THE BINOMIAL OPTION PRICING MODEL

The simplest model for pricing derivative securities is the binomial model. It generalizes the one period "up-down" model of Chapter 1 to a multi-period setting, assuming that the price of the underlying asset follows a random walk.

In the binomial model, there are N trading periods and $N+1$ trading dates, t_0 , t_1 , ... t_N when it is possible to invest in a risky security with price S_n , $n = 0, 1, \dots N$, and a riskless bond with one-period yield R. The price varies according to the rule

$$S_{n+1} = S_n \, H_{n+1} \quad , \; 0 \le n \le N-1 \; , \tag{1}$$

where H_{n+1} is a random variable such that

$$H_{n+1} = \begin{cases} U & \text{with probability } p \\ \\ D & \text{with probability } q \end{cases} \tag{2}$$

with $p + q = 1$.

The situation can be visualized in terms of a *binomial tree*, shown in Figure 1. Each node of the tree is labeled by a pair of integers $(n\,, j)$, $n = 0 \; , \; \dots N$, and $j = 0 \; , \; \dots n$. We use the convention that node $(n\,, j)$ leads to nodes $(n+1\,, j)$ and $(n+1\,, j+1)$ at the next trading date with the "up" move corresponding to $(n+1\,, j+1)$ and the "down" move to $(n+1\,, j)$. The price of the underlying asset at the node $(n\,, j)$ is therefore

$$S_n^j = S_0^0 \, U^j \, D^{n-j} \; .$$

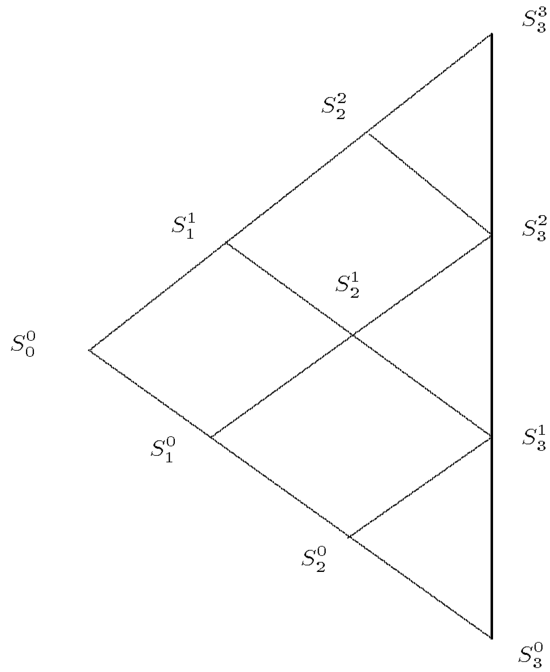

FIGURE 1

Assume first that the risky asset is a stock which pays no dividends for $0 \leq n \leq N$. Let us determine a probability measure on the set of paths $(S_0 , S_1 , \ldots , S_n , S_N)$ which makes the model arbitrage-free.[1] Since the asset price divided by the returns on riskless investment must be a a martingale under the pricing measure, we should have

$$S_n = \frac{1}{1+R} E\{ S_{n+1} \mid S_n \}$$

or,

$$1 = \frac{1}{1+R} (P_U U + P_D D) , \tag{3}$$

where P_D and P_U represent risk-neutral onditional probabilities for up or down moves given the spot price S_n. Using equation (3) and the constraint $P_U + P_D = 1$, we can determine the probabilities P_U and P_D in terms of the parameters of the model. They are given by

$$P_U = \frac{1+R - D}{U - D} , \quad P_D = \frac{U - (1+R)}{U - D} \tag{4}$$

[1] The terminologies probability measure, probability distribution, probability assignment, etc. are equivalent for us.

In particular, P_U and P_D are independent of the spot price. Thus, we have

Proposition 1. *Given the parameters U, D and R, there is a unique probability measure which makes the binomial model arbitrage-free. This probability is the one for which the successive price shocks H_n , $n = 1$, ... N are independent and identically distributed with*

$$\text{Prob.}\{ H_n = U \} = \frac{1+R - D}{U - D}$$

and

$$\text{Prob.}\{ H_n = D \} = \frac{U - (1+R)}{U - D} .$$

1. Recursion relation for pricing contingent claims

We give a general method for calculating the value of a "European-style contingent-claim", i.e. a derivative security which gives the holder the right to a payoff contingent on the value of the stock at some fixed date in the future. This date is taken to be $t_n = t_N$ and the payoff is represented by a known function of the stock price $F(S_N^j)$.[2] We denote the value of the contingent claim at the trading date n conditional on $S_n = S_n^j$ — i.e. its value at the node (n, j)— by

$$V_n^j = V_n(S_n^j) .$$

Since this derivative security has no intermediate cash-flows or coupons, its value must satisfy

$$V_n^j = \frac{1}{1+R} E\{ V_{n+1}(S_{n+1}) \mid S_n = S_n^j \}$$

or,

$$\begin{cases} V_n^j = \frac{1}{1+R} \left[P_U V_{n+1}^{j+1} + P_D V_{n+1}^j \right] \\ \\ V_N^j = F(S_N^j) \end{cases} \tag{5}$$

This recursion relation determines the arbitrage-free value of the contingent claim inductively, from its values at maturity. From the values at maturity, one can derive the values at the date t_{N-1} and proceed backwards in time until the present date. This procedure is sometimes called "rolling back the tree". Despite its apparent simplicity, relation (5) is very important. It will "resurface" under different forms in the course of these lectures.

