AMERICAN-STYLE OPTIONS, EARLY EXERCISE AND TIME-OPTIONALITY[1]

We have discussed so far derivative securities that can be exercised only at specific dates. In this lecture, we discuss the issue of **time-optionality**, i.e. the valuation of derivative securities that give the holder or the issuer the right to exercise some option at an unknown date in the future.

The simplest derivatives with time-optionality are American-style options. Exchange-traded equity options on the CBOT and AMEX are American-style. Other examples of securities with time-optionality are encountered in debt markets. U.S. Treasury bonds issued before the 1980's were *callable*: the government had the right to repay the face-value of the bond after a certain date thereby stopping coupon payments. (Callable bonds have not been issued in recent years; see F. Fabozzi: *Bond market strategies*). Corporate debentures usually contain some kind of time-optionality provision. For instance, companies issue bonds that are callable after a certain number of years. *Convertible bonds*, usually issued by corporations, are debt securities which can be converted into company stock after a certain time. Callable and convertible bonds are said to have *embedded options*. Mortgages constitute another class of securities with time-optionality. The mortgage issuer (homeowner) has the right to increase monthly payments to reduce interest payments or to pay the mortgage in full by refinancing the loan at a lower interest rate. Similarly, commercial loans (credit cards, lines of credit) have time-optionality since they can be paid off faster or refinanced.

1. American-Style options

The holder of an American style option maturing at time T has the right to exercise the option at any time before the expiration date. Suppose that such option is written on a stock or on an index which pays continuous dividends (stock index of currencies being the two major examples).

The rationale for the early exercise of American options is simple: by exercising, the holder can claim the dividends provided by holding the underlying security (in the case of a call), or the interest income that derives from investing the proceeds from selling the underlying security (puts).

To fix ideas, assume that the option is a call. If the option is exercised at time τ ($\tau \leq T$), the payoff is

$$max\ (S_\tau - K\ ,\ 0)\ , \tag{1}$$

[1] Preliminary version. A more complete version will be available later in the week.

where S_τ is the value of the index and K is the strike price. According to arbitrage pricing theory, the fair value (present value) of this payoff today is

$$\mathbf{E}\left\{ e^{-r\,\tau}\, Max\,(S_\tau\ -\ K\ ,\ 0\,)\right\}\ , \tag{2}$$

where $\mathbf{E}$ represents the expectation operator with respect to an arbitrage-free probability defined on the set of forward paths $\{\, S_t\ , 0\ \leq\ t\ \leq\ T\ \}$. Since the holder has the right to exercise at *any* time, then the fair value of this option is given by

$$V_{am}\ =\ \sup_{0\,\leq\,\tau\,\leq\,T}\ \mathbf{E}\left\{ e^{-r\,\tau}\, Max\,(S_\tau\ -\ K\ ,\ 0\,)\right\}\ , \tag{3}$$

where the supremum is taken over the class of all possible *exercise times.*[2] An exercise time is a particular case of a *stopping time*, or decision rule contingent on information up to time τ. Mathematically, we have

Definition 1: *A* stopping time *τ is a function taking values on $[0,T]$ such that the event $\{\ \tau\ =\ t\}$, i.e. the decision to "stop at time t", is determined by the path $\{\, S_u\ ,\ 0\ \leq\ u\ \leq\ t\ \}$.*

The main problems of interest are the evaluation of the supremum in (3) and the *optimal exercise time* which realizes the supremum. This stopping time is also called the *rational* exercise time because it maximizes the present value of the cash-flows received by the holder. By knowing the optimal exercise time (as a function of the time to expiration and observed market prices) the holder knows whether the option has a premium above the intrinsic value, $max\,[S-K,\ 0]$, and hence whether it should be exercised or not.

2. The early-exercise premium

What is the additional value (early-exercise premium) of an American option compared to a European option with the same intrinsic payoff? The first observation to make is that if the fair value $V(S,T)$ of the European-style option, $F(S)$,[3] satisfies

$$V(S,T)\ \geq\ F(S)\ \equiv\ \text{intrinsic value} \tag{4}$$

for all S, then the American option has no early-exercise premium. In fact, the value of the American option, $V_{am}(S,T)$, satisfies (cf. (3))

$$V_{am}(S,T)\ \geq\ V(S,T)\ . \tag{5}$$

[2] Over-the-counter American-style derivatives may be exercisable on a smaller set of dates, say, only after 1 year. In this case the supremum in (3) should be replaced by $\tau\ \in\ \Theta$, where Θ represents the subset of possible exercise dates.

[3] $F(S)\ =\ max[S-K\ ,\ 0]$ (calls) or $F(S)\ =\ max[K-S\ ,\ 0]$ (puts).

