asset for $\$K$ at time T, provided that the spot price goes above $\$H$ between now and the maturity date.

Similar definitions apply to puts. Barrier options are especially used in foreign-exchange derivatives markets. The London Financial Times of November 16. 1995 reported that exotic options now constitute around 10% of the currency option business.[6] Barrier options, which are relatively simple variations on the European put and call enjoy a great popularity.

A first observation regarding barrier options is that are much cheaper than standard options. The optionality feature can be targeted more precisely by introducing a barrier. This is illustrated in the following example described to me by a trader.

> **Example:** A large multinational corporation based in Europe must convert its U.S. business revenue into DEM periodically. Given the weakness of the Dollar with respect to the Deutschemark in the past years and drop of the Dollar at the beginning of 1995, the company fears a decrease in revenues in DEM terms. Its treasury department could have anticipated the problem by purchasing standard options but did not do this. Ideally, the company would like to have an at-the-money DEM call/Dollar put with six months to expiration. If the spot exchange rate is 1.4225 DEM/USD, the value of a dollar put with strike 1.42 expiring in 180 days is \$ 0.0391 per dollar notional. On a \$ 100 million notional, the cost of this option is therefore approximately \$ 3,910,000.[7] On th other hand, suppose that the company purchases now a down-an-out dollar put (or, equivalently, an up-and-out Mark call) with a knockout barrier at 1.27 DEM/USD. The value of this option is instead \$ 0.01181 per dollar notional, or \$ 1,181,000 to the nearest \$ 1,000. (We will derive below a pricing formula for knockout options.) Thus,

[6] G. Bowley: "*New Breed of exotics thrives*", *LFT*, Nov. 16, Supplement on derivatives.

[7] We used a volatility of 13.00% a U.S. deposit rate of 5.80% and a German deposit rate of 4.00%. The result was rounded to the nearest \$ 10,000.

the knockout option with a 1.27 barrier is nearly 4 times cheaper than the vanilla. Therefore, if the treasurer believes that the dollar will not drop below \$ 1.27 over the next six months, the knockout option provides a cheaper alternative with the "same" terminal payoff.

The option described in the above example, which knocks out when the option is *in-the-money* is often called a **reverse knock-out**. The difference between in-the-money and out-of-the-money barriers is significant because the former have discontinuous payoffs at expiration. Thus, Delta-hedging reverse knock-in and knock-out options may lead to significant pin risk, similar to the one encountered in digitals. In contrast, options with out-of-the-money barriers do not seem to be very interesting from a hedging perspective. We will therefore discuss primarily barrier options which knock in or out when the option is in-the-money.

Knock-in and knock-out options are related by the simple formula

$$\text{KI} + \text{KO} = \text{Vanilla}\,. \tag{27}$$

This formula is self-evident: the holder of a portfolio consisting of one knock-in call and one knock-out call with same strike, barrier and maturity will effectively hold a call at maturity regardless of whether the barrier was crossed or not. We can therefore reduce the question pricing barrier options to the pricing of knockouts.

We note that in some cases, the structure of barrier options is more complicated. We note two cases that were mentioned to us by professional traders:

- **Double knock-in** or **double knock-out** options, which have two barriers;
- **knock-out options with rebate**. The holder receives a "consolation prize" in the form of a cash rebate on the premium paid if the option knocks out.[8]

3.1 Pricing barrier options

Barrier options are priced by solving a boundary-value problem similar to the one for American digitals. In the case of an up-and-out call, the value of this derivative security is determined recursively by solving the problem:

$$V_n^j = e^{-r_n\,dt}\left[P_U^{(n)}\,V_{n+1}^{j+1} + P_D^{(n)}\,V_{n+1}^{j}\right] \quad \text{if } S_n^j < H\,, \tag{28a}$$

where $P_U^{(n)}$ and $P_D^{(n)}$ are risk-neutral probabilities,

$$V_n^j = 0 \quad \text{if } S_n^j \geq H\,, \tag{28b}$$

and

[8] This option consists of a regular knockout option with an attached American digital option.

$$V_N^j \ = \ \text{Max}\left[S_N^j \ - \ K \, , 0 \right] \ \ \text{if} \ \ S_N^j \ \leq \ K \ . \tag{28c}$$