[2] We will use the expression "date n" instead of "date t_n" sometimes.

The value of the derivative security can be obtained in closed form. For this, we think of the arbitrage-free price as the discounted expected cash-flows of the security, so that

$$V_n^j \;=\; \frac{1}{(1\;+\;R\;)^{N-n}}\;\; E\{\;F(\;S_N\;)\;|\;S_n\;=\;S_n^j\;\}$$

$$=\; \frac{1}{(1\;+\;R\;)^{N-n}}\;\sum_{k=0}^{N}\text{Prob.}\{\;S_N\;=\;S_N^k\;|\;S_n\;=\;S_n^j\;\}\cdot\;F(\;S_N^k\;)\;. \tag{6}$$

Let us calculate the conditional probabilities appearing in the latter equation. For this, we must "count" the number of paths going from S_n^j to S_N^k. Notice that if a path starts at S_n^j at time n it can end up at most at position S_N^{j+N-n}. This will happen only if $H_i \;=\; U$ for $i = \;n+1\; ...\; N-1$. Similarly, the lowest possible value is S_N^j, which corresponds to $H_i \;=\; D$ for all i. For any k such that $j \;<\; k \;<\; N-n+j$, the path will end up at S_N^k if and only if $H_i \;=\; U$ for $k-j$ periods out of $N-n+j$ and $H_i \;=\; D$ for the rest of the periods. It is important to notice that only the *number* of "up-shocks" matters and not the order in which they ocurr. Therefore, since the random variables H_i are independent, we have

$$\text{Prob.}\{\;S_N\;=\;S_N^k\;|\;S_n\;=\;S_n^j\;\}\;=\;\binom{N-n}{k-j}\cdot\;P_U^{k-j}\;P_D^{N-n-k+j}\;, \tag{7}$$

where

$$\binom{N-n}{k-j}\;=\;\frac{(N-n)!}{(k-j)!\;(N-n-k+j)!}$$

is the number of combinations of $k-j$ elements in a set of $N-n$ elements. Thus, in the binomial model, the Arrow-Debreu conditional probability distribution for S_N given S_n is the *multinomial distribution*, well-known from elementary Probability Theory.

The value (6) can be rewritten using equation (7) as

$$V_n^j \;=\; \frac{1}{(1\;+\;R\;)^{N-n}}\;\sum_{k=j}^{N-n+j}\binom{N-n}{k-j}\;\cdot P_U^{k-j}\;P_D^{N-n-k+j}\;\cdot\;F(\;S_N^k\;)\;.$$

In particular, if $n = \;j\;=\;0$, we obtain the calue in explicit form:

$$V_0^0 \;=\; \frac{1}{(1\;+\;R\;)^{N}}\;\sum_{k=0}^{N}\binom{N}{k}\;\cdot P_U^{k}\;P_D^{N-k}\;\cdot\;F(\;S_N^k\;)\;. \tag{8b}$$

2. Delta-hedging and the replicating portfolio

The pricing formula can be derived in a different way, using replicating portfolios.

Suppose that at the n^{th} trading date, the stock price is S_n^j and that an agent holds (long or short) a portfolio of Δ_n^j shares and B_n^j dollars in an interest-bearing account, with total value

$$V_n^j \;=\; \Delta_n^j\, S_n^j \;+\; B_n^j \;.$$

The value of the agent's holdings after the next trading period would be

$$\begin{cases} \Delta_n^j\, S_{n+1}^{j+1} \;+\; B_n^j\,(1+R) \;\equiv\; V_{n+1}^{j+1} & \text{in the ``up'' state} \\[2ex] \Delta_n^j\, S_{n+1}^{j} \;+\; B_n^j\,(1+R) \;\equiv\; V_{n+1}^{j} & \text{in the ``down'' state} \end{cases}$$

A relation between $\left(\Delta_n^j\, B_n^j\right)$ and the subsequent portfolio values is obtained by solving this system of equations. The result is:

$$\Delta_n^j \;=\; \frac{V_{n+1}^{j+1} \;-\; V_{n+1}^{j}}{S_{n+1}^{j+1} \;-\; S_{n+1}^{j}} \quad \text{shares} \tag{9a}$$

and

$$B_n^j \;=\; -\,\frac{1}{1\,+\,R}\,\frac{S_{n+1}^{j}\,V_{n+1}^{j+1} \;-\; S_{n+1}^{j+1}\,V_{n+1}^{j}}{S_{n+1}^{j+1} \;-\; S_{n+1}^{j}} \tag{9b}$$

Straightforward algebra shows that

$$\begin{aligned} V_n^j \;&=\; \frac{1}{1\,+\,R}\left[\,\frac{S_n^j(1+R) \;-\; S_{n+1}^{j}}{S_{n+1}^{j+1} \;-\; S_{n+1}^{j}}\cdot V_{n+1}^{j+1} \;+\; \frac{S_{n+1}^{j+1} \;-\; S_n^j(1+R)}{S_{n+1}^{j+1} \;-\; S_{n+1}^{j}}\cdot V_{n+1}^{j}\,\right] \\ &=\; \frac{1}{1\,+\,R}\left[\,P_U\,V_{n+1}^{j+1} \;+\; P_D\,V_{n+1}^{j}\right] \end{aligned} \tag{10}$$

where P_U and P_D are given in (4). Notice the similarity between this last equation and the recursion relation of the previous section.