We conclude from the latter inequality and (4) that the holder of the American option will achieve a higher return by selling the option at the fair value than by exercising it. In particular, this implies that $\tau = T$ and hence that $V_{am}(S,T) \ = \ V(S,T)$.

Conversely, if an American option has no early-exercise premium, then $\tau \ = \ T$ and $V_{am} \ = \ V$. But then, since

$$V_{am}(S,T) \ \geq \ F(S) \ ,$$

we conclude that the value of the European-style option must be greater than the option's intrinsic value for all S. From these considerations, we deduce

Proposition 1. *The following statements are equivalent:*

(i) $V_{am}(S,T) \ > \ V(S,T)$

(ii) $F(S) \ > \ e^{-rT}\,F(e^{(r-q)T}S)$ for some S .

Proof: Suppose that the option is a call. Recall (say, from the Black & Scholes formula or from general principles) that the fair value of a European call option satisfies

$$C(S,K;T) \ \propto \ S\,e^{-qT} \ - \ K\,e^{-rT} \ \propto \ e^{-rT}\,F(e^{(r-q)T}S \ \ , \quad S \ \gg \ K \ .$$

(The value of a deep-in-the-money call is asymptotic to the value of a forward.) Therefore, if (ii) holds, we will have $C(S,K;T) \ < \ F(S)$ for some level of spot and hence the American option will have an early-exercise premium. Conversely, assume that $C(S,K;T) \ < \ F(S)$ for some level of spot. Then, since we have $C(S,K;T) \ > \ e^{-rT}\,F(e^{(r-q)T}S)$, then (ii) holds. Puts can be analyzed similarly (this is left as an exercise).

This Proposition has the following useful

Corollary: *Assuming a riskless rate r and a continuous dividend rate q,*

(i) Calls have early exercise premium if and only if $q \ > \ 0$

(ii) Puts have early exercise premium if and only if $r \ > \ 0$.

In particular, American-style call options on stocks that pay no dividends have no early-exercise premium. This is by far the most important example, because it applies to listed stock options (assuming no dividends). Puts on stocks have early-exercise premium since interest rates are generally non-zero. Options on futures always have an early exercise premium (exercise). The Corollary can be illustrated using payoff diagrams, as shown in Figures 1 and 2 below.

3. Pricing American Options using the binomial model: the dynamic programming equation

In the simplest formulation, the no-arbitrage probability can be taken to be either the binomial random walk (binomial tree) or its lognormal approximation. Even with such simple models, and assuming a constant volatility, and constant interest and dividend rates, the premium of an American option cannot be expressed in closed-form. The premium and optimal exercise time must be evaluated numerically.

Assume that the volatility, dividend rate and interest rate are given, and consider a binomial tree with parameters

$$U = \frac{e^{\sigma\sqrt{dt} + r\,dt - q\,dt}}{cosh(\sigma\sqrt{dt})},$$

$$D = \frac{e^{-\sigma\sqrt{dt} + r\,dt - q\,dt}}{cosh(\sigma\sqrt{dt})}$$

and

$$P_U = P_D = \frac{1}{2},$$

where dt represents the duration between hedge adjustments. Recall that the value of the underlying index at the node (n,j) is given by

$$S_n^j = S_0^0\,U^j\,D^{n-j} \quad , 0 \le j \le n \le N.$$

Using this model, we shall calculate the value of an American option recursively. Let V_n^j represent the value of the option at the node (n,j), and assume that V_{n+1}^{j+1} and V_{n+1}^j, the values at the two "offspring" of node (n,j), have been determined. From the analysis of the binomial model, we know that, at time n, there exists a portfolio of stocks and bonds that will replicate the two "cash-flows" V_{n+1}^{j+1} and V_{n+1}^j at time $n+1$. The value of this replicating portfolio is

$$e^{-r\,dt}\left\{ P_U\,V_{n+1}^{j+1} + P_D\,V_{n+1}^j \right\}. \qquad (6)$$

Hence, the fair value of the option, *conditionally on that it not be exercised* is (6). However, the holder of the option can exercise at time n it and earn the intrinsic value $F(S_n^j)$. Clearly, the decision to exercise should be made according to whether the value of holding the option until the next period, given by (6), exceeds or not the intrinsic value $F(S_n^j)$. Because of this, the value of the option at time n if $S_n = S_n^j$ is

$$V_n^j \;=\; Max\left\{\; F(S_n^j)\;,\; e^{-r\,dt}\left[P_U\,V_{n+1}^{j+1} \;+\; P_D\,V_{n+1}^{j}\right]\right\}\;. \tag{7}$$

This equation is known as a **dynamic programming equation**.[4] Once the values of the option at the expiration date T are specified, *viz.*,

$$V_N^j \;=\; F(S_N^j)\;. \tag{8}$$

equation (7) can be solved recursively, as with the case of the linear relation used for pricing of European options.

