ANALYSIS OF THE BLACK-SCHOLES FORMULA

The appeal of the Black-Scholes formula lies in the simple relation expressed between the price of the underlying asset and of options written on it. Another important feature is that it gives a *dynamical* relation between the underlying instrument, the option value and the *hedge-ratio in the cash market*(Δ) that can be used to offset the risk of the option. The formula is the basis for designing dynamical hedging strategies for a variety of financial derivatives.[1]

Recall that the Black-Scholes formulas for pricing calls and puts are that

$$C(S,K;T) \;=\; S\,N(d_1) \;-\; K\,e^{-rT}\,N(d_2) \tag{1}$$

and

$$P(S,K;T) \;=\; K\,e^{-rT}\,N(-d_2) \;-\; S\,N(-d_1)\;. \tag{2}$$

Here S is spot price, K is the strike price, T is the time-to-maturity, r is the interest rate and σ is the volatility. The "percentiles" d_i are given by

$$\begin{cases} d_1 \;=\; \frac{1}{\sigma\sqrt{T}}\ln\left(\frac{S\,e^{rT}}{K}\right) \;+\; \frac{1}{2}\,\sigma\,\sqrt{T} \\[2ex] d_2 \;=\; \frac{1}{\sigma\sqrt{T}}\ln\left(\frac{S\,e^{rT}}{K}\right) \;-\; \frac{1}{2}\,\sigma\,\sqrt{T}\;. \end{cases} \tag{3}$$

More generally, the no-arbitrage value of a European-style derivative security maturing at in T years with stock-contingent value $F(S_T)$ at expiration is

$$\begin{aligned} V(S,T) \;&=\; e^{-rT}\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} F\left(S\;e^{\,z\,\sigma\,\sqrt{T}\,+\,(r\,-\,\sigma^2/2)\,T}\right)\;e^{-\frac{z^2}{2}}\,dz \\[2ex] &=\; e^{-rT}\,\mathbf{E}^{(l)}\,\{F(S_T)\} \end{aligned} \tag{4}$$

where $\mathbf{E}^{(l)}$ is the expectation value operator for the lognormal distribution with variance $\sigma^2\,T$ and drift $(r - \frac{1}{2}\sigma^2)\,T$. The latter formula can be used to price option portfolios or other European-style contingent claims.

[1]It is remarkable that equity options have existed for one hundred years but the Black-Scholes formula was discovered only in 1973. This is recognized as a major breakthrough in Modern Finance.

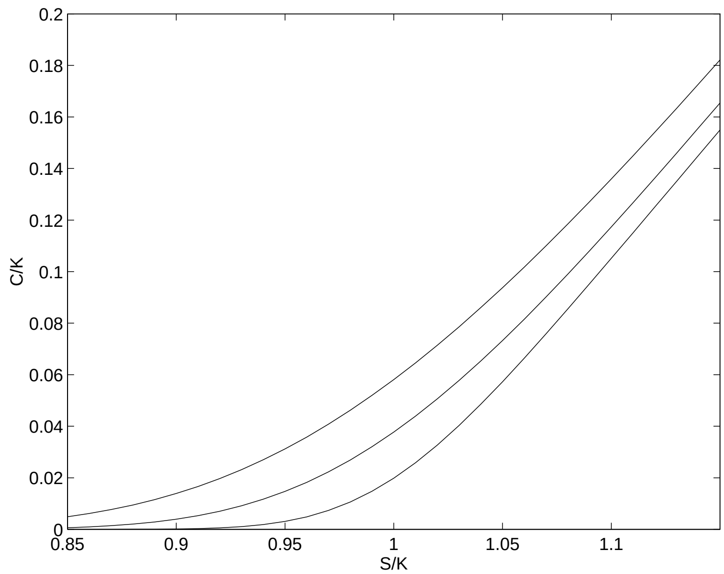

Figure 1

Figure 1 shows the Black-Scholes value of call option for different maturities (1, 3 and 9 months, with 15% volatility and 6% interest rate. Notice that the value of the call approaches zero for $S \ll K$ (the call is deep **out-of-the-money**) and is asymptotic to $S - K\,e^{-rT}$ — the value of a forward contract at price K — for $S \gg K$ (the call is deep **in-the-money**). Thus, a deep-in-the money call is essentially equivalent to a forward contract, whereas a deep out-of-the-money call is equivalent to a leveraged bet on the rise of the stock price.