In the case of an up-and-in call, the boundary conditions (28b) and (28c) are replaced by

$$V_n^j \ = \ \tilde{V}_n^j \quad \text{if} \ \ S_n^j \ \geq \ H \ , \tag{29a}$$

where $\tilde{V}_n^j$ represents the value of a vanilla call at the node (n, j), and

$$V_N^j \ = \ 0 \ \ \text{if} \ \ S_N^j \ \leq \ K \ . \tag{29b}$$

The validity of these equations follows from (i) the terms of the barrier options, which determine their value at the barrier and at maturity, and (ii) the absence of arbitrage, which implies (28a). The values of barrier-puts are determined by making obvious modifications.

Next, we consider the pricing assuming that S_t is a lognormal random walk with constant $\sigma \, , q$ and r constant (geometric Brownian Motion). As in the case of the American binary option, we will need some auxiliary results on the properties of Brownian motion with drift.

Lemma 2. *Let $Z_t^\mu \ , t \ \geq \ 0$ represent a Brownian motion with drift μ. Then, if A and B are positive numbers with $B \ \leq \ A$,*

$$\mathbf{P} \left\{ \underset{0 \leq \ t \ \leq \ T}{Max} \ Z_t^\mu \ \geq \ A \ \ \text{and} \ \ Z_T^\mu \ \in \ (B \, , B + dB) \right\}$$

$$= \ \frac{1}{\sqrt{2\pi \, T}} \, e^{- \ \frac{(2A-B)^2}{2T}} \ e^{\, B\mu \, - \, \frac{1}{2}\mu^2 \, T} \ dB \quad , \qquad dB \ \ll \ 1 \ . \tag{30}$$

We shall prove this Lemma later.

To apply this result, let τ denote the first time that the lognormal walk (20) hit the level $S = H$. Then, the value of a down-and-out put with strike price K, knockout at H $(H < K)$ and maturity T satisfies

$$P_{KO} \ (S, T \ ; \ K, H \ ) \ = \ e^{-r\,T} \, \mathbf{E} \left\{ Max \ [K \ - \ S_T \, , 0] \ ; \ \tau \ > \ T \ \right\}$$

$$= \ e^{-r\,T} \, \mathbf{E} \left\{ Max \ [K \ - \ S_T \, , 0] \ ; \ \underset{0 \leq \ t \ \leq T}{Min} \ S_t \ > \ H \ \right\} \tag{31}$$

This last expression can be rewritten as

$$e^{-r\,T}\,\mathbf{E}\left\{ K \,-\, S_T \;;\; H \,<\, S_T \,<\, K \,;\, \underset{0\leq\, t\,\leq T}{Min}\; S_t \,>\, H \right\}$$

$$= \; e^{-r\,T}\,\mathbf{E}\,\{ K \,-\, S_T \;;\; H \,<\, S_T \,<\, K \,\}$$

$$- \; e^{-r\,T}\,\mathbf{E}\left\{ K \,-\, S_T \;;\; H \,<\, S_T \,<\, K \,;\, \underset{0\leq\, t\,\leq T}{Min}\; S_t \,\leq\, H \right\}$$

$$= \; e^{-r\,T}\,\mathbf{E}\,\{ K \,-\, S_T \;;\; S_T \,<\, K \,\}$$

$$- \, e^{-r\,T}\,\mathbf{E}\,\{ K \,-\, S_T \;;\; S_T \,<\, H \,\}$$

$$- \, K\,e^{-r\,T}\,\mathbf{P}\left\{ \, H \,<\, S_T \,<\, K \,;\, \underset{0\leq\, t\,\leq T}{Min}\; S_t \,\leq\, H \right\}$$

$$+ \, e^{-r\,T}\,\mathbf{E}\left\{ S_T \;;\; H \,<\, S_T \,<\, K \,;\, \underset{0\leq\, t\,\leq T}{Min}\; S_t \,\leq\, H \right\} . \qquad (32)$$