Assume now that V_n^j is a function defined on the nodes of the binomial tree which satisfies the recursion relation (5) with final condition $V_N^j \;=\; F(S_N^j)$. Consider the following **trading strategy**:

(i) at time n take the position $\left(\Delta_n^j\, B_n^j\right)$.

(ii) at subsequent times, maintain a number of shares in the portfolio equal to

$$\Delta_k^j \;=\; \frac{V_{k+1}^{j+1} \;-\; V_{k+1}^{j}}{S_{k+1}^{j+1} \;-\; S_{k+1}^{j}} \quad , \qquad n \;\leq\; k \;\leq N. \tag{11}$$

Equation (10) shows that the value of the agent's portfolio (stocks plus money-market account) will be equal to V_n^j, regardless of price movements, and that the share adjustments are such that the money-market account will be equal to $B_k^j \;=\; V_k^j \;-\; \Delta_k^j\, S_k^j$ for all $k \;\geq\; n$. The situation is described by the following diagram:

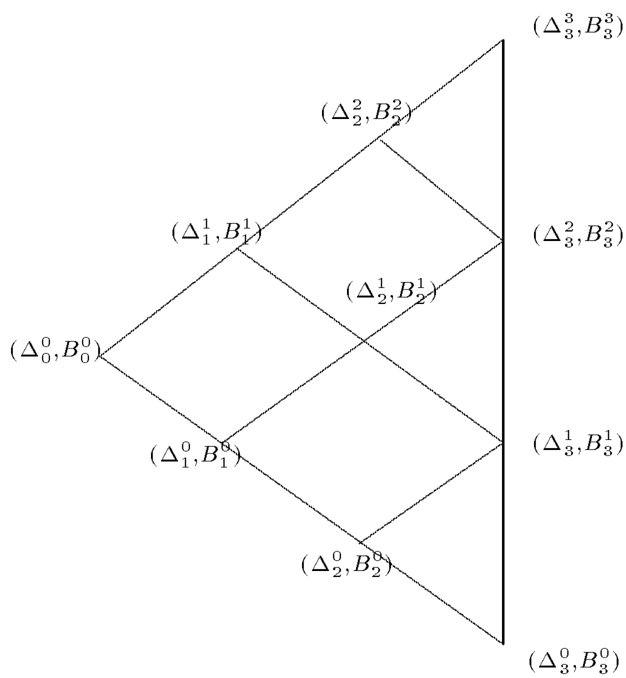

FIGURE 2. The replicating strategy is such that $B_n^j \;=\; V_n^j \;-\; \Delta_n^j\, S_n^j$ at each node of the tree. The final portfolio value is $F(S_N^j)$. (Here $N = 3$).

We conclude that

Proposition 2. *Assume that* $\left\{ V_k^j \;;\; n \;\leq k \;\leq\; N \;,\; k \;\leq\; j \;\leq\; n \right\}$ *satisfies the recursion relation (5). Then, an agent who takes an initial position in shares and bonds given by (9a) and (9b) and subsequently maintains* Δ_k^j *shares in his portfolio at time* t_k *if the spot price is* S_k^j*, financing this position with a money-market account, will have a a portfolio at time* t_N *worth* $F(S_N)$*. Thus, in the absence of arbitrage opportunities, if the spot price at time* t_n *is* S_n^j*, the fair value of a contingent claim with payoff* $F(S_N)$ *is exactly* V_N^j*.*

3. Example: Pricing European puts and calls

Let us calculate the fair price a European-style call option maturing after N periods using the binomial model. We shall use the payoff function

$$F(S_N) \ = \ \mathrm{Max} \ (\ S_N \ - \ K \ , \ 0 \ ) \ ,$$

where K is the strike price. According to (8), the arbitrage-free value of the call is (with $S_0^0 \ = \ S$)

$$C(\ S \ , \ K \ ; \ N \ ) \ = \ \frac{1}{(1 \ + \ R \ )^N} \ \sum_{k=0}^{N} \binom{N}{k} \cdot P_U^k \ P_D^{N-k} \ \cdot \ \mathrm{Max}(\ S_N^k \ - \ K \ , 0 \ )$$

$$= \ \frac{1}{(1 \ + \ R \ )^N} \ \sum_{S_N^k > \ K}^{N} \binom{N}{k} \cdot P_U^k \ P_D^{N-k} \ \cdot \ (\ S_N^k \ - \ K \ )$$

$$= \ S \cdot \left\{ \sum_{k > \ k_0}^{N} \binom{N}{k} \cdot Q_U^k \ Q_D^{N-k} \ \right\} \ - \ \frac{K}{(1 \ + \ R \ )^N} \cdot \left\{ \sum_{k > \ k_0}^{N} \binom{N}{k} \cdot P_U^k \ P_D^{N-k} \right\} \ , \qquad (12)$$

where

$$Q_U \ = \ \frac{U}{1+R} P_U, \qquad Q_D \ = \ \frac{D}{1+R} P_D$$

and where k_0 is the smallest integer which is greater or equal to

$$\ln(\frac{K}{S \, D^N}) \ / \ \ln(\frac{U}{D}) \ . \qquad (13)$$

The numerical evaluation of the call price can be made using tabular values for the multinomial distribution (notice that $Q_U \ + \ Q_D \ = \ 1$ from (3)), but, in practice, solving the recursion (5) numerically is recommended.