The *premium* $C(S,K;T)$ is often "decomposed" into three parts:

- the "intrinsic value" $Max\,(S-K,0)$
- the "interest rate premium", $K\left(1 - e^{-rT}\right)$ if $S > K\,e^{-rT}$
- the "risk premium" $C(S,K;T) - Max\left(S - Ke^{-rT},0\right)$.

Clearly, the risk premium is sensitive to the value of the volatility used to price the call.

1. Delta

From the analysis of the binomial model, we know that the Delta of the equivalent portfolio (or, more simply, the Delta of the derivative security) satisfies

$$\Delta_n^j = \frac{V_{n+1}^{j+1} - V_{n+1}^j}{S_{n+1}^{j+1} - S_{n+1}^j}\,, \tag{5}$$

which may be viewed as a "discrete derivative" of the value function $V_n^\bullet$ with respect to the spot price. Hence, we expect the Delta of a derivative security to be equal to $\partial V/\partial S$ in the lognormal approximation. In fact, we have

Proposition 1. *Consider a European-style derivative security with payoff $F(S_T)$. Then, under the assumptions of the binomial model and its lognormal approximation, we have*

$$\lim_{dt \to 0} \Delta_0^0 \;\equiv\; \Delta(S,T) \;=\; \frac{\partial V(S,T)}{\partial S}\,, \tag{6}$$

where $V(S,T)$ is the value of the derivative security given in (4).

Proof. Assume, for simplicity, that F is a smooth function and let $\mathbf{E}^b$ denote the expectation operator with respect to the *binomial* risk-neutral probability measure. Then, from (5), if $dt \ll 1$, we have

$$\begin{aligned}
\Delta_0^0 \;&=\; \frac{e^{-rT}}{S\,(U-D)}\,\mathbf{E}^b\,\{\,F(S_N\,)\,|S_1 = SU\,\} \;-\; \frac{e^{-rT}}{S\,(U-D)}\,\mathbf{E}^b\,\{\,F(S_N\,)\,|S_1 = SD\,\} \\
&=\; \frac{e^{-rT}}{S\,(U-D)}\,\mathbf{E}^b\,\{\,F(S_{N-1}\,U\,)\,|S_0 = S\,\} \;-\; \frac{e^{-rT}}{S\,(U-D)}\,\mathbf{E}^b\,\{\,F(S_{N-1}\,D\,)\,|S_0 = S\,\} \\
&=\; \frac{e^{-rT}}{S\,(U-D)}\,\mathbf{E}^b\,\{F'(S_{N-1})\,S_{N-1}/S\,\} \;+\; \text{lower-order terms}\,.
\end{aligned} \tag{7}$$

To obtain the last equality, we applied the Intermediate Value Theorem to the function F. Passing to the limit as $dt \to 0$, we obtain

$$\begin{aligned}
\Delta(S,T) \;&=\; e^{-rT}\,\mathbf{E}^l\,\{F'(S_T)\,S_T/S\,\} \\
&=\; e^{-rT}\,\mathbf{E}^l\left\{\frac{\partial}{\partial S}F(S_T)\right\} \\
&=\; e^{-rT}\,\frac{\partial}{\partial S}\,\mathbf{E}^l\,\{F(S_T)\} \\
&=\; \frac{\partial}{\partial S}\,V(S,T)\,.
\end{aligned}$$

(Here, we used the fact that $S_T = S\, e^{\sigma\, Z\, \sqrt{T}\, +\, (r - 1/2\sigma^2)\, T}$ and thus $\partial S_T / \partial S = S_T / S$.) Q.E.D.