Notice that the first term corresponds to the value of a standard European put with strike K. The second term can be calculated easily using the same reasoning as in the derivation of the Black-Scholes formula. To calculate the two remaining terms, we will use the result of Lemma 2. Introducing the parameters

$$A_H \;\equiv\; \frac{1}{\sigma}\ln\left(\frac{H}{S}\right) \;,$$

$$A_K \;\equiv\; \frac{1}{\sigma}\ln\left(\frac{K}{S}\right) \;,$$

and

$$\mu \;\equiv\; \frac{r-q}{\sigma} - \;\frac{1}{2}\,\sigma \;,$$

we have, using Lemma 2,

$$\mathbf{P}\left\{ H < S_T < K \, ; \, \underset{0 \leq t \leq T}{Min} S_t \leq H \right\} = \mathbf{P}\left\{ A_H < Z_T^{\mu} < A_K \, ; \, \underset{0 \leq t \leq T}{Min} Z_t^{\mu} \leq A_H \right\}$$

$$= \mathbf{P}\left\{ -A_K < Z_T^{-\mu} < -A_H \, ; \, \underset{0 \leq t \leq T}{Max} Z_t^{-\mu} \geq -A_H \right\}$$

$$= \int_{-A_K}^{-A_H} e^{-\frac{(-2A_H - B)^2}{2T}} e^{-B\mu - \frac{1}{2}\mu^2 T} \frac{dB}{\sqrt{2\pi T}} \, . \tag{33}$$

Here, we use the fact that $-Z_t - \mu t$ and $Z_t - \mu t$ have the same probability distribution. Similarly,

$$\mathbf{E}\left\{ S_T \, ; \, H < S_T < K \, ; \, \underset{0 \leq t \leq T}{Min} S_t \leq H \right\} =$$

$$S \cdot \mathbf{E}\left\{ e^{-\sigma Z_T^{-\mu}} \, ; \, -A_K < Z_T^{-\mu} < -A_H \, ; \, \underset{0 \leq t \leq T}{Max} Z_t^{-\mu} \leq -A_H \right\}$$

$$= S \cdot \int_{-A_K}^{-A_H} e^{-\sigma B} \cdot e^{-\frac{(2A_H + B)^2}{2T}} e^{-B\mu - \frac{1}{2}\mu^2 T} \frac{dB}{\sqrt{2\pi T}} \, . \tag{34}$$

Calculating explicitly the two integrals in (33) and (34) and using the Black-Scholes formula to calculate the first two terms in (32), we arrive at the final result

$$P_{KO}(S, T; K, H) =$$

$$K e^{-rT} \cdot N(-d_2^K) - S e^{-qT} \cdot N(-d_1^K)$$

$$- K e^{-rT} \cdot N(-d_2^H) + S e^{-qT} \cdot N(-d_1^H)$$

$$- K e^{-rT} \left(\frac{H}{S}\right)^{\left(\frac{2(r-q)}{\sigma^2} - 1\right)} \cdot \{ N(d_4) - N(d_5) \}$$

$$+ \, S \, e^{-q\,T} \left(\frac{H}{S}\right)^{\left(\frac{2(r-q)}{\sigma^2}\right)+1} \cdot \{ N(d_6) \; - \; N(d_7) \; \} \;, \tag{35}$$

where

$$d_1^K \;=\; \frac{1}{\sigma\,\sqrt{T}} \left\{ \ln\left(\frac{S}{K}\right) \;+\; \left(r \;-\; q \;+\; \frac{1}{2}\sigma^2\right) \, T \; \right\} \;,$$

$$d_2^K \;=\; \frac{1}{\sigma\,\sqrt{T}} \left\{ \ln\left(\frac{S}{K}\right) \;+\; \left(r \;-\; q \;-\; \frac{1}{2}\sigma^2\right) \, T \; \right\} \;,$$

$$d_1^H \;=\; \frac{1}{\sigma\,\sqrt{T}} \left\{ \ln\left(\frac{S}{H}\right) \;+\; \left(r \;-\; q \;+\; \frac{1}{2}\sigma^2\right) \, T \; \right\} \;,$$

$$d_2^H \;=\; \frac{1}{\sigma\,\sqrt{T}} \left\{ \ln\left(\frac{S}{H}\right) \;+\; \left(r \;-\; q \;-\; \frac{1}{2}\sigma^2\right) \, T \; \right\} \;,$$

