Market Assets

- Underlyings: stocks, bonds, foreign exchange
- Derivatives
  - forward and futures contracts
  - vanilla options (puts and calls)
  - exotic options
Stocks
Share of ownership of tangible company (“equity”)
Value depends on future earnings prospects
at least market’s opinion of what other people’s opinion will be
(“irrational exuberance” …)

Efficient market hypothesis:
Current price encodes all information available anywhere.
(If everyone thinks it will go up, it will already be up.)
Price changes depend on arrival of unanticipated information
→ suggests random walk models

We can anticipate magnitude of future movements, not sign.
Past history is good guide to statistical properties of future
movements (unless option trading modifies properties).
Bonds
Commitment to pay specific amount of cash at specific future time $T$
Issued by corporations (default risk) and government (no risk)
Present value depends on interest rate $r$ between now and $T$.
$r$ constant and known: present value $B_0 = e^{-rT}B_T$
E.g. overnight rate between large corporations (“repo rate”) (Short-term option pricing not too sensitive to $r$.)

Interesting issues:

- Dynamics of interest rates for different times (yield curve). Interest rate fluctuates “randomly”, like equities but many more “degrees of freedom”. Mathematically challenging.

- Credit spreads: additional interest demanded for borrower risk of default. Big events, small probability.
Forward and futures contracts
Agreement to buy asset for specific price at specific time.
No choice for either party.

strike (delivery) price : $K$
expiration date: $T$
value of contract: $F$
market price of underlying asset: $S$

At $t = T$, price of contract in terms of price of underlying:

$$F_T = S_T - K$$

Often just exchange this amount of cash.
Futures: standardized and traded on exchange
Payoff function of futures contract:

Long Forward

Short Forward
Determine present value $F_0$ by no-arbitrage reasoning:
You promise to deliver one ounce gold time $T$ in future for $K$.
You have risk that gold might go up between now and $T$.
Solution: go buy gold now, invest difference at risk-free rate.

Purchase one ounce gold now: cost $S_0$
Borrow $K e^{-rT}$
Net cash from pocket: $S_0 - Ke^{-rT}$

At expiry:
Customer gives you delivery price $K$
You pay off the loan (loan amount has grown to $K$)
You give him the gold
You are clear
The *only* proper price to charge for this contract is the cost of setting up the hedging portfolio:

\[ F_0 = S_0 - K e^{-rT} \]

• If contract is being bought for more than this, then sell a contract, buy gold and borrow cash as above, and pocket a riskless profit.

• If contract is being sold for less than this, then buy a contract, sell short one ounce of gold, and loan out the cash difference. At time \( T \) you have a riskless profit.

Delivery price \( K = e^{rT} S_0 \) often set so \( F_0 = 0 \)

Futures price is proxy for current price (often more liquid)

Currency futures: include foreign interest
Options
Buyer has the *right* but not the *obligation* to execute the deal

Call option: right to purchase asset for price $K$
Put option: right to sell asset for price $K$

Buy or sell to whoever holds the other side.
Value of option at expiry in terms of underlying

Call: \[ C_T = \max\{ S_T - K, 0 \} \]

Put: \[ P_T = \max\{ K - S_T, 0 \} \].
Long side: never negative because no obligation
corners are convex because choose better value

Long call, short put: *bullish* exposure (make money if price ↑)
Long put, short call: *bearish* exposure (make money if price ↓)

*Speculate:* Use options to enhance risk and reward (?)
*Hedge:* Use options to decrease exposure to preexisting risk

Evaluate option value using no-arbitrage pricing
Harder than forwards because payoff is nonlinear
Major distinction:

**European:** Can exercise *only* at expiry time
   Analytic expression for solution

**American:** Can exercise anytime up to expiry (most are this).
   Optimal control/decision problem
   Gives PDE with free boundary (hard)

Various types of exotic options:

*Spreads, straddles, etc:* superpositions of vanilla
   any payoff function $V(S, T)$

**Bermudan:** Exercise at finitely many specified times

**Asian:** Payoff based on time average of asset value

**Lookback:** Payoff based on *max* or *min* of value

**Barrier:** Become worthless/valuable if price crosses a level
Put/call parity:
Consider
- European call option
- European put option
- futures contract
all on same asset with same strike price $K$
Examine payoff: at $t = T$,

$$C - P = F.$$ 

Therefore also true at earlier times:

$$C = P + S - Ke^{-rT}$$

Can get either call or put value from the other
Useful to decompose more complicated instruments
Continuum limit
Above procedure valid for any \( N \), any set of node prices
Real world has (more or less) continuous time
continuous price evolution (actually, “tick” size)

Take \( N \to \infty \) for continuous time sampling
Take node spacing \( \to 0 \) for continuous price sampling

If model is reasonable, solution should converge to something
independent of details of tree