The dynamical programming (DP) equation can be used to determine if the option should be exercised, given the spot price and the time to expiration. More precisely, the nodes of the tree can be divided into two classes, according to whether

$$V_n^j \;>\; F(S_n^j)\;, \qquad \text{i.e.}$$

$$Max\left\{F(S_n^j)\;,\; e^{-r\,dt}\left[P_U\,V_{n+1}^{j+1} \;+\; P_D\,V_{n+1}^{j}\right]\right\} \;=\; e^{-r\,dt}\left[P_U\,V_{n+1}^{j+1} \;+\; P_D\,V_{n+1}^{j}\right]\;, \tag{9a}$$

or

$$V_n^j \;=\; F(S_n^j)\;, \qquad \text{i.e.}$$

$$Max\left\{F(S_n^j)\;,\; e^{-r\,dt}\left[P_U\,V_{n+1}^{j+1} \;+\; P_D\,V_{n+1}^{j}\right]\right\} \;=\; F(S_n^j)\;. \tag{9b}$$

Nodes satisfying (9a) correspond to spot levels S_n^j where the option should not be exercised. On the other hand, if (9b) is satisfied at a node, the option should be exercised at time $t_n \;=\; n\,dt$ In particular, the optimal stopping time is given by

$$\tau^* \;=\; Min\left\{\,n\,dt \;:\; V_n^j \;=\; F(S_n^j)\,\right\}\;. \tag{10}$$

In Figures 3 and 4 below we illustrate the solution of the dynamical programming equation for puts and calls using a tree with 10 periods.

Insert figures

[4] Because the optimal exercise decision is determined dynamically.

The *exercise region*, i.e. the set of pairs $(S_n^j, n\,dt)$ such that $V_n^j = F(S_n^j)$, has a simple geometry, as indicated in the above figures. For call options, we have

$$V_n^j = F(S_n^j) \implies V_n^k = F(S_n^k) \qquad \forall k > j \ ,$$

whereas for puts, we have

$$V_n^j = F(S_n^j) \implies V_n^k = F(S_n^k) \qquad \forall k < j \ .$$

These monotonicity properties follow from the convexity of the option premium with respect to the variable S.

4. Hedging

The Delta of an American option is given by the difference-quotient

$$\Delta_n^j = \frac{V_{n+1}^{j+1} - V_{n+1}^j}{S_n^{j+1} - S_{n+1}^j} \ . \tag{11}$$

where V_n^j is the solution of (7)-(8). Suppose that a trader writes (sells) an American option and decides to hedge his exposure using a self-financing portfolio of stocks and bonds with Delta given by (11). More precisely, assume that

- at time $t = 0$ an individual sells an American-style option,
- he implements thereafter a dynamic hedge which consists in holding $\Delta_n^\bullet$ shares and $B_n^\bullet = V_n^\bullet - \Delta_n^\bullet \cdot S_n^\bullet$ in bonds at time $t_n = n\,dt$, $n \geq 0$.[5]
- the buyer exercises the option at some time $\tau \leq T$.

We claim that this strategy is riskless for the seller of the option, regardless of when the option is exercised. To see this, notice that the dynamical programming equation ensures that the value of the replicating portfolio at time t_n, *before the holder exercises*, is equal to $V_n^\bullet$. Since the solution of the dynamical programming equation satisfies

$$V_n^j \geq F(S_n^j) \ , \ \forall j \ , \ \forall n \ ,$$

the value of the replicating portfolio will always be at least equal to the intrinsic value of the option (the liability faced by the seller of the option). Therefore, the strategy is effectively riskless.

[5] The superscript $\bullet$ represents an arbitrary level $0 \leq j \leq n$.

5. Characterization of the solution for $dt \ll 1$: free-boundary problem for the Black-Scholes equation

In the limit $dt \ll 1$, the solution of the recursion relation (7)-(8) can be expressed as a partial differential equation with a *free boundary condition.* This free boundary represents the geometric boundary of the set where the value of the option is equal to the intrinsic value (exercise region). This section provides a discussion of the limit $dt \ll 1$ of the binomial pricing of American options. This characterization in terms of partial differential equations is useful to construct more advanced numerical schemes for pricing and hedging American options as well as to understand the Gamma-risk at the exercise boundary.[6]

From the previous analysis, we know that there is an exercise region, where $V_n^j = F(S_n^j)$ and its complement in the (S,t)-plane, where the dynamic programming equation reduces to the linear relation

$$V_n^j = e^{-r\,dt}\left[\, P_U\, V_{n+1}^{j+1} + P_D\, V_{n+1}^j \,\right] . \tag{12}$$