Portfolio Delta. The composition of the equivalent portfolio $(\Delta \, , B)$ for the call option can be derived in closed form. One way to do this is to observe, using (11), that the *equity component of the portfolio*, $E_k^j \ = \Delta_k^j \ S_k^j$, is a eterministic linear function of $V_{n+1}^{\cdot}$ and hence satisfies a recursion relation of type (5). Namely, we have

$$\begin{cases} E_n^j = \frac{1}{1+R} \cdot \left[P_U E_{n+1}^{j+1} + P_D E_{n+1}^j \right] \\ \\ E_N^j = S_N^j \quad \text{if } S_N^j \geq K \quad \text{and} \quad 0 \text{ if } S_N^j < K . \end{cases} \tag{14}$$

Therefore, we have

$$\Delta = \Delta_0^0 = E_0^0/S = \frac{1}{S(1 + R)^N} \sum_{k > k_0}^{N} \binom{N}{k} \cdot S_N^k P_U^k P_D^{N-k}$$

$$= \sum_{k > k_0}^{N} \binom{N}{k} Q_U^k Q_D^{N-k} . \tag{15}$$

Money-market account. Similarly, the *cash component* of the equivalent portfolio satisfies the recursion relation (5) with final values $B_N^j = -K$ if $S_N^j \geq K$ and $B_N^j = 0$ if $S_N^j < K$. We conclude that

$$B = B_0^0 = -\frac{K}{(1 + R)^N} \cdot \left\{ \sum_{k > k_0}^{N} \binom{N}{k} \cdot P_U^k P_D^{N-k} \right\} \tag{16}$$

It should be clear form these calculations that

• the equivalent portfolio for a call consists of a long equity position and a short cash position

• the roles of stock and cash are symmetric: a call on the stock is equivalent to a "put on cash" (exchange stock for cash when the latter is undervalued with respect to the stock).[3]

• Δ approaches unity for $S \gg K$ ($k_0 \approx 0$)

• Δ approaches zero for $S \ll K$ ($k_0 \approx N$)

Puts The value for a European put option predicted by the model is

$$P(S , K ; N) =$$

$$\frac{K}{(1 + R)^N} \cdot \left\{ \sum_{k < k_0}^{N} \binom{N}{k} \cdot P_U^k P_D^{N-k} \right\} - S \left\{ \sum_{k < k_0}^{N} \binom{N}{k} \cdot Q_U^k Q_D^{N-k} \right\} , \tag{17}$$

[3] This remark is important, for instance, in the case of currency options.

This formula follows from direct calculation or from put-call parity. Notice also that (17) can be obtained from (12) by exchanging S by $K/(1+R)^N$ and the probabilities $P_\bullet$ and $Q_\bullet$, consistently with the remark made above. (Thus, we have yet another way of deriving the value of a put from that of a call).

The equivalent portfolio of a put is

$$\Delta = -\sum_{k< k_0}^{N} \binom{N}{k} \cdot Q_U^k \, Q_D^{N-k}$$

$$B = \frac{K}{(1 + R)^N} \cdot \left\{ \sum_{k< k_0}^{N} \binom{N}{k} \cdot P_U^k \, P_D^{N-k} \right\} .$$

It is important to notice that (independently of the model used)

$$\Delta\,(\text{put}) - \Delta\,(\text{call}) = 1 .$$

To proceed futher with the analysis, we need to study how the parameters of the model are adjusted.

3. Adjusting the parameters of the tree : Volatility

The parameters N and R depend on the time-interval between successive portfolio adjustments and the interest rate for short-term deposits over the contract's lifetime. Let T represent the duration of the contract in years and let r denote the annualized interest rate. For simplicity, we assume equal periods between adjustments. The duration of each period is then

$$dt = \frac{T}{N}$$

and the interest-rate for one period is therefore

$$R = e^{r\,dt} - 1 \approx r\,dt .$$

The parameters U and D model the " variability" of the price of the underlying asset. For instance, if the asset were riskless we would have, trivially,

$$U = D = 1 + R = e^{r\,dt} .$$

In order to reflect in the model the market's expectations for the asset's variability, we will consider the *mean and the variance of the asset's returns under the arbitrage-free probability measure.* The annual yield (compound return) of a risky security over T years is, by definition,

$$Y \;=\; \frac{1}{T}\ln\left(\frac{S_N}{S_0}\right) \;. \tag{18}$$

(Thus $Y \;=\; r$ for a riskless bond). From Proposition 1, we know that Y is a sum of independent and identically distributed random variables under the arbitrage-free probability measure. In fact, we have

$$\begin{aligned}\ln(\frac{S_N}{S_0}) \;&=\; \sum_{n=1}^{N} \ln(\frac{S_n}{S_{n-1}}) \\ &=\; \sum_{n=1}^{N} \ln(H_n) \;\;.\end{aligned}$$

