Market Assets

- Underlyings: stocks, bonds, foreign exchange
- Derivatives
  - forward and futures contracts
  - vanilla options (puts and calls)
  - exotic options

Stocks

Share of ownership of tangible company ("equity")
Value depends on future earnings prospects
at least market's opinion of what other people's opinion will be
("irrational exuberance" ...)

Efficient market hypothesis:
Current price encodes all information available anywhere.
(If everyone thinks it will go up, it will already be up.)
Price changes depend on arrival of unanticipated information
$\rightarrow$ suggests random walk models

We can anticipate *magnitude* of future movements, not *sign*.
Past history is good guide to *statistical properties* of future movements (unless option trading modifies properties).

Bonds

Committment to pay specific amount of cash
at specific future time T
Issued by corporations (default risk) and government (no risk)
Present value depends on interest rate r between now and T.
r constant and known: present value $B_0 = e^{-rT} B_T$
E.g. overnight rate between large corporations ("repo rate")
(Short-term option pricing not too sensitive to r.)

Interesting issues:

- Dynamics of interest rates for different times (*yield curve*). Interest rate fluctutates "randomly", like equities but many more "degrees of freedom". Mathematically challenging.

- *Credit spreads:* additional interest demanded for borrower risk of default. Big events, small probability.

Forward and futures contracts

Agreement to buy asset for specific price at specific time.
No choice for either party.

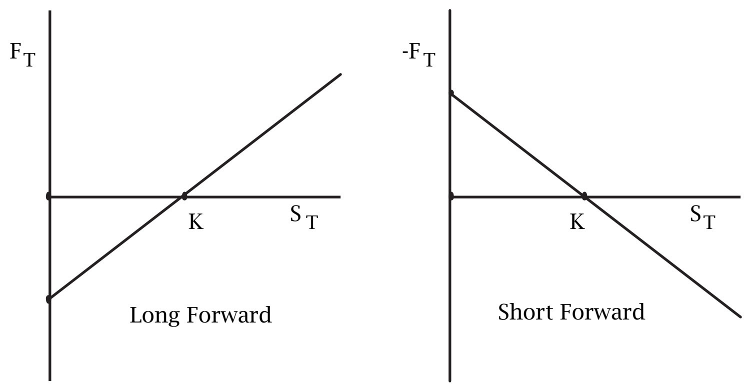
$$\begin{aligned} \text{strike (delivery) price}: &\quad K \\ \text{expiration date}: &\quad T \\ \text{value of contract}: &\quad F \\ \text{market price of underlying asset}: &\quad S \end{aligned}$$

At $t = T$, price of contract in terms of price of underlying:

$$F_T = S_T - K$$

Often just exchange this amount of cash.
Futures: standardized and traded on exchange

Payoff function of futures contract:



Determine present value F_0 by no-arbitrage reasoning:
You promise to deliver one ounce gold time T in future for K.
You have risk that gold might go up between now and T.
Solution: go buy gold now, invest difference at risk-free rate.

Purchase one ounce gold now: cost S_0

Borrow Ke^{-rT}

Net cash from pocket: $S_0 - Ke^{-rT}$

At expiry:

Customer gives you delivery price K

You pay off the loan (loan amount has grown to K)

You give him the gold

You are clear

The *only* proper price to charge for this contract is the cost of setting up the hedging portfolio:

$$F_0 = S_0 - K e^{-rT}$$

- If contract is being bought for more than this, then sell a contract, buy gold and borrow cash as above, and pocket a riskless profit.
- If contract is being sold for less than this, then buy a contract, sell short one ounce of gold, and loan out the cash difference. At time T you have a riskless profit.