This is an important result. For a given interest rate and volatility, the discounted expectation with respect to the risk-neutral lognormal measure gives the theoretical value of the derivative security. The *sensitivity* of this theoretical value with respect to the spot price, i.e, $\partial V / \partial S$, gives *hedge-ratio*, or number of units of the underlying security which, combined with a short position in the derivative, will offset immediate market risk. Thus,

$$\text{Long}\, \Delta \text{ units of the underlying asset} \quad \text{short 1 derivative} \iff \text{market-neutral}\,. \tag{8}$$

Similarly,

$$\text{Short}\, \Delta \text{ units of the underlying security}\,, \text{ long 1 derivative} \iff \text{market-neutral} \tag{9}$$

These rules follow from Proposition 2 of the previous chapter, where we showed that that the above portfolios were equivalent, at any given date, to rolling over a position in riskless bonds for one period.

Remark. The actual number of cash instruments needed to hedge the market exposure depends on the notional amount specified in the contract. For instance, stock option contracts traded on CBOT and AMEX are written for a notional amount of **100 shares**. Prices are quoted on a per-share basis. Therefore, the number of shares required to achieve a "market-neutral" position is $\pm\Delta \times 100$.

Another example worth considering is the case of *exchange-rate options*. Exchange-rate options for dollars against Yen or Deutschemark (DEM) are generally quoted in *percentage of dollar notional*. On the other hand, exchange rates and option strikes are commonly quoted using the value of a dollar in local currency: e.g. 1 dollar = 1.4134 DEM, dollar put with strike 1.40 DEM, etc. In the valuation of option premia and Deltas, there is a choice of which currency should be used as the "local" currency and which currency is the "underlying asset". For instance, if the foreign currency (say DEM) is chosen as the underlying asset (this is natural for a US-based investor) then the Delta represents the amount of DEM spot which must be carried in the hedge per DEM notional. The amount of DEM *per dollar notional* is obtained by multiplying by (marks/dollar), e.g, by 1.4134. Currency options will be considered in more detail in other lectures. (See Figlewski, Silber and Subhramanyam: *Financial Options*.)

Option Deltas. To derive an expression for the Delta of a call, we differentiate (1) with respect to S. The result is

$$
\begin{aligned}
\Delta_{call}(S,K;T) &= \frac{\partial}{\partial S} C(S,K;T) \\
&= N(d_1) \; + \; S\,N'(d_1)\,\frac{\partial d_1}{\partial S} \; - \; K\,e^{-rT}\,N'(d_2)\,\frac{\partial d_2}{\partial S} \\
&= N(d_1)\ . \qquad (10)
\end{aligned}
$$

The last equality is obtained after some algebra[2], using the formulas

$$
N'(d) \;=\; \frac{1}{\sqrt{2}\pi}\, e^{-\frac{d^2}{2}}
$$

and

$$
\frac{\partial d_i}{\partial S} \;=\; \frac{1}{S\,\sigma\,\sqrt{T}}.
$$

Another (more complicated but also interesting) way to derive the expression for Δ is to pass to the limit in the formula for the Delta for the binomial option pricing.

The Delta for a put is obtained immediately from put-call parity. In fact, since

$$
P(S,K;T) \;=\; C(S,K;T) \;-\; S \;+\; K\,e^{-rT}\ ,
$$

we have

$$
\begin{aligned}
\Delta_{put} &= \Delta_{call} \; - \; 1 \;\; \Delta_{put}(S,K;T) \\
&= \; N(d_1) \; - \; 1 \\
&= \; N(-d_1) \qquad (11)
\end{aligned}
$$

In Figure 2 we present the graph of $\Delta_{call}(S,K;T)$ for the same parameter values as in Figure 1 ((—) = 6 months, (*)= 3 months, (- -)= 1 month). Notice that Δ is an increasing function of S (for both puts and calls).

[2] This calculation is suggested to those readers which are not familiar with the Black-Scholes formula.