This relation is identical to the one satisfied by European-style derivative securities. It is therefore not surprising that the value of the American option should satisfy the Black-Scholes differential equation in the complement of the exercise region for $dt \ll 1$. More precisely, $V(S,t)$ represent the limiting value of the American option outside the exercise region, i.e.,

$$V(S,t) = \lim_{S_n^j = S\ ,\ (N-n)\,dt = t\ ,\ dt \to 0} V_n^j .$$

Here, t represents the time-to-expiration of the option. Then, according to the analysis of European contingent claims, we know that $V(S,t)$ satisfies the equation

$$\frac{\partial V(S,t)}{\partial t} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V(S,t)}{\partial S^2} + (r-q)\,S\,\frac{\partial V(S,t)}{\partial S} - r\,V(S,t) \tag{13}$$

for (S,t) in the complement of the exercise region. (A rigorous proof of this result will be given later.) However, the precise location of the boundary of the exercise region is unknown. The following proposition characterizes the behavior of $V(S,t)$ along the exercise region.

Proposition 2. *Let $\mathcal{B}$ represent the boundary of the exercise region, i.e. the region where $V(S,t) = F(S)$. Then*

[6] PDE solvers, usually based on finite-difference schemes can be used to obtain a more accurate characterization of the exercise boundary (see Wilmott, Dewynne, Howison: *Option Pricing*). We shall discuss the PDE approach later in the course.

$$V(S,t) \;=\; F(S) \;\text{ for }\; S \in \mathcal{B} \tag{14}$$

$$\frac{\partial V(S,t)}{\partial S} \;=\; F'(S) \;\text{ for }\; S \in \mathcal{B} \tag{15}$$

Also,

$$\Theta(S,t) \;=\; -\frac{\partial V(S,t)}{\partial} \;=\; 0 \;\text{ for }\; S \in \mathcal{B} \;. \tag{16}$$

Remark: For call options, (15) reads $\frac{\partial V(S,t)}{\partial S} = +1$. For puts, the boundary condition is $\frac{\partial V(S,t)}{\partial S} = -1$. In words, the graph of the function $V(S,t)$ is *tangent to graph of the intrinsic value* along the exercise boundary.

Proof of Proposition 2: Let For each t, let $S(t)$ represent the point at which

$$V(S(t),t) \;=\; F(S(t)) \;. \tag{17}$$

(Hence, $(S(t),t) \in \mathcal{B}$.) Since the value of the option cannot be less than the intrinsic value we have, along $\mathcal{B}$,

$$\frac{\partial V(S,t)}{\partial S}\Big|_{S=S(t)} \begin{cases} \geqq \; -1 \text{ for puts} \\ \\ \leqq \; +1 \text{ for calls} \;. \end{cases} \tag{18}$$

These inequalities show that the option premium should be *locally convex* along $\mathcal{B}$. However, considering the valuation problem for the view of the hedger of the option, we see that the option premium cannot have a jump in the first derivative, unless this jump is such that the premium is *locally concave.* The reason for this is that delta hedging across such a point (which would correspond to having infinite Gamma) would produce a loss to the short position if Gamma were infinitely negative. We conclude from this that the option premium must be locally convex *and* locally concave. The only possiblity is that it is locally "flat", i.e. that the derivative with repect to S is continuous across $\mathcal{B}$. The fact that the option has no time-decay across the free-boundary can be deduced immediately from this property and eq. (17). In fact, differentiating (17) with respect to t, we find that

$$\frac{\partial V(S(t),t)}{\partial S}\,\dot{S}(t) \;+\; \frac{\partial V(S(t),t)}{\partial t} \;=\; -\,\dot{S}(t) \;,$$

where $\dot{S}(t)$ represents the derivative of $S(t)$ with respect to t. Therefore,

$$\frac{\partial V(S(t),t)}{\partial t} = 0 \ .$$

The vanishing of Θ along the free boundary means that the agent who sells the American option and hedges using Delta will be *Gamma-neutral* along the free-surface. In fact, the Γ of the option satisfies, from the Black-Scholes equation,

$$\begin{aligned} 0 &= \frac{\partial V(S(t),t)}{\partial t} \\ &= \frac{1}{2}\,\sigma^2\, S(t)^2\, \Gamma(S(t),t) \ + \ (r-q)\, S(t)\, \Delta(S(t),t) \ - \ r\, V(S(t),t) \ . \end{aligned}$$

Therefore, if the dynamic hedge is such that the portfolio Delta is zero and the total value of the portfolio is zero, then $\Gamma \ = \ 0$ along the exercise boundary.