The mean and variance of the T-year yield are, respectively,

$$\begin{aligned}\mu \;=\; \mathrm{E}\{Y\} \;&=\; \frac{1}{T}\, N\, \mathrm{E}\{\ln\, H_1\,\} \\ &=\; \frac{1}{dt}\;[\,(\ln U)\;P_U \;+\; (\ln D)\;P_D\,]\end{aligned} \tag{19}$$

and

$$\begin{aligned}\mathrm{Var}\; Y \;&=\; \frac{1}{T^2}\, N\;\mathrm{Var}\,\ln(H_1) \\ &=\; \frac{1}{T\;dt}\,\{\,[\,(\ln U)^2\;P_U \;+\; (\ln D)^2\;P_D\,] \;-\; [\,(\ln U)\;P_U \;+\; (\ln D)\;P_D\,]^2\,\} \\ &=\; \frac{1}{T\;dt}\left[\ln\,(\frac{U}{D}\,)\,\right]^2 P_U\;P_D\end{aligned} \tag{20}$$

In particular, the variance of the one-year yield, obtained by setting $T \;=\; 1$ in the last equation, is

$$\sigma^2 \;\equiv\; \frac{1}{dt}\,\left[\ln\,(\frac{U}{D}\,)\,\right]^2\; P_U\; P_D\;. \tag{21}$$

Definition. *The standard deviation of the 1-year yield under the risk-neutral measure is known as the* **volatility** (*of the underlying asset*).

The average yield and the volatility can be thought of as input parameters for the model which reflect the value that the market assigns to future price oscillations. Note however that the no-arbitrage assumption implies that

$$\mathbf{E}\,\{\,S_n\,\}\;=\;S_0\,e^{r\,n\,dt}\quad,1\;\leq\;n\;\leq\;N\;.$$

In particular, by Jensen's inequality[4], we must have

$$\begin{aligned}\mu\;&=\;\frac{1}{dt}\,\mathbf{E}\left\{\ln\left(\frac{S_1}{S_0}\right)\right\}\\ &\leq\;\frac{1}{dt}\ln\left(\mathbf{E}\left\{\frac{S_1}{S_0}\right\}\right)\\ &=\;r\;.\end{aligned}$$

Hence, the no-arbitrage condition imposes a constraint on the value of μ. We shall return to this point shortly.

Calibration of volatility. Let us determine parameters U and D which are consistent with a given volatility σ. For this, it is convenient to define new parameters

$$\begin{cases} U\;=\;U'\;e^{r\,dt}\\ \\ D\;=\;D'\;e^{r\,dt}\;. \end{cases} \tag{22}$$

The no-arbitrage condition (3) then becomes

$$U'\;P_U\;+\;D'\;P_D\;=\;1\;. \tag{23}$$

We set

[4]Jensen's inequality states that, for any convex function f, and any random variable X, we have $f\,(\,\mathbf{E}\,X\,)\leq\;\mathbf{E}\;(\,f(X)\,)$.

$$\frac{U}{D} \; = \; \frac{U'}{D'} \; = \; e^{\, 2 \, \rho \sqrt{dt}} \; ,$$

where $\rho \; \geq \; 0$. Using (23) and the fact that P_U and P_D are probabilities, we obtain

$$\begin{cases} P_U \; + \; P_D \; = \; 1 \\ \\ P_U \; P_D \; = \; \frac{\sigma^2}{4 \; \rho^2} \; . \end{cases}$$

This system of equations has positive solutions if and only if $\rho \; \geq \; \sigma$. They are given by

$$P_U \; = \; \frac{1}{2} \; [\; 1 \; \pm \; \sqrt{ 1 \; - \; \frac{\sigma^2}{\rho^2} } \;] \tag{24a}$$

and

$$P_D \; = \; \frac{1}{2} \; [\; 1 \; \mp \; \sqrt{ 1 \; - \; \frac{\sigma^2}{\rho^2} } \;] \; . \tag{24b}$$

To find the corresponding values of U' and D' we use the formulas

$$P_U \; = \; \frac{1 \; - \; D'}{U' \; - \; D'} \quad , \quad P_D \; = \; \frac{U' \; - \; 1}{U' \; - \; D'} \; ,$$

which follow from (22) and (4). Accordingly, we have

$$U' \; = \; \frac{e^{\rho \sqrt{dt}}}{P_D \; e^{-\rho \sqrt{dt}} \; + \; P_U \; e^{\rho \sqrt{dt}}} \tag{25a}$$

and

$$D' \; = \; \frac{e^{- \; \rho \sqrt{dt}}}{P_D \; e^{-\rho \sqrt{dt}} \; + \; P_U \; e^{\rho \sqrt{dt}}} \; . \tag{25b}$$

Thus, the parameters U and D are given by

$$U \; = \; \frac{e^{\rho \sqrt{dt} + r \, dt}}{P_D \; e^{-\rho \sqrt{dt}} \; + \; P_U \; e^{\rho \sqrt{dt}}} \tag{26a}$$

and

$$D \;=\; \frac{e^{-\,\rho\sqrt{dt}+r\,dt}}{P_D\; e^{-\rho\sqrt{dt}} \;+\; P_U\; e^{\rho\sqrt{dt}}}\;. \tag{26b}$$