$$d_4 \;=\; \frac{1}{\sigma\,\sqrt{T}} \left\{ \ln\left(\frac{H}{S}\right) \;+\; \left(r \;-\; q \;-\; \frac{1}{2}\sigma^2\right) \, T \; \right\} \;,$$

$$d_5 \;=\; \frac{1}{\sigma\,\sqrt{T}} \left\{ \ln\left(\frac{H^2}{S\,K}\right) \;+\; \left(r \;-\; q \;-\; \frac{1}{2}\sigma^2\right) \, T \; \right\} \;,$$

$$d_6 \;=\; \frac{1}{\sigma\,\sqrt{T}} \left\{ \ln\left(\frac{H}{S}\right) \;+\; \left(r \;-\; q \;+\; \frac{1}{2}\sigma^2\right) \, T \; \right\} \;,$$

and

$$d_7 \;=\; \frac{1}{\sigma\,\sqrt{T}} \left\{ \ln\left(\frac{H^2}{S\,K}\right) \;+\; \left(r \;-\; q \;+\; \frac{1}{2}\sigma^2\right) \, T \; \right\} \;.$$

The formula for an up-and-out call is obtained immediately by a change of numeraire: an up-and-out call on the risky asset with strike K is nothing but a down-and-out put on cash, with the underlying asset viewed as the unit of account.

The pricing formulas for up-and-out puts/ down-and-out calls are obtained using very similar techniques. We leave them as an exercise for the interested reader.

Finally, the fair values of knock-in options can be obtained using the parity relation (27). For instance, using (27) and (35) we find that the value of a down-and-in put is

$$P_{KI}(S,T;K,H) \;=\; +\,K\,e^{-r\,T}\cdot N(-d_2^H) \;-\; S\,e^{-q\,T}\cdot N(-d_1^H)$$

$$+\,K\,e^{-r\,T}\left(\frac{H}{S}\right)^{\left(\frac{2(r-q)}{\sigma^2}-1\right)}\cdot\{\,N(d_4)\;-\;N(d_5)\,\}$$

$$-\,S\,e^{-r\,T}\left(\frac{H}{S}\right)^{\left(\frac{2(r-q)}{\sigma^2}+1\right)}\cdot\{N(d_6)\;-\;N(d_7)\,\}\;,$$

To end this section we present some numerical values for a particular barrier option (this example was mentioned earlier).

Example: A reverse-knockout dollar put / DEM call.

USD interest rate: 5.85 %

DEM interest rate: 4.00 %

volatility: 13.00 %

Strike: 1.42 DEM/USD

Knockout at: 1.27 DEM/USD

180 days to maturity

Spot	1.28	1.30	1.32	1.34	1.36	1.38	1.40	1.4225	1.44
Val.	.0015	.0043	.0068	.0090	.0105	.0115	.0116	.0118	.0117
Δ	.1893	.1823	.1623	.1326	.0968	.0588	.0219	-.0148	-.0111

90 days to maturity

Spot	1.28	1.30	1.32	1.34	1.36	1.38	1.40	1.4225	1.44
Val.	.0035	.0102	.0133	.0193	.0211	.0210	.0196	.0169	.0146
Δ	.4511	.4084	.3185	.2012	.0782	-.0315	-.1160	-.0174	-.0194

30 days to maturity

Spot	1.28	1.30	1.32	1.34	1.36	1.38	1.40	1.4225	1.44
Val.	.0107	.0298	.0411	.0438	.0398	.0320	.0232	.0145	.0135
Δ	1.3610	1.0327	.4989	-.0250	-.0390	-.5579	-.5668	-.4645	-.3544

7 days to maturity

Spot.	1.28	1.30	1.32	1.34	1.36	1.38	1.40	1.4225	1.44
Val.	.0308	.0713	.0737	.0600	.0445	.0411	.0169	.0067	.0025
Δ	3.6459	1.3226	-.4863	-.9666	-.9900	-.9417	-.7798	-.4676	-.2282

3.2. Hedging barrier options.

The risk-management of barrier options should take into consideration the mixed-Gamma exposure of these instruments (for reverse-knockouts and knock-ins) as well as the pin risk at the barrier.

