Mathematics in Finance

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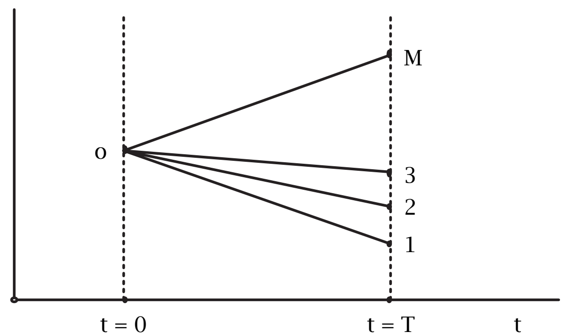
1. Pricing by the no-arbitrage principle
2. Multi-period pricing
3. An overview of real securities
4. Continuum limit: The Black-Scholes equation
5. Random walks and stochastic calculus

One-period model:

now $t = 0$: certain, one state

$t = T$: uncertain: M possible states of world



$$\Omega = \{1, 2, \ldots, M\}$$

Examples:

- Coin flip: heads or tails
- Lottery ticket: few specified payoffs
- Earthquake: destroys house or not
- Weather: divide temperature into bands
- Stock: several future prices

No opinion about *probability*
Only need list of *possible* events
(Or: only care about zero *vs* nonzero probability)

Random variable: function $\Omega \to \mathbb{R}$
$f(j)$ = value of variable f if state j happens
$\big(f(1), \ldots, f(M)\big)^{\mathrm{T}} \equiv$ vector in $\mathbb{R}^M$.

Econo-jargon:

- *Endowment:* How much I get from outside source

- *Consumption:* How much I actually have,
  as modified by trading activity

Economics: measured in abstract units
For us: measured in dollars

Endowment and consumption are both random variables
We use trading securities to tailor consumption

Examples:

- Coin flip, lottery: My endowment is independent of result.
- Earthquake: house falls down is negative endowment
- Weather: energy companies have exposure
- Stock motion: endowment is independent of result

Tailoring consumption by trading:

- Bet on coin, buy lottery ticket
- Purchase insurance contract
- Energy companies trade temperature derivatives
- Invest in stock

Utility function $U(c)$ Measures "value" to me
of different pattern of consumption outcomes

Concave: Second million less valuable than first million

$$U(\alpha c_1 + (1-\alpha)c_2) \;>\; \alpha U(c_1) \;+\; (1-\alpha)U(c_2)$$

⇒ Risk-aversion: prefer more uniform distribution

$$U\begin{pmatrix}1\\1\end{pmatrix} \quad > \quad U\begin{pmatrix}0\\2\end{pmatrix}$$

(Depending on weights: if $U(0,2) = U(2,0)$, say)

All we care about in this course: $U(x)$ *increasing*
More money is better than less money

We completely *eliminate* risk, don't need to measure

Security: A contract that pays different amounts in different states of the world.

$$\text{Random variable:} \qquad d = \big(d(1), \ldots, d(M)\big)^{\mathrm{T}} \in \mathbb{R}^M$$

Payoff in different states of the world

- Coin flip: $d = (1, -1)$
- Insurance contract: $d = (\ 0,\ 100{,}000\ )$
- Temperature (and other) derivatives: custom-specified
- Stock share: $d = (\ 90,\ 100,\ 110\ )$, say

Investment θ: Your payoff is

$$\theta\, d \;=\; \Big(\, \theta d_1, \,\ldots,\, \theta d_M \,\Big)$$

You choose θ at $t = 0$, then see what happens

θ can be any positive or negative number

$$\theta > 0: \;\textit{long} \qquad\qquad \theta < 0: \;\textit{short}$$

Someone else has to take $-\theta$

- Bet on coin flip: heads or tails
- Purchase insurance contract or derivative (or sell to neighbor)
- Buy some shares of stock
  Short sale: borrow shares, sell, buy later to give back

Every security can be bought long or short (no margin).

Market: List of N securities

Payout matrix $(N \times M)$

$$D = \begin{pmatrix} d_1(1) & \cdots & d_1(M) \\ \vdots & & \vdots \\ d_N(1) & \cdots & d_N(M) \end{pmatrix} \qquad \theta = \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_N \end{pmatrix}$$

$$\text{payout} \quad D^{\mathrm{T}}\theta = \begin{pmatrix} \theta_1 d_1(1) + \cdots + \theta_N d_N(1) \\ \vdots \\ \theta_1 d_1(M) + \cdots + \theta_N d_N(M) \end{pmatrix}$$

consumption $c = e + D^{\mathrm{T}}\theta$

Prices

$$p = \begin{pmatrix} p_1 \\ \vdots \\ p_N \end{pmatrix}$$

Portfolio cost $p^{\mathsf{T}}\theta$ (not random)

$$\bar{D} = \begin{pmatrix} -p & D \end{pmatrix} = \begin{pmatrix} -p_1 & d_1(1) & \cdots & d_1(M) \\ \vdots & \vdots & & \vdots \\ -p_N & d_N(1) & \cdots & d_N(M) \end{pmatrix}$$

Complete description of market

"Price" means you can buy or sell arbitrary amounts

Prices determined by mysterious and complicated mechanisms

We want to relate prices to payouts

All of Financial Mathematics:

"No free lunch" $\Longrightarrow$

- Inequality constraints on p_j
- When new security is added with specifed payouts depending on existing securities, determine *exactly* what new price must be ("derivative pricing")

A possible idea:
If you have probabilities, compute “fair value”

- Flip of fair coin: $p = 0$
- Insurance cost: depends on probability of earthquake
- Stock price: depends on your opinion about future values

All these formulations have *risk*
When we can eliminate risk (not always),
the risk-free price is *always* the correct one.

Probabilities do not matter

Example: Two-asset market

Bond: A generic risk-free asset; payout always = 1

Stock: A generic risky asset

$$\begin{pmatrix} B_1 \\ \vdots \\ B_M \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \qquad\qquad \begin{pmatrix} S_1 \\ \vdots \\ S_M \end{pmatrix}$$

Interest rate r: *discount factor* $B_0 = e^{-rT}$

S_0 arbitrary

$$\bar{D} = \begin{pmatrix} -e^{-rT} & 1 & \cdots & 1 \\ -S_0 & S_1 & \cdots & S_M \end{pmatrix}$$

Arbitrage: $\theta \in \mathbb{R}^N$ so that

$$\bar{D}^{\mathsf{T}}\theta \geq 0 \qquad \text{and} \qquad \bar{D}^{\mathsf{T}}\theta \neq 0$$

- $p^{\mathsf{T}}\theta \leq 0$: no initial cost
  $D^{\mathsf{T}}\theta \geq 0$: never lose any money
  $D^{\mathsf{T}}\theta \neq 0$: possibility to win some money

or

- $p^{\mathsf{T}}\theta < 0$: gain money when you implement
  $D^{\mathsf{T}}\theta \geq 0$: never lose any money

If the market were such that arbitrage possibilities existed, then everyone would rush to take the good deal.

Prices would respond (in mysterious way) to one-sided demand.

Arbitrage possibility would disappear.

$\Longrightarrow$ *Arbitrage strategies cannot exist*

Impossible to make profit greater than risk-free rate, without taking on risk.

Gives constraints on prices in terms of payouts

State-price vector: $\psi > 0$ so $p = D\psi$

$\psi \in \mathbb{R}^M$: Assigns a weight to each state.
Prices determined by payouts via ψ

Theorem:
Non-existence of arbitrage $\iff$ existence of ψ

Proof:
$\Leftarrow$ $\quad p^T\theta = \psi^T D^T \theta$.
Any component of $D^T\theta > 0 \Longrightarrow p^T\theta > 0$
(You can win but it will cost you.)

$\Longrightarrow$

Sets in $\mathbb{R}^{M+1}$ (initial and final consumption)

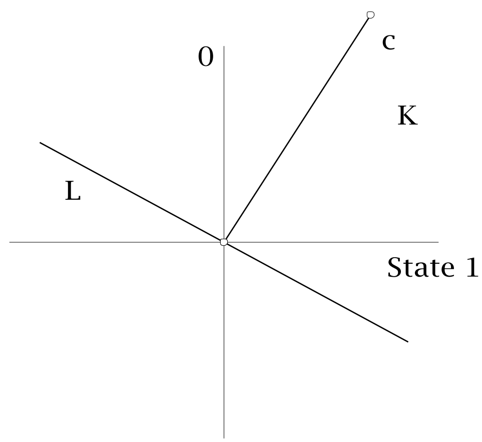
Set of arbitrage payoffs: positive cone

$$K = \{ x \in \mathbb{R}^{M+1} \mid x \geq 0 \text{ and } x \neq 0 \}$$

Set of attainable payoffs: linear space, dim= $\mathrm{rank}(\bar{D})$

$$L = \{ x \in \mathbb{R}^{M+1} \mid x = \bar{D}^{\mathrm{T}}\theta \text{ for some } \theta \in \mathbb{R}^{N} \}.$$

No arbitrage $\Longrightarrow K \cap L = \emptyset$


c
0
K
L
State 1

Separating hyperplane theorem:
There exists a normal vector c so

$$
\begin{aligned}
c^{\mathrm{T}} x &= 0 \qquad \text{all } x \in L \\
c^{\mathrm{T}} x &> 0 \qquad \text{all } x \in K
\end{aligned}
$$

Separate into now (x_0 scalar), and future ($\tilde{x} \in \mathbb{R}^M$)
$c^{\mathrm{T}} x = a x_0 + b^{\mathrm{T}} \tilde{x}$
$a > 0$ and $b > 0$ since K includes positive coordinate axes
$\forall\, \theta \in \mathbb{R}^N$, $-a p^{\mathrm{T}} \theta + b^{\mathrm{T}} D^{\mathrm{T}} \theta = 0$
$(ap - Db)^{\mathrm{T}} \theta = 0$
$\psi = b/a$ is state-price vector

QED

Theorem:

$\mathrm{rank}(D) \geq M \quad \Longrightarrow \quad \mathrm{rank}(\bar{D}) \geq M \quad \Longrightarrow \quad \psi$ is unique

Proof: $\dim(L) = \mathrm{rank}(\bar{D})$

$\mathrm{rank}(D) = M \quad \Longrightarrow$ market is *complete*

Can achieve any combination of payoffs

Requires $N \geq M$. At least as many securities as states.

Arbitrage always gives *inequality* constraints

Prices are uniquely determined when add securities
to a market that is already complete

- Coin flip: |price of game| should be < max payoff
- Insurance: should cost something but less than house

Suppose our market has a bond, discount factor $B_0 = e^{-rT}$

$$e^{-rT} = B_1\psi_1 + \cdots + B_M\psi_M = \psi_1 + \cdots + \psi_M$$

$q = e^{rT}\psi$ can be interpreted as "risk-neutral probabilities"

For random variable f, define "expectation"

$$\mathbb{E}_Q[f] = q_1 f(1) + \cdots + q_M f(M)$$

Any security in market can be priced by the formula

$$\begin{aligned} p &= e^{-rT}\,\mathbb{E}_Q[d] \\ &= \text{"discounted expectation in risk-neutral measure"} \end{aligned}$$

If q is unique (ψ is