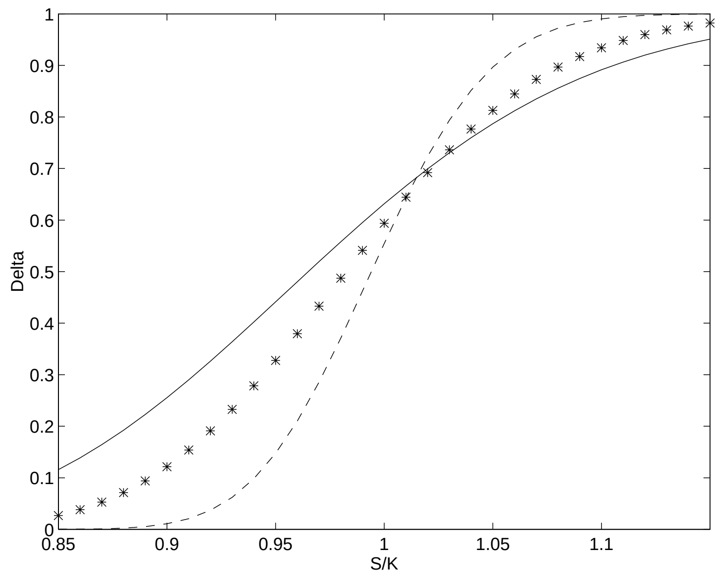

Figure 2

Example. 1. Calculate the Delta of a 6-month call option on an asset that pays no dividends, assuming a volatility of 16%, an interest rate of 6% and a strike price equal to 90% of spot.

Solution: Using the formula for d_1 given in (3), we find that $d_1 = 1.1348$. Hence $\Delta = N(1.1348) \approx 0.8719$

2. Practical Delta Hedging

The holder of a portfolio of options with different maturities and strikes can dynamically hedge his exposure to price movements using Δ. To fix ideas, suppose that the portfolio consists of M different types of options (put/call, strike, maturity) with n_j options of type j, $1 \leq j \leq M$. At some initial time ($t = 0$), we denote the M maturities by T_1, T_2, $...T_M$, and the strikes by K_1, K_2, $...K_M$. The number of options of each type is denoted by n_j, $j = 1...M$.

According to equation (9), the holder of this portfolio can hedge dynamically the risk due to price changes by holding

$$\Delta_t = - \sum_{j=1\ ,\ t\ T_j}^{M} n_j\, \Delta(S_t, K_j, T_j - t)\ ,$$

units of the underlying asset for times $t = t_n = n\, dt$, $...n = 0,\ 1,\ 2,\ ...\max\{T_j\}/dt$. (Here, dt represents a small interval of time which, in practice, must be determined by the hedger.) Assuming that the model assumptions were correct and, in particular, that **volatility was**

correctly estimated, the above strategy would have approximately riskless returns over time.

Example 2. Assume that $S = 100\,\sigma = 0.16$, $r = 0.07$. Calculate the portfolio Delta for

+5 100 Calls expiring in 60 days

-3 90 Puts expiring in 90 days

-4 85 Puts expiring in 120 days

Solution: Individual calculation of the Deltas for each option gives:

60 day call with strike 100	Delta = 0.5826
90 day put with strike 90	Delta = -0.0571
120 day put with strike 85	Delta = -0.0194

Combined Delta : $\Delta = 5\times\ 0.5826 + 3\times\ 0.0571 + 4\times 0.0194 = 3.1619$

Example 3. Calculate the Delta of the above portfolio if $\sigma = 30\%$

Solution:

60 day call with strike 100	Delta = 0.5617
90 day put with strike 90	Delta = -0.1857
120 day put with strike 85	Delta = -0.1228

Combined Delta : $\Delta = 5\times\ 0.5617 + 3\times\ 0.1857 + 4\times\ 0.1228 = 3.8568$

Several factors may contribute to making Δ-hedging risky in practice. Namely,

- The lognormal assumption may not be valid (model risk)
- The hedge may not be done frequently enough to prevent losses due to price movements (hedge-slippage risk)

The first type of risk is complex to control — its discussion is outside the scope of the basic Black-Scholes model. Model risk is the most fundamental risk in option risk-management and will be studied in more detail later, as we relax some of the assumptions of the Black-Scholes model.[3] One important dimension of model risk is the *mis-specification of the volatility parameter* σ.

[3] It is important for us to recognize, even at this early stage, the non-parametric nature of financial markets. The random-walk model is consistent with the no-arbitrage hypothesis but, markets being fundamentally incomplete, it is not sufficient to encompass changes in volatility expectations due to the arrival of new information.

Hedge-slippage risk, on the other hand, stems from the fact that if the portfolio is not rebalanced frequently enough, the strategy is no longer riskless. Changes in the value of the hedge portfolio are then governed by higher-order derivatives of the value function with respect to the spot price. This question is of crucial importance in applications. In fact, even if we believe that the underlying price follows approximately a binomial process under the risk-neutral probability, rehedging at every time-step is virtually impossible. Hence, the agent is obliged to rebalance the portfolio after "macroscopic", rather than "microscopic" time-intervals.

3. Gamma: the convexity factor

Definition. *Let $V(S,T)$ represent the value of a derivative security in the lognormal approximation. The Gamma (Γ) of the derivative security is the sensitivity of Δ with respect to S i.e.,*

$$\Gamma(S,T) \;=\; \frac{\partial^2 V(S,T)}{\partial S^2}\,. \tag{12}$$

The concept of Gamma is important when the position cannot be adjusted *exactly* after each "microscopic" time period dt. One way of analyzing the problem is to assume that rehedging is not done at the smallest micro-shocks, but instead at some intermediate time-scale δt, such that $dt \ll \delta t \ll T$. The accuracy of the hedge then depends on the rate of change of Δ as S changes, which is precisely Gamma.

To better understand the influence of convexity in hedging, it is useful to visualize the profit/loss from which arises from hedging in terms of the graph of $V(S,T)$. Given that the spot price is S, an agent that is short the derivative security and that takes a position in the equivalent portfolio will have a profit/loss in the cash market that varies on a straight line **tangent to the graph of the value function**. On the other hand, the value of the derivative security will vary along the graph of V. The effect of risk-neutral valuation is to achieve a position in the cash market after time δt which *on average* equal to the value of the derivative at that new time/ price level. However, unless the curvature of $V(S,T)$ is identically zero (in which case the contingent claim is equivalent to a static portfolio of cash instruments), there are instances in which

$$V(S_{t+\delta t}, T-(t+\delta t)) \;>\; S_{t+\delta t}\,\Delta_t \;+\; B_t\, e^{r\,\delta t} \tag{13a}$$

and other instances in which

$$V(S_{t+\delta t}, T-(t+\delta t)) \;<\; S_{t+\delta t}\,\Delta_t \;+ B_t\, e^{r\,\delta t} \tag{13b}$$

according to the magnitude of the price change of the period δt. It is graphically clear that inequality (13a) will hold for *large* price movements and inequality (13b) will hold for *small* price movements when V is convex (and thus $\Gamma > 0$).

Option Gamma The primary example of a **positive-Gamma position** is a long option position. In fact, according to the Black-Scholes formula, we have

$$\Gamma_{call}(S,K;T) = \Gamma_{put}(S,K;T) = \frac{1}{S\sqrt{2\pi\,\sigma^2\,T}}\,e^{-\frac{d_1^2}{2}}\,, \tag{14}$$

a formula which follows immediately by differentiating $\Delta(S,K,T) = N(d_1)$ with respect to S. The graph of Gamma as a function of the spot price is given in Figure 3. The parameter values are as in Figures 1 and 2.

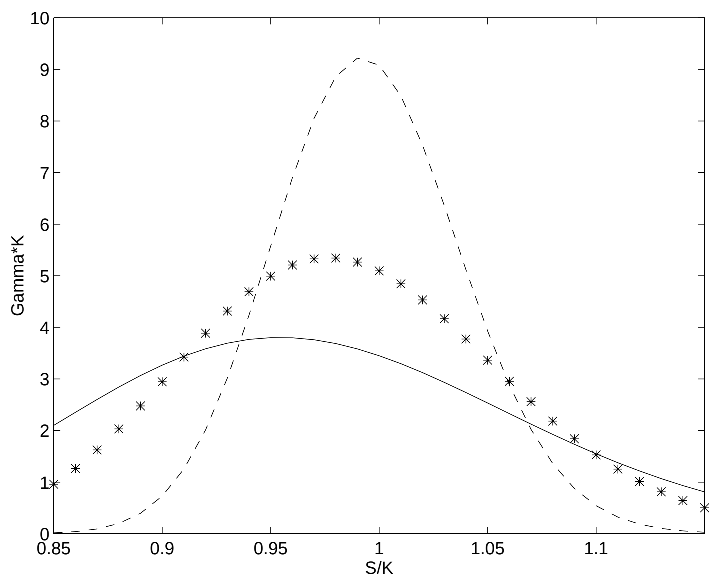

Figure 3

We conclude that

Proposition 2.*(i) The holder of an option who is short Δ units of the underlying asset will achieve a positive cash-flow if subsequently the price movement is sufficiently large and a negative cash-flow if the price movement is sufficiently small.*

(ii) The writer of an option who is long Δ units of the underlying security will achieve a positive cash-flow if subsequently the price movement is sufficiently small and negative cash-flow if the price movement is sufficiently large.

The use of Gamma becomes particularly relevant when managing option portfolios. Agents can, for instance, "buy" or "sell" Gamma in order to achieve a position which is consistent with their views for the near-future.[4] Buying or selling Gamma should be understood, of course, as buying or selling options.

Example 4. Calculate the Gamma of the portfolio of Examples 2 and 3.

Solution: Using formula (14), we find that if σ=0.16,

60 day call with strike 100 Gamma$\approx$ 0.05

90 day put with strike 90 Gamma$\approx$ 0.02

120 day put with strike 85 Gamma$\approx$ negligible

Combined Gamma of the portfolio : Γ = 5$\times$ 0.05 - 3$\times$ 0.02 = 0.19

If $\sigma = 0.30$, then

60 day call with strike 100 Gamma$\approx$ 0.04

90 day put with strike 90 Gamma$\approx$ negligible

120 day put with strike 85 Gamma$\approx$ 0.01

Combined Gamma of the portfolio : Γ = 5$\times$ 0.04 - 4$\times$ 0.01 = 0.16

4. Theta: the time-decay factor

The expression "an option is a wasting asset" is part of the options trading lore and is commonly used in option trading manuals. Consider, for instance, the case of call options. Since the interest-rate premium and the risk-premium are non-negative, the option is worth more than its intrinsic value at any time before expiration[5].

To evaluate the time-decay of the option, we can differentiate the option price with respect to T.

Definition. *The Theta (Θ) of a European-style contingent claim with value function $V(S,T)$ is defined as*

$$\Theta(S,T) \;=\; -\frac{\partial V\,(S\;T)}{\partial T}\;. \tag{15}$$

[4] Needless to say, the value of Gamma depends crucially on the volatility parameter — more on this later.

[5] We are assuming here that the underlying asset pays no dividends. Otherwise, this statement is not generally true.

The negative sign in this definition is due to the fact that T represents the time-to-maturity. Thus, Θ represents the variation in the fair value of a contingent claim for a small increment of time (decrease in time-to-expiration).

Proposition 3. *For any European-style contingent claim, we have*

$$\Theta(S,T) \;=\; -\,\frac{1}{2}\,\sigma^2\,S^2\,\Gamma(S,T) \;-\; r\,S\,\Delta(S,T) + \; r\,V(S,T) \tag{16}$$

Proof: Consider the basic formula

$$V(S,T) \;=\; e^{-rT}\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} F\left(S\;e^{\,z\,\sigma\,\sqrt{T}\,+\,(r\,-\,\sigma^2/2)\,T}\right)\;e^{-\,\frac{z^2}{2}}\,dz\;.$$

Differentiating with respect to T, we obtain

$$\frac{\partial V\,(S\;T)}{\partial T} \;=\;$$

$$-\,r\,V(S,T) \;+\; e^{-rT}\int_{-\infty}^{+\infty}\frac{\partial}{\partial T}\,F\left(S\;e^{\,z\,\sigma\,\sqrt{T}\,+\,(r\,-\,\sigma^2/2)\,T}\right)\;e^{-\,\frac{z^2}{2}}\,\frac{dz}{\sqrt{2\pi}}\;. \tag{17}$$

We can calculate the T-derivative of the function inside the integral sign. The result is

$$F'\left(S\;e^{\,z\,\sigma\,\sqrt{T}\,+\,(r\,-\,\sigma^2/2)\,T}\right)\cdot S\;e^{\,z\,\sigma\,\sqrt{T}\,+\,(r\,-\,\sigma^2/2)\,T}\cdot\left(\frac{z\,\sigma}{2\,\sqrt{T}}\;+\;r\;-\;\sigma^2/2\right)$$

$$\begin{aligned}
&=\; r\,S\,\frac{\partial}{\partial S}F\left(S\;e^{\,z\,\sigma\,\sqrt{T}\,+\,(r\,-\,\sigma^2/2)\,T}\right)\\
&+\; S\,\frac{\partial}{\partial S}F\left(S\;e^{\,z\,\sigma\,\sqrt{T}\,+\,(r\,-\,\sigma^2/2)\,T}\right)\cdot\left(\frac{z\,\sigma}{2\,\sqrt{T}}\;-\;\sigma^2/2\right)\;.
\end{aligned} \tag{18}$$

Let us substitute this expression into (17). This gives

$$\frac{\partial V\,(S\;T)}{\partial T} \;=\; -\,r\,V(S,T) \;+\; r\,S\,\Delta(S,T) \;+$$

$$e^{-rT}\int_{-\infty}^{+\infty} S\,\frac{\partial}{\partial S}F\,\left(S\;e^{\;z\;\sigma\;\sqrt{T}\;+\;(r\;-\;\sigma^2/2)\;T}\;\right)\cdot\left(\frac{z\,\sigma}{2\,\sqrt{T}}\;-\;\sigma^2/2\right)\;\;e^{-\;\frac{z^2}{2}}\;\frac{dz}{\sqrt{2\pi}}\;. \quad (19)$$

The desired result is now obtained by integration by parts, using the fact that

$$\frac{d}{dz}\,e^{-\;\frac{z^2}{2}} \;=\; -\,z\,e^{-\;\frac{z^2}{2}}\;.$$

To wit, we can rewrite the last integral in (19) as

$$e^{-rT}\int_{-\infty}^{+\infty} S\,\frac{\partial}{\partial z}\frac{\partial}{\partial S}F\,\left(S\;e^{\;z\;\sigma\;\sqrt{T}\;+\;(r\;-\;\sigma^2/2)\;T}\;\right)\cdot\frac{\sigma}{2\,\sqrt{T}}\;e^{-\;\frac{z^2}{2}}\;\frac{dz}{\sqrt{2\pi}}$$

$$-\;e^{-rT}\int_{-\infty}^{+\infty} S\,\frac{\partial}{\partial S}F\,\left(S\;e^{\;z\;\sigma\;\sqrt{T}\;+\;(r\;-\;\sigma^2/2)\;T}\;\right)\cdot\frac{\sigma^2}{2}\,e^{-\;\frac{z^2}{2}}\;\frac{dz}{\sqrt{2\pi}}$$

$$=\;e^{-rT}\int_{-\infty}^{+\infty} S\,\frac{\partial}{\partial S}\frac{\partial}{\partial z}F\,\left(S\;e^{\;z\;\sigma\;\sqrt{T}\;+\;(r\;-\;\sigma^2/2)\;T}\;\right)\cdot\frac{\sigma}{2\,\sqrt{T}}\;e^{-\;\frac{z^2}{2}}\;\frac{dz}{\sqrt{2\pi}}$$