The risk-exposure of a reverse knockout put option can be understood intuitively as follows: from equation (27) the holder of this option is

- Long a standard put with strike K
- short an American digital option with barrier at H which pays one put with strike K upon hitting the barrier (in other words, a knock-in put)

Ignoring the difference between a knock-in put and an American digital option with payoff $H - K$ at the barrier is not a bad approximation near the expiration date. We then see immediately that the option has mixed Gamma exposure: far away from the barrier, the standard put dominates and the holder of the knockout is long Gamma/Vega, whereas near $S = H$ the "digital" dominates and the holder is short Gamma/Vega.

The seller that wishes to hedge faces the mirror-image position: short Vega and Gamma near the strike or in-the-money and long Gamma/Vega closer to the barrier. However, *at the barrier* the Gamma risk is complex: if the spot price is just below the barrier, the hedger must adjust his Delta in order to "earn time decay" (his liability is that of a standard put if the options fails to knock-out). The Delta increases without bounds near the barrier. On the other hand, if the option does not knock-out, the large Delta position may be detrimental in case of a large market because this would lead to a loss in th spot market.

Example. Consider the option described in the previous section, assuming that an agent sold the option with 180 days to expiration, when the spot price was 1.4225, at 0.0118 per dollar notional. If seven days before expiration the spot trades at 1.28 DEM/USD and the agent Delta-hedges according to the above tables, his spot position in USD would be long \$ 3.6459 per dollar notional. A drop of 0.03 in the exchange rate in one day will result in a loss of \$ 0.1094 per dollar notional. To make this more concrete, assume that the notional amount is \$ 100,000,000 dollars. The premium collected for the option was \$ 1,180,000. The spot position, on the other hand is (a whopping) \$ 364,559,000. The loss in the dollar position if the market moves suddenly down by 0.03 through the barrier would be \$ 10,940,000 ! To have a better idea of the likelihood that this happens, note that this represents a 2.4% move of spot in one day. At an annual volatility of 13%, this would be a three-standard deviation move in one day. The event has low probability but is not impossible. Would you risk a loss of \$ 9 million given the odds?[9] Moreover, let us mention the important point of **liquidity**. A selling order of nearly 400 million dollars as the exchange-rate goes through the barrier may cause a further drop in the dollar as there will be few buyers and many sellers. This will have dire consequences for the "hedger".[10]

4. Proofs of Lemmas 1 and 2

This section contains sketches of the proofs of the two Lemmas used to derive closed-form solutions for barrier options and American digitals.

4.1 A consequence of the invariance of Brownian Motion under reflections.

Lemma 3. *Let Z_t denote standard Brownian motion on the interval $[0,T]$. Then, for all $A > 0$ and $B < A$, we have*

$$\mathbf{P}\left\{ \max_{0 \le t \le T} Z_T > A \,;\, Z_T \in (B\,,\, B + dB\,) \right\} = \frac{1}{\sqrt{2\pi T}}\, e^{-\frac{(2A-B)^2}{2T}}\, dB\,. \qquad (36)$$

[9] One should also take into account that the annualized volatility may very well underestimate the daily move of the exchange rate.

[10] To find out more about the risk-management of exotic options, see N. Taleb: *Dynamic Hedging* (1995, manuscript in preparation), where, in particular, the liquidity issue in the trading of barrier options is discussed in great depth.

Proof: Consider a simple random walk defined by

$$X_n = X_{n-1} \pm \sqrt{dt} \quad , n = 1, 2, ..., N$$

where dt represents a small positive number. The probabilities for $+\sqrt{dt}$ and $-\sqrt{dt}$ are assumed to be 1/2. Set

$$A' \equiv \left[\frac{A}{\sqrt{dt}} \right] \sqrt{dt} \ ,$$

where $[X]$ represents the *integer part* of X. Therefore, A' represents the largest integer multiple of $\sqrt{dt}$ which is $\leq A$.