Discrete formula \( V_0 = e^{-rT}(qV_2 + (1 - q)V_2) \) will converge to
Black-Scholes partial differential equation for \( V(S, t) \)
Solve PDE with known values \( V(S, T) \), extract \( V(S_0, 0) \)
Need to specify tree more systematically
Choose tree to have constant up/down ratios at each node
Children of \( S_{i,j} \) are \( S_{i+1,j} \) and \( S_{i+1,j+1} \)

\[
S_{i+1,j+1} = u \, S_{i,j} \quad S_{i+1,j} = d \, S_{i,j}
\]

\( u, d \) uniform across tree, but depend on \( N \)

Proportional increments rather than fixed arithmetic jumps:

- maybe more reasonable:
  size of motion should scale with value of price

- mathematical problems if price becomes negative
  (might not happen if \( T \) is small)
Time levels equally spaced: \( k = T/N \)

Two degrees of freedom:

\[
\begin{align*}
  u &= \exp\left[ sk + \frac{1}{2}h \right] \\
  d &= \exp\left[ sk - \frac{1}{2}h \right].
\end{align*}
\]

“spread” \( h \):

\[
\frac{u - d}{d} = e^h - 1 \approx h
\]

“drift” \( s \):

\[
\sqrt{ud} = e^{sk}
\]

Want to take \( h \to 0 \) and \( s \) constant as \( k \to 0 \).

- Value of \( s \) will not matter (compensated by \( q \))
- Relationship of \( h \) to \( k \) will be critical

Option value depends on some properties of tree but not others.
Pricing on tree: same strategy as before

risk-neutral probabilities

\[ q = \frac{e^{rk} - d}{u - d} = \frac{e^{(r-s)k} - e^{-h/2}}{e^{h/2} - e^{-h/2}} \]

inequality constraint

\[ -\frac{1}{2} h < (r - s)k < \frac{1}{2} h. \]

iteration formula for option value

\[ V_{i,j} = e^{-rk} \left( q V_{i+1,j+1} + (1 - q) V_{i+1,j} \right) \]

Values \( V(S, T) \) given at \( t = T \): starting values

\[ V_{N,j} = V(S_{N,j}, N) \]
To take limit, need to specify relationship $h$ to $k$

Take

$$k = \lambda h^2, \quad \lambda \text{ constant as } h, k \to 0$$

or

$$h = \sqrt{k/\lambda}$$

How do we know that $k \sim h^2$ is right scaling?
To confirm, check final result in limit $\lambda \to 0, \infty$.

(Note: inequality constraint always satisfied when $h$ small.)
Aside:
convergence of finite-difference method to PDE in simple problem

Consider staggered grid of points

\[ t_i = ik, \quad i = 0, 1, \ldots \]
\[ x_{i,j} = \left( \frac{1}{2} i + j \right) h, \quad -\infty < j < \infty \]

Point \((i, j)\) has “ancestors” \((i - 1, j)\), \((i - 1, j + 1)\)

Define \(u_{i,j}\) by

1. \(u_{0,j} = f(jh)\), smooth function \(f(x)\)
2. \(u_{i,j} = \frac{1}{2} \left( u_{i-1,j} + u_{i-1,j+1} \right)\)
**Theorem:** As $h, k \to 0$ with $k = \lambda h^2$, then $u_{i,j} \to u(x_{i,j}, t_i)$, where $u(x, t)$ is solution of initial-value PDE

$$u(x, 0) = f(x)$$

$$\frac{\partial u}{\partial t} = \frac{1}{4\lambda} \frac{\partial^2 u}{\partial x^2}$$

**Proof:** Based on *Lax equivalence theorem* for finite-difference schemes

Consistency + Stability $\implies$ Convergence
Stability: obvious since $u_i$ is smoothing of $u_{i-1}$.

Consistency: replace $u_{i,j} = u(x_{i,j}, t_i)$ exactly, with $u_t = \frac{1}{4\lambda} u_{xx}$ see by how much difference formula fails to be satisfied:

Make local expansions about point $(i-1, j + \frac{1}{2})$.

$$u_{i,j} \sim u + ku_t + \cdots$$

$$u_{i-1,j} \sim u - \frac{h}{2} u_x + \frac{1}{2} \left( \frac{h}{2} \right)^2 u_{xx} + \cdots$$

$$u_{i-1,j+1} \sim u + \frac{h}{2} u_x + \frac{1}{2} \left( \frac{h}{2} \right)^2 u_{xx} + \cdots$$
\[ u_{i,j} - \frac{1}{2} (u_{i-1,j} + u_{i-1,j+1}) \sim k u_t - \left( \frac{h}{2} \right)^2 u_{xx} + \cdots \]
\[ \sim k \left( u_t - \frac{1}{4\lambda} u_{xx} \right) + \cdots \]

So consistency requires

\[ u_t = \frac{1}{4\lambda} u_{xx} \]

QED

Not to hard to make this fully rigorous

end of Aside
Option-pricing problem:
Assume smooth solution, make local asymptotic expansions about suitable intermediate point

\[ V_{i,j} = V - kv_t + \frac{1}{2}k^2V_{tt} + \cdots \]
\[ V_{i+1,j} = V - h_-V_S + \frac{1}{2}h_-^2V_{SS} + \cdots \]
\[ V_{i+1,j+1} = V + h_+V_S + \frac{1}{2}h_+^2V_{SS} + \cdots \]

Compute consistency just as for simple model problem
The *Black-Scholes equation for derivative pricing*:

\[ V_t + \frac{1}{2} \sigma^2 S^2 V_{SS} + r S V_S - r V = 0. \]

\[ \frac{1}{2} \sigma^2 = \frac{1}{8 \lambda} \]

Satisfied by value of *every* derivative security on $S$. (As long as no early exercise or other complication)

- Backwards parabolic: *terminal* data at $t = T$
- Linear: asset values add

$\sigma^2$ appears as a diffusion coefficient related to how much stock price “jumps around”
s has vanished!

$\sigma$ tells us how to construct grid: $h = 2\sigma \sqrt{k}$

How do we know $\sigma$?

How do we know other grids might not give different PDE? (stochastic calculus helps here)
Solution of Black-Scholes equation

new independent variables

\[ \chi = \log \frac{S}{S_{\text{ref}}}, \quad \tau = \sigma^2(T - t) \]

\( \tau \) is time remaining to expiration

New dependent function

\[ V(S, t) = V_{\text{ref}} u\left( \log \frac{S}{S_{\text{ref}}}, \sigma^2(T - t) \right) \]

Get

\[ u_{\tau} = \frac{1}{2} u_{xx} + \beta u_x - \gamma u, \]

with constant coefficients

\[ \beta = \frac{r - \frac{1}{2} \sigma^2}{\sigma^2}, \quad \gamma = \frac{r}{\sigma^2}. \]
Easy to solve initial-value problem using Green’s functions

Black-Scholes formula for European call with strike $K$:

$$C(S, t) = S N\left(\frac{\log(S/K) + \left(r + \frac{1}{2} \sigma^2\right) (T - t)}{\sigma \sqrt{T - t}}\right)$$

$$- K e^{-r(T-t)} N\left(\frac{\log(S/K) + \left(r - \frac{1}{2} \sigma^2\right) (T - t)}{\sigma \sqrt{T - t}}\right)$$

Get European put by put-call parity
Compare with market-quoted option values
Impossible to find single volatility value
that works for all strikes
(even at just one maturity date)

*Historical volatility:* The volatility estimated from statistical analysis of past price motions.

*Implied volatility:* The volatility you have to use in the Black-Scholes pricing formula to get the market price.
Dynamic Hedging:

\[ \Delta = \frac{V_2 - V_1}{S_2 - S_1} \rightarrow \frac{\partial V}{\partial S} \]

How much you have to hold at each instant to maintain hedge
Jumps around as much as stock price

Vanilla puts and calls as \( t \not\sim T \):

\begin{itemize}
  \item \( \Delta \rightarrow \pm 1 \) if option is expiring in the money
  \item \( \Delta \rightarrow 0 \) if expiring out of the money
\end{itemize}

You end with exactly what you need
American option
At each moment, a decision problem:
Should I exercise, or should I hold?

Option value always ≥ value of exercise
(e.g. \( V(S, t) \geq \Lambda(S, t) = \max\{S - K, 0\} \) for American call)

Tree modification for early exercise:
Compare computed value at each node against exercise value
Use the larger of the two.
This affects values at all nodes earlier in time.
Black-Scholes equation with early exercise:

\[-V_t \geq \frac{1}{2} \sigma^2 S^2 V_{SS} + r S V_S - r V \quad \text{and} \quad V(S,t) \geq \Lambda(S,t)\]

And at least one of them is an equality

A variational inequality, or obstacle problem

Free boundary problem:
critical stock price $S_*(t)$ where you should exercise
American put: exercise for $S < S_*$
American call: exercise for $S > S_*$, if stock pays dividends.

Physical analogs: oxygen consumption problem, Stefan problem
Stochastic models
Binomial tree  \( \rightarrow \) random walk
Assign probabilities to up and down moves
These probabilities disappear from final solution

Stock motion has probability distribution
Changes in \( \log S(t) \) are serially independent

\[
\log \frac{S(t)}{S(0)} \text{ is Gaussian}
\]

\[
\text{mean} = \mu t \\
\text{variance} = \sigma^2 t
\]

Model from Bachelier (1900) (for arithmetic motion)
First mathematics of Brownian motion (before Einstein)
Stochastic differential equation

\[ dS = \mu S \, dt + \sigma S \, dX, \]

\(dX\) is increment of standard Brownian

Theory is elegant, obscures messy reality