We conclude that

Proposition 3. *Given dt, there exists a one-parameter family of arbitrage-free binomial trees which is consistent with a given volatility value σ.*

Average yield. So far, we have not incorporated the average yield μ into the model. From equations (24), (26a) and (26b) we obtain

$$\mu \;=\; r \;+\; \frac{\rho}{\sqrt{dt}}\cdot\,(\;P_U \;-\; P_D\;) \;-\; \frac{1}{dt}\,\ln\,\left[\;P_D\; e^{-\rho\sqrt{dt}} \;+\; P_U\; e^{\rho\sqrt{dt}}\;\right] \tag{27}$$

Expanding the logarithm in powers of $\sqrt{dt}$ in this last equation and using the fact that

$$P_U \;-\; P_D \;=\; \pm\,\sqrt{\,1\;-\;\frac{\sigma^2}{\rho^2}\,}\;,$$

we obtain

$$\mu \;=\; r \;-\; \frac{1}{2}\,\sigma^2 \;+\; O\;(\;(P_U \;-\; P_D)\rho^3\;(dt)^{1/2})\;)\;. \tag{28}$$

Thus, in the limit $dt \;\ll\; T$, we have

$$\mu \;\approx\; r \;-\; \frac{1}{2}\,\sigma^2\;. \tag{29}$$

In this (important) limit, the average yield of the underlying asset is essentially determined by the interest rate and the volatility.

The most common common choices for U and D for implementating of the binomial model are the following:

1: *Symmetric probabilities.* Take $\rho \;=\; \sigma$. Then

$$P_U \;=\; P_D \;=\; \frac{1}{2} \tag{30a}$$

and

$$U \;=\; \frac{e^{\sigma\sqrt{dt}+r\,dt}}{\cosh\,(\sigma\;\sqrt{dt})} \;\approx\; e^{\sigma\sqrt{dt}+\left(r\,-\,\frac{1}{2}\,\sigma^2\,\right)dt}$$

$$D \;=\; \frac{e^{-\,\sigma\,\sqrt{dt}\,+\,r\,dt}}{\cosh\,(\sigma\,\sqrt{dt})} \;\approx\; e^{-\,\sigma\,\sqrt{dt}\,+\,\left(r\,-\,\frac{1}{2}\,\sigma^2\,\right)\,dt} \tag{30b}$$

It will be shown below that the approximation of $cosh(\sigma\,\sqrt{dt})$ made in these two formulas has a negligible effect on the value of derivative securities if $dt \ll T$.

2. *Specifying a subjective mean yeld.* We set

$$\begin{cases} U \;=\; e^{\sigma\,\sqrt{dt}\,+\,\nu\,dt} \\ \\ D \;=\; e^{-\,\sigma\,\sqrt{dt}\,+\,\nu\,dt} \end{cases} \tag{31a}$$

and

$$P_U \;=\; \frac{1}{2}\left(1 \;-\; \frac{r\;-\;\nu\;-\;\frac{1}{2}\sigma^2}{\sigma}\,\sqrt{dt}\,\right)$$

$$P_D \;=\; \frac{1}{2}\left(1 \;-\; \frac{r\;-\;\nu\;-\;\frac{1}{2}\sigma^2}{\sigma}\,\sqrt{dt}\,\right)\;. \tag{31b}$$

The probabilities in (31b) were obtained by expanding formula (4) in powers of dt and keeping terms up to order $\sqrt{dt}$.

The parameter ν can be interpreted as the "subjective" annual mean yield under the "subjective" probabilities $p_U \;=\; p_D \;= 1/2$. Observe that the expectation of S_1 under the *risk-neutral* probabilities is is

$$\mathbf{E}\,(S_1) \;=\; S_0\,(1\;+\;r\,dt)\;+\;\text{lower order terms}\;,$$

consistently with the arbitrage-free condition to this order of approximation.

Example 1: Suppose volatility is 15%, interest rate is 6% and you wish to construct a binomial tree with 10 periods to value a 6-month instrument. Then

$$dt \;=\; 0.5/10 \;=\; 0.05$$

Using the method of **1**, we have

$$\begin{cases} U \;=\; e^{\sigma\,\sqrt{dt}\,+\,r\,dt} \;=\; e^{0.15\times\sqrt{0.05}\,+0.06\times 0.05} \;=\; 1.0372 \\ \\ D \;=\; e^{-0.15\times\sqrt{0.05}\,+0.06\times 0.05} \;=\; 0.9700 \end{cases}$$

In this case the risk-neutral probabilities are $P_U = P_D = 0.5$.

4. The limit for $dt \to 0$: Lognormal approximation

The limiting behavior of the model as $dt \to 0$ is of particular interest.