Assume that a given path, or realization, of the random walk is such that $X_n = A'$ for some $n \leq N$ and that $X_m < A'$ for $m < n$. We observe that the path which coincides with this realization of the random walk for $m \leq n$ and which is *reflected about the line* $X = A'$ for $m > n$ occurs with the same probability as the original one (namely $\left(\frac{1}{2}\right)^N$). Therefore, we conclude that for $B < A'$

$$\mathbf{P} \left\{ \max_{1 \leq j \leq N} X_j \geq A' ; X_N = B \right\} = \mathbf{P} \left\{ \max_{1 \leq j \leq N} X_j \geq A' ; X_N = 2A' - B \right\}$$

$$= \mathbf{P} \left\{ X_N = 2A' - B \right\} . \qquad (37)$$

(This last equality holds because $2A' - B > A'$.)

Let $T = N\,dt$. By the Central Limit Theorem, the joint distribution of the random variables $X_{\frac{[nt]}{n}}$ approaches that or a Brownian Motion as $dt \to 0$, $N \to +\infty$. Therefore, if we replace formally $\max_{1 \leq j \leq N} X_j$ by $\max_{0 \leq t \leq T} Z_t$, X_N by Z_T and A' by A, we conclude from (37) that equation (36) holds. The Lemma is proved.[11]

Notice that we have established the analogue of Lemma 2 in the case $\mu = 0$.

4.2 The case $\mu \neq 0$.

To prove Lemma 2, we will need the following result about Brownian Motion with drift:

[11] For a rigorous proof of the formal passage to the limit (37) $\Longrightarrow$ (36), see for instance Billingsley: *"Convergence of Probability Measures"* , Wiley, 1968.

Theorem *(Cameron - Martin). Let* $(F(z_1\,,\,z_2\,,\,...\,,\,z_n)$ *be a continuous function. Then,*

$$\mathbf{E}\,\left\{\,F\,(\,Z^{\mu}_{t_1}\,,\,Z^{\mu}_{t_2}\,,...\,,\,Z^{\mu}_{t_n}\,)\,\right\}\;=$$

$$\mathbf{E}\,\left\{\,F\,(\,Z_{t_1}\,,\,Z_{t_2}\,,...\,,\,Z_{t_n}\,)\cdot\;e^{\mu\,Z_{t_n}\;-\;\frac{1}{2}\,\mu^2\,t_n}\,\right\}\;. \tag{38}$$

This result states that the expectation of a function of Brownian motion with drift is equal to the expectation of the same function of regular Brownian multiplied by an exponential factor, namely

$$e^{\mu\,Z_T\;-\;\frac{1}{2}\,\mu^2\,T}\;.$$

Proof of the Theorem: Define the increments of the Brownian path

$$Y^{\mu}_j\;=\;Z^{\mu}_{t_j}\;-\;Z^{\mu}_{t_{j-1}}$$

$$=\left(\,Z_{t_j}\;-\;Z_{t_{j-1}}\right)\;+\;\mu\,(t_j\;-\;t_{j-1})\;.$$

Also, set

$$G\,(y_1\,,\,y_2\,,\,...\,,\,y_n)\;\equiv\;F\,(y_1\,,\,y_1\,+\,y_2\,,\,...\,,\,y_1\,+\,y_2\,+\,...\,y_n)$$

Using the explicit form of the Gaussian distribution and the fact that the increments Y_j are independent random variables with mean $\mu\,(t_j - t_{j-1})$ and variance $t_j - t_{j-1}$, we obtain

$$\mathbf{E}\,\left\{\,F\,(\,Z^{\mu}_{t_1}\,,\,Z^{\mu}_{t_2}\,,...\,,\,Z^{\mu}_{t_n}\,)\,\right\}\;=$$

$$\mathbf{E}\,\left\{\,G\,(\,Y^{\mu}_1\,,\,Y^{\mu}_2\,,...\,,\,Y^{\mu}_n\,)\,\right\}$$

$$=\int\limits_{\mathbf{R}^n} G\,(y_1\,,\,y_2\,,\,...\,,\,y_n)\cdot\exp\left\{-\sum_{j=1}^{n}\frac{\left(y_j\;-\mu\;(t_j\;-\;t_{j-1})\right)^2}{2\,(t_j\;-\;t_{j-1})}\right\}\frac{dy_1\,dy_2\,...dy_n}{(t_1\;-\;t_0)\cdot\;...\;(t_n\;-\;t_{n-1})}$$