Delivery price $K = e^{rT} S_0$ often set so $F_0 = 0$
Futures price is proxy for current price (often more liquid)
Currency futures: include foreign interest

Options

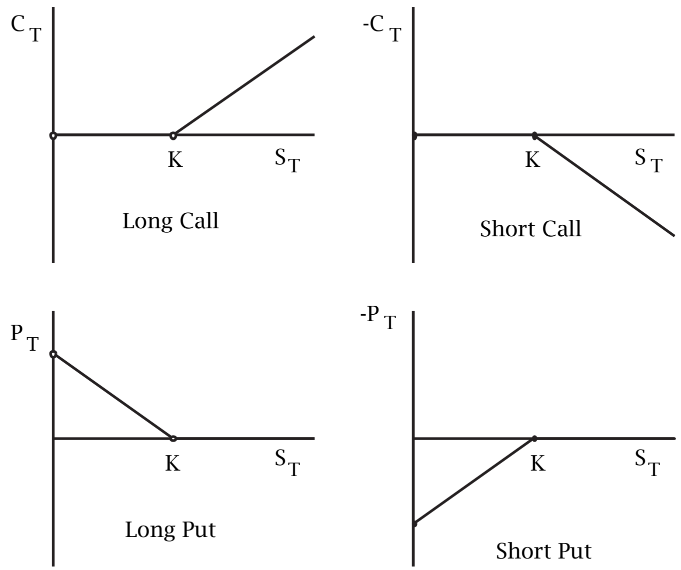
Buyer has the *right* but not the *obligation* to execute the deal

Call option:	right to purchase asset for price K
Put option:	right to sell asset for price K

Buy or sell to whoever holds the other side.

Value of option at expiry in terms of underlying

$$\begin{aligned} \text{Call:} \qquad C_T &= \max\{\, S_T - K,\ 0 \,\} \\ \text{Put:} \qquad P_T &= \max\{\, K - S_T,\ 0 \,\}. \end{aligned}$$


C_T
K
S_T
Long Call
$-C_T$
K
S_T
Short Call
P_T
K
S_T
Long Put
$-P_T$
K
S_T
Short Put

Long side: never negative because no obligation
corners are convex because choose better value

Long call, short put: *bullish* exposure (make money if price $\nearrow$)
Long put, short call: *bearish* exposure (make money if price $\searrow$)

Speculate: Use options to enhance risk and reward (?)
Hedge: Use options to decrease exposure to preexisting risk

Evaluate option value using no-arbitrage pricing
Harder than forwards because payoff is nonlinear

Major distinction:

European: Can exercise *only* at expiry time

Analytic expression for solution

American: Can exercise anytime up to expiry (most are this).

Optimal control/decision problem

Gives PDE with free boundary (hard)

Various types of exotic options:

Spreads, straddles, etc: superpositions of vanilla

any payoff function $V(S,T)$

Bermudan: Exercise at finitely many specified times

Asian: Payoff based on time average of asset value

Lookback: Payoff based on *max* or *min* of value

Barrier: Become worthless/valuable if price crosses a level

Put/call parity:

Consider

European call option

European put option

futures contract

all on same asset with same strike price K

Examine payoff: at $t = T$,

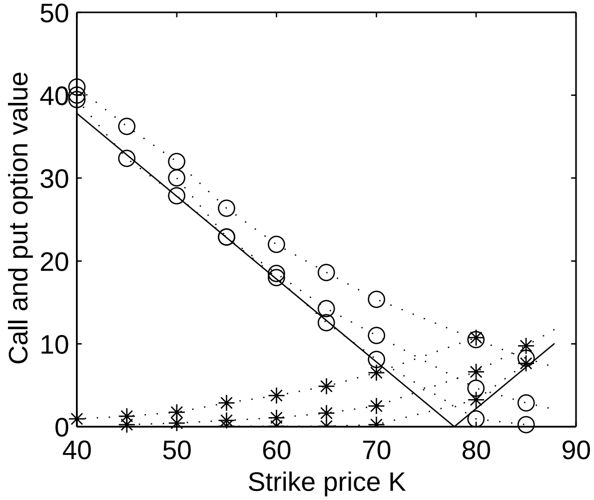
$$C - P = F.$$

Therefore also true at earlier times:

$$C = P + S - Ke^{-rT}$$

Can get either call or put value from the other

Useful to decompose more complicated instruments



Call and put option value
Strike price K
50
40
30
20
10
0
40
50
60
70
80
90

Continuum limit

Above procedure valid for any N, any set of node prices

Real world has (more or less) continuous time

continuous price evolution (actually, “tick” size)

Take $N \to \infty$ for continuous time sampling

Take node spacing $\to 0$ for continuous price sampling

If model is reasonable, solution should converge to something independent of details of tree

Discrete formula $V_0 = e^{-rT}(qV_2 + (1-q)V_2)$ will converge to *Black-Scholes partial differential equation* for $V(S,t)$

Solve PDE with known values $V(S,T)$, extract $V(S_0,0)$

Need to specify tree more systematically
Choose tree to have constant up/down *ratios* at each node
Children of $S_{i,j}$ are $S_{i+1,j}$ and $S_{i+1,j+1}$

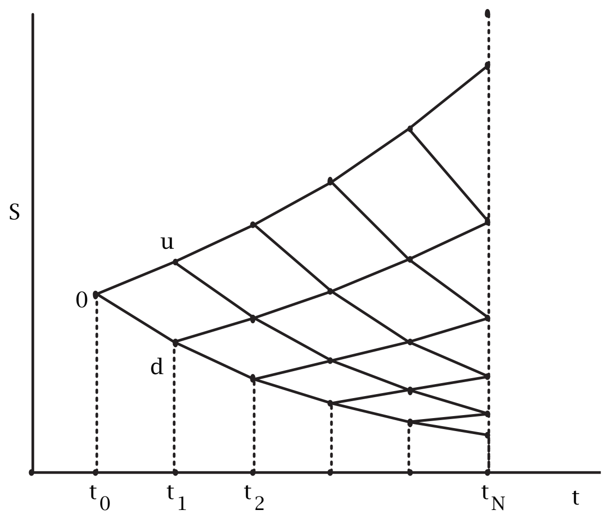
$$S_{i+1,j+1} \; = \; u\, S_{i,j} \qquad S_{i+1,j} \; = \; d\, S_{i,j}$$

u, d uniform across tree, but depend on N

Proportional increments rather than fixed arithmetic jumps:

- maybe more reasonable:
  size of motion should scale with value of price

- mathematical problems if price becomes negative
  (might not happen if T is small)


S
u
0
d
t_0
t_1
t_2
t_N
t

Time levels equally spaced: $k = T/N$

Two degrees of freedom:

$$u \;=\; \exp\left[sk + \tfrac{1}{2}h\right] \qquad d \;=\; \exp\left[sk - \tfrac{1}{2}h\right].$$

$$\text{“spread” } h: \qquad \frac{u-d}{d} \;=\; e^h - 1 \;\approx\; h$$

$$\text{“drift” } s: \qquad \sqrt{ud} \;=\; e^{sk}$$

Want to take $h \to 0$ and s constant as $k \to 0$.

- Value of s will not matter (compensated by q)
- Relationship of h to k will be critical

Option value depends on some properties of tree but not others

Pricing on tree: same strategy as before

risk-neutral probabilities

$$q \;=\; \frac{e^{rk} - d}{u - d} \;=\; \frac{e^{(r-s)k} - e^{-h/2}}{e^{h/2} - e^{-h/2}}$$

inequality constraint

$$-\tfrac{1}{2}h \;<\; (r - s)k \;<\; \tfrac{1}{2}h.$$

iteration formula for option value

$$V_{i,j} \;=\; e^{-rk}\Big(q\, V_{i+1,j+1} \;+\; (1 - q)\, V_{i+1,j} \Big)$$

Values $V(S, T)$ given at $t = T$: starting values

$$V_{N,j} \;=\; V(S_{N,j}, N)$$

To take limit, need to specify relationship h to k
Take

$$k = \lambda h^2, \qquad \lambda \text{ constant as } h, k \to 0$$

or

$$h = \sqrt{k/\lambda}$$

How do we know that $k \sim h^2$ is right scaling?
To confirm, check final result in limit $\lambda \to 0, \infty$.

(Note: inequality constraint always satisfied when h small.)

Aside:
convergence of finite-difference method to PDE in simple problem

Consider staggered grid of points

$$\begin{aligned} t_i &= i\,k, \qquad i = 0, 1, \ldots \\ x_{i,j} &= \left(\tfrac{1}{2} i + j\right) h, \qquad -\infty < j < \infty \end{aligned}$$

Point (i, j) has "ancestors" $(i-1, j)$, $(i-1, j+1)$
Define $u_{i,j}$ by

1. $u_{0,j} = f(jh)$, smooth function $f(x)$
2. $u_{i,j} = \frac{1}{2}\left(u_{i-1,j} + u_{i-1,j+1}\right)$

Theorem: As $h, k \to 0$ with $k = \lambda h^2$, then $u_{i,j} \to u(x_{i,j}, t_i)$, where $u(x,t)$ is solution of initial-value PDE

$$
\begin{aligned}
u(x,0) &= f(x) \\
\frac{\partial u}{\partial t} &= \frac{1}{4\lambda} \frac{\partial^2 u}{\partial x^2}
\end{aligned}
$$

Proof: Based on *Lax equivalence theorem* for finite-difference schemes

$$\text{Consistency} \quad + \quad \text{Stability} \quad \Longrightarrow \quad \text{Convergence}$$

Stability: obvious since $u_{i,\cdot}$ is smoothing of $u_{i-1,\cdot}$.

Consistency: replace $u_{i,j} = u(x_{i,j}, t_i)$ exactly, with $u_t = \frac{1}{4\lambda} u_{xx}$ see by how much difference formula fails to be satisfied:

Make local expansions about point $\left(i-1, j+\frac{1}{2}\right)$.

$$
\begin{aligned}
u_{i,j} &\sim u + k u_t + \cdots \\
u_{i-1,j} &\sim u - \frac{h}{2} u_x + \frac{1}{2}\left(\frac{h}{2}\right)^2 u_{xx} + \cdots \\
u_{i-1,j+1} &\sim u + \frac{h}{2} u_x + \frac{1}{2}\left(\frac{h}{2}\right)^2 u_{xx} + \cdots
\end{aligned}
$$

$$
\begin{aligned}
u_{i,j} \; - \; \tfrac{1}{2}\left(u_{i-1,j} + u_{i-1,j+1}\right) \; &\sim \; k\,u_t \; - \; \left(\frac{h}{2}\right)^2 u_{xx} \; + \; \cdots \\
&\sim \; k\left(u_t - \frac{1}{4\lambda} u_{xx}\right) \; + \; \cdots
\end{aligned}
$$

So consistency requires

$$
u_t \; = \; \frac{1}{4\lambda}\, u_{xx}
$$

QED

Not to hard to make this fully rigorous

end of Aside

Option-pricing problem:

Assume smooth solution, make local asymptotic expansions about suitable intermediate point

$$
\begin{aligned}
V_{i,j} &= V - k v_t + \tfrac{1}{2}k^2 V_{tt} + \cdots \\
V_{i+1,j} &= V - h_- V_S + \tfrac{1}{2}h_-^2 V_{SS} + \cdots \\
V_{i+1,j+1} &= V + h_+ V_S + \tfrac{1}{2}h_+^2 V_{SS} + \cdots
\end{aligned}
$$

Compute consistency just as for simple model problem

The *Black-Scholes equation for derivative pricing:*

$$V_t + \tfrac{1}{2}\sigma^2 S^2 V_{SS} + r S V_S - r V = 0.$$

$$\frac{1}{2}\sigma^2 = \frac{1}{8\lambda}$$

Satisfied by value of *every* derivative security on S.
(As long as no early exercise or other complication)

- Backwards parabolic: *terminal* data at $t = T$
- Linear: asset values add

σ^2 appears as a diffusion coefficient
related to how much stock price "jumps around"

s has vanished!
σ tells us how to construct grid: $h = 2\sigma\sqrt{k}$

How do we know σ?

How do we know other grids might not give different PDE?
(stochastic calculus helps here)

Solution of Black-Scholes equation

new independent variables

$$x = \log \frac{S}{S_{\text{ref}}}, \qquad \tau = \sigma^2 (T - t)$$

τ is time remaining to expiration

New dependent function

$$V(S,t) = V_{\text{ref}}\, u\left(\log \frac{S}{S_{\text{ref}}},\, \sigma^2 (T-t) \right)$$

Get

$$u_\tau = \tfrac{1}{2} u_{xx} + \beta u_x - \gamma u,$$

with constant coefficients

$$\beta = \frac{r - \frac{1}{2}\sigma^2}{\sigma^2}, \qquad \gamma = \frac{r}{\sigma^2}.$$

Easy to solve initial-value problem using Green's functions

Black-Scholes formula for European call with strike K:

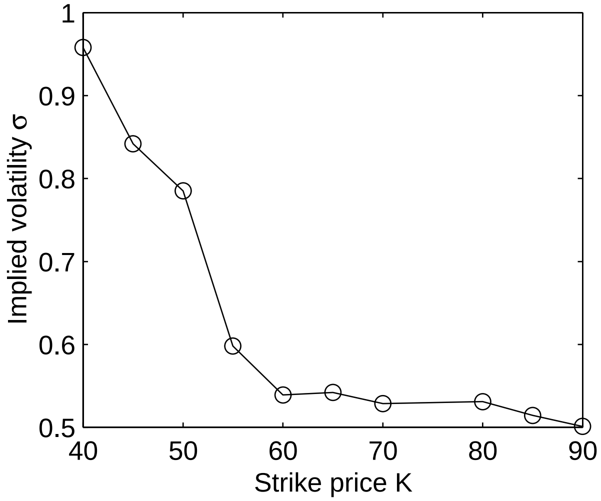
$$
\begin{aligned}
C(S,t) \;=\; & S\,N\left(\frac{\log(S/K) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right) \\
& - \; K\,e^{-r(T-t)}\,N\left(\frac{\log(S/K) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right)
\end{aligned}
$$

Get European put by put-call parity

Compare with market-quoted option values
Impossible to find single volatility value
that works for all strikes
(even at just one maturity date)

Historical volatility: The volatility estimated from statistical analysis of past price motions.

Implied volatility: The volatility you have to use in the Black-Scholes pricing formula to get the market price.



1
0.9
0.8
0.7
0.6
0.5
Implied volatility σ
40
50
60
70
80
90
Strike price K

Dynamic Hedging:

$$\Delta = \frac{V_2 - V_1}{S_2 - S_1} \to \frac{\partial V}{\partial S}$$

How much you have to hold at each instant to maintain hedge
Jumps around as much as stock price

Vanilla puts and calls as $t \nearrow T$:

- $\Delta \to \pm 1$ if option is expiring in the money
- $\Delta \to 0$ if expiring out of the money

You end with exactly what you need

American option

At each moment, a *decision problem:*
Should I exercise, or should I hold??

Option value always ≥ value of exercise
(e.g. $V(S,t) \geq \Lambda(S,t) = \max\{S-K, 0\}$ for American call)

Tree modification for early exercise:
Compare computed value at each node against exercise value
Use the larger of the two.
This affects values at all nodes *earlier* in time.

Black-Scholes equation with early exercise:

$$-V_t \;\geq\; \tfrac{1}{2}\sigma^2 S^2 V_{SS} \;+\; rSV_S \;-\; rV \qquad \text{and} \qquad V(S,t) \;\geq\; \Lambda(S,t)$$

And at least one of them is an equality

A *variational inequality*, or *obstacle problem*

Free boundary problem:
critical stock price $S_*(t)$ where you should exercise
American put: exercise for $S < S_*$
American call: exercise for $S > S_*$, if stock pays dividends.

Physical analogs: oxygen consumption problem, Stefan problem

Stochastic models

Binomial tree $\Rightarrow$ random walk

Assign probabilities to up and down moves

These probabilities disappear from final solution

Stock motion has probability distribution

Changes in $\log S(t)$ are serially independent

$$\log \frac{S(t)}{S(0)} \text{ is Gaussian}$$

$$\begin{aligned} \text{mean} &= \mu t \\ \text{variance} &= \sigma^2 t \end{aligned}$$

Model from Bachelier (1900) (for arithmetic motion)

First mathematics of Brownian motion (before Einstein)

Stochastic differential equation

$$dS = \mu S \, dt + \sigma S \, dX,$$

dX is increment of standard Brownian

Theory is elegant, obscures messy reality