Recall that, under the no-arbitrage probability, the logarithm of the price of the underlying asset is a sum of independent and identically distributed random variables:

$$Y = \frac{1}{T} \sum_{j=1}^{N} \ln H_j$$

Since the parameters of the model have been adjusted so that, to leading order for $dt \ll T$,

$$\mathrm{E}\{Y\} = r - \frac{1}{2}\sigma^2 \quad \text{and} \quad \mathrm{Var}\, Y = \frac{\sigma^2}{T},$$

we conclude from the Central Limit Theorem that the asymptotic probability distribution of the random variable Y as $dt \to 0$ is a Gaussian with mean $r - \sigma^2/2$ and variance σ^2/T. Hence

$$\ln\left(\frac{S_N}{S_0}\right) = T\, Y$$

$$\longleftrightarrow \mathbf{N}\left[\, (r - \sigma^2/2)\, T\, ,\ \sigma^2\, T \,\right] \qquad \Delta t \ll 1\,, \tag{32}$$

where $\mathbf{N}(a, b)$ is the normal distribution with mean a and variance b. In particular, the price of the underlying asset at time t is *lognormally distributed* i.e.

$$S_t = S_0\, e^{\sigma\sqrt{t}\, Z \,+\, (r - 1/2\sigma^2)\, t}\,, \tag{33}$$

where Z is $\mathbf{N}(0, 1)$.

Consider now a European-style derivative security with a payoff function $F(S)$ maturing in T years. Its value under the arbitrage-free probability measure is given by

$$V = e^{-r\,T}\, \mathrm{E}\,\{\, F(\, S_N\,)\,\}$$

Therefore, by the Central Limit Theorem, under mild regularity assumptions on the function F (linear growth of $F(S)$ is sufficient) we have

$$\lim_{dt \to 0} V \; = \; e^{-r\,T} \, \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F\left(S \, e^{\, z \, \sigma \, \sqrt{T} \;\; (r \, - \, \sigma^2/2) \; T}\right) \; e^{-\,\frac{z^2}{2}} \, dz \; , \qquad (34)$$

where S is the spot price and we used the explicit form of the normal probability distribution. The differences in the fair prices which result from the normal approximation for the binomial model can be estimated numerically. They are generally small, of order $\sigma\sqrt{dt}$ for $dt \; \ll \; T$.[5]

5. The Black-Scholes Formula

We apply the latter result to option pricing. Assuming a volatility σ and an interest rate r, the lognormal approximation to the binomial pricing model gives

$$C(S, K; T) \; = \; e^{-r\,T} \, \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{Max}\left(e^{\,\sigma\,\sqrt{T}\; z \; + \; (r \, - \, \sigma^2/2) \; T} \; - \; K \; , \; 0\right) \; e^{-\,\frac{z^2}{2}} \, dz$$

$$= \; e^{-r\,T} \, \frac{1}{\sqrt{2\pi}} \int_{d_2}^{+\infty} S \, e^{\;\sigma\,\sqrt{T} \;\; (r \, - \, \sigma^2/2) \; T} \; e^{-\,\frac{z^2}{2}} \, dz \; - K \, e^{-r\,T} \, \frac{1}{\sqrt{2\pi}} \int_{d_2}^{+\infty} e^{-\,\frac{z^2}{2}} \, dz \; , \qquad (35)$$

where the lower limit of integration is $d_2 \; = \; \frac{1}{\sigma\,\sqrt{T}} \ln\left(\frac{S\,e^{r\,T}}{K}\right) \; - \; \frac{1}{2}\,\sigma\,\sqrt{T}$. After making a change of variables in the first integral in (35), we obtain the celebrated **Black-Scholes formula**:

$$C(S, K; T) \; = \; S \, N(d_1) \; - \; K \, e^{-r\,T} \, N(d_2) \; , \qquad (36)$$

where

$$N(Z) \; = \; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{Z} e^{-\,\frac{z^2}{2}} \, dz \qquad (37a)$$

[5] This error estimate will be made precise later.

is the *cumulative normal distribution* or *error function*,[6]

$$d_1 \;=\; \frac{1}{\sigma\sqrt{T}} \ln\left(\frac{S\,e^{rT}}{K}\right) \;+\; \frac{1}{2}\,\sigma\,\sqrt{T}\;. \tag{37b}$$

and

$$d_2 \;=\; \frac{1}{\sigma\sqrt{T}} \ln\left(\frac{S\,e^{rT}}{K}\right) \;-\; \frac{1}{2}\,\sigma\,\sqrt{T}\;. \tag{37c}$$

Puts can be valued in the same way. Using put-call parity and the relation

$$N(-Z) \;=\; 1 \;-\; N(Z)\;,$$

we find that

$$P(S,K,T) \;=\; K\,e^{-rT}\,N(-d_2) \;-\; S\,N(-d_1)\;.$$

The reader should note the strong parallel between the Black-Scholes formula and the binomial option pricing formula in (12). In fact, the Black-Scholes formula could have been derived by passing to the limit in (12) using the so-called *de Moivre-Laplace theorem* (Central Limit Theorem for standard random walk) to estimate the sums by integrals.

Example 2: A volatility of 15% corresponds to a daily standard deviation of $\sigma_{daily} = 0.15/\sqrt{365} = 0.008 = 0.8\%$. Sometimes people take a 250-day year (removing weekends). The estimate for the daily standard deviation is then $\sigma_{daily} = 0.15/\sqrt{250} \approx 1\%$. It is useful to memorize this particular volatility value to get a "feeling" for the relation between daily movements and annualized volatility.