$$= \int_{\mathbf{R}^n} \tilde{G}(y_1, y_2, \ldots, y_n) \cdot \exp \left\{ - \sum_{j=1}^{n} \frac{y_j^2}{2(t_j - t_{j-1})} \right\} \cdot \frac{dy_1\, dy_2 \ldots dy_n}{(t_1 - t_0) \cdot \ldots (t_n - t_{n-1})}, \tag{39}$$

where

$$\tilde{G}(y_1, y_2, \ldots, y_n) = G(y_1, y_2, \ldots, y_n) \cdot e^{\mu \sum_{j=1}^{n} y_j - \frac{1}{2}\mu^2 t_n}.$$

Making a change of variables, we find that the last integral in (39) is equal to

$$\mathbf{E} \left\{ F(Z_{t_1}, Z_{t_2}, \ldots, Z_{t_n}) \cdot e^{\mu Z_{t_n} - \frac{1}{2}\mu^2 t_n} \right\},$$

which is what we wanted to show. This concludes the proof of the Theorem.

We are now ready for the

Proof of Lemma 2: Using an approximation argument which we omit, it can be shown that the above theorem can be also applied to the *functional* of the path

$$F(Z_\cdot^\mu) \equiv \underset{0 \le t \le T}{Max} Z_t^\mu .$$

(This is a continuous function of the path that can be approximated in a suitable sense by continuous functions of n variables, as in the previous theorem[12])

Applying the Theorem to this functional, we conclude that if $C < A$, then

$$\mathbf{P} \left\{ \underset{0 \le t \le T}{Max} Z_t^\mu \ge A \,;\, Z_T^\mu < C \right\} =$$

$$\mathbf{E} \left\{ \underset{0 \le t \le T}{Max} Z_t \ge A \,;\, Z_T < C \,;\, e^{\mu Z_T - \frac{1}{2}\mu^2 T} \right\}$$

$$= \int_0^C \mathbf{E} \left\{ \underset{0 \le t \le T}{Max} Z_t \ge A \,;\, Z_T = B \right\} \cdot e^{\mu B - \frac{1}{2}\mu^2 T}\, dB$$

$$= \int_0^C e^{-\frac{(2A-B)^2}{2T}} e^{\mu B - \frac{1}{2}\mu^2 T} \frac{dB}{\sqrt{2\pi T}}, \tag{40}$$

[12]See Billingsley, *ibid.*

where we used Lemma 3 to derive the last equality. Equation (30) follows immediately by differentiating both sides of (40).

Finally, we prove Lemma 1 on the distribution of the first-passage time for Brownian motion with drift.

Proof of Lemma 1. Using the notation of Lemmas 1 and 2, we find that

$$\begin{aligned}
\mathbf{P}\,\{\,\tau_A \;<\; T\} \;&=\; \mathbf{P}\left\{\underset{0\;\leq\;t\;\leq\;T}{Max}\;Z_t^\mu \;\geq\; A\right\} \\
&= \int_0^A \mathbf{P}\left\{\underset{0\;\leq\;t\;\leq\;T}{Max}\;Z_t^\mu \;\geq\; A\;;\;Z_T^\mu \;=\; B\right\} dB \\
&= \int_0^A \mathbf{P}\,\{Z_T^\mu \;=\; B\}\;dB \;-\; \int_0^A \mathbf{P}\left\{\underset{0\;\leq\;t\;\leq\;T}{Max}\;Z_t^\mu \;<\; A\;;\;Z_T^\mu \;=\; B\right\} dB \\
&= \int_0^A e^{-\frac{B^2}{2T}}\,\frac{dB}{\sqrt{2\pi\,T}} \;-\; \int_0^A e^{-\frac{(2A-B)^2}{2T}}\,\frac{dB}{\sqrt{2\pi\,T}}\;. \qquad (41)
\end{aligned}$$

The conclusion of Lemma 1 follows by evaluating this last expression in terms of the cumulative normal distribution.