$$-\;e^{-rT}\int_{-\infty}^{+\infty} S\,\frac{\partial}{\partial S}F\,\left(S\;e^{\;z\;\sigma\;\sqrt{T}\;+\;(r\;-\;\sigma^2/2)\;T}\;\right)\cdot\frac{\sigma^2}{2}\,e^{-\;\frac{z^2}{2}}\;\frac{dz}{\sqrt{2\pi}}$$

$$=\;e^{-rT}\int_{-\infty}^{+\infty} S\,\frac{\partial}{\partial S}\,S\,\frac{\partial}{\partial S}F\,\left(S\;e^{\;z\;\sigma\;\sqrt{T}\;+\;(r\;-\;\sigma^2/2)\;T}\;\right)\cdot\frac{\sigma^2}{2}\,e^{-\;\frac{z^2}{2}}\;\frac{dz}{\sqrt{2\pi}}$$

$$-\;e^{-rT}\int_{-\infty}^{+\infty} S\,\frac{\partial}{\partial S}F\,\left(S\;e^{\;z\;\sigma\;\sqrt{T}\;+\;(r\;-\;\sigma^2/2)\;T}\;\right)\cdot\frac{\sigma^2}{2}\,e^{-\;\frac{z^2}{2}}\;\frac{dz}{\sqrt{2\pi}}$$

$$= e^{-rT} \int_{-\infty}^{+\infty} S^2 \frac{\partial^2}{\partial S^2} F \left(S\, e^{\, z\, \sigma\, \sqrt{T} \,+\, (r \,-\, \sigma^2/2)\, T} \right) \cdot \frac{\sigma^2}{2}\, e^{-\,\frac{z^2}{2}}\, \frac{dz}{\sqrt{2\pi}}$$

$$= \frac{\sigma^2}{2}\, S^2\, \Gamma(S,T)\ . \tag{20}$$

Combining (20) with (19), we conclude that equation (16) holds. Q.E.D.

As an application, consider the case of an agent hedging a position in a derivative security. We can assume, without loss of generality that, initially, both $\Delta(S,T)$ and $V(S,T)$ are zero, since this can be achieved by assuming a position in shares and riskless bonds. In this case, equation (16) reduces to

$$\Theta(S,T) \ = \ -\, \frac{1}{2}\, \sigma^2\, S^2\, \Gamma(S,T) \tag{21}$$

In other words, *if a portfolio of derivative securities and cash instruments has zero initial cost and is Delta-neutral, the time-decay factor* Θ *and the convexity factor* Γ *multiplied by* $\frac{\sigma^2}{2}\, S^2$ *are exactly opposite to each other.*

This is just a reformulation of Proposition 2: the *owner of Gamma* (who is net long options) is subject to time-decay value if the spot does not move but benefits from price movements. Conversely, the *seller of Gamma* (who is net short options) is subject to hedge-slippage risk if the spot price moves but gains if spot does not move.

5. The Black-Scholes partial differential equation

Proposition 3 can be formulated as a dynamical evolution equation for the value of a derivative security in the lognormal approximation. In fact, from (16), we have

Proposition 4. *Consider a European derivative security, contingent on the value of a security that pays no dividends, with final value* $F(S_T)$*, where* T *is the time-to-maturity. In the lognormal approximation, its value* $V(S,T)$ *satisfies the* **Black-Scholes partial differential equation**

$$\frac{\partial V(S,T)}{\partial T} \ = \ \frac{\sigma^2}{2}\, S^2\, \frac{\partial^2 V(S,T)}{\partial S^2} \ + \ r\, S\, \frac{\partial V(S,T)}{\partial S} \ - r\, V(S,T) \tag{22}$$

with

$$V(S,0) \; = \; F(S) \; .$$

This equation will be studied in the context of Continuous-time Finance. We will also provide a direct derivation of the Black-Scholes equation from the recursion relation for contingent-claim pricing on the binomial lattice,

$$V_n^j \; = \; \frac{1}{1+R} \left[P_U \, V_{n+1}^{j+1} \; + \; P_D \, V_{n+1}^j \right] \; ,$$

which is the "discrete analogue" of (22).