Example 3: Suppose that the interest rate is 6% and assume a volatility of 15%. The value of a 3-month European call option on an stock that pays no dividends, with strike at 90% of the share price, is calculated as follows:

$$d_1 \;=\; \tfrac{1}{0.15\times\sqrt{0.25}} \cdot \ln\left(e^{0.25\times 0.06}/0.9\right) \;+\; 0.5\times\,0.15\times\sqrt{0.25} \;=\; 1.6415$$

$$d_2 \;=\; \tfrac{1}{0.15\times\sqrt{0.25}} \cdot \ln\left(e^{0.25\times 0.06}/0.9\right) \;-\; 0.5\times\,0.15\times\sqrt{0.25} \;=\; 1..5673$$

$$N(1.6415) \;=\; 0.9497\;\;,\quad N(1.5673) \;=\; 0.9414\;\;,\quad e^{-0.25\times 0.06} \;=\; 0.9851$$

[6] Values of the error function are available in standard scientific calculators and software packages such as MATLAB.

Call value = $0.9497 - 0.9 \times 0.9851 \times 0.9414 = 0.1150 \approx 11.5\%$ of the share value.

The key variable which enters the Black-Scholes formula is the volatility parameter. Needless to say, the value of an option can vary significantly according to the volatility. Figure 2 shows the Black-Scholes value of a call as a function of the spot price using different volatilites. The next lecture analyzes in greater detail the Black-Scholes formula and its implications. In particular, we will focus on the effect of volatility on the value of an option on the composition of the equivalent portfolio.

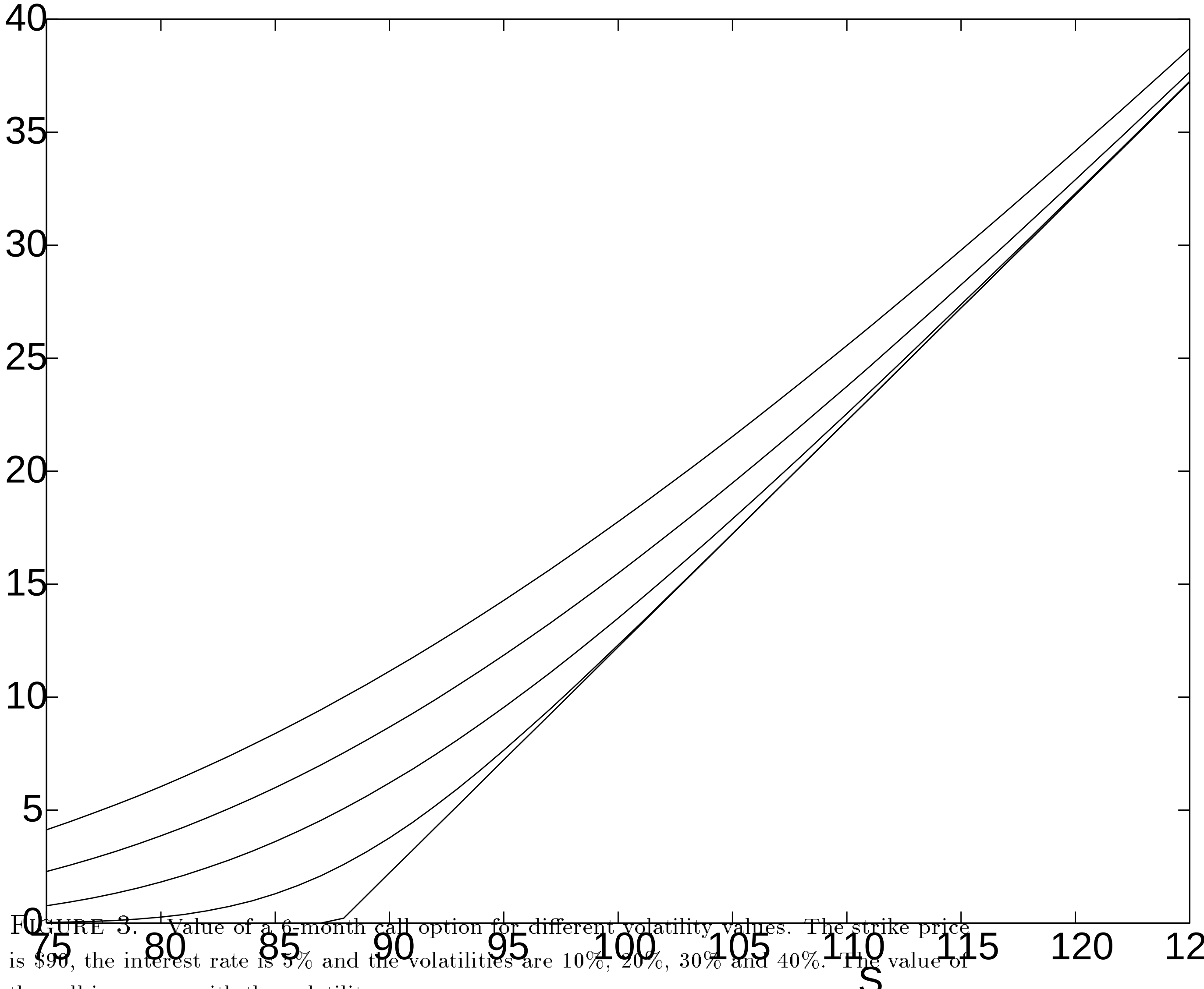

FIGURE 3. Value of a 6-month call option for different volatility values. The strike price is \$90, the interest rate is 5% and the volatilities are 10%, 20%, 30% and 40%. The value of the call increases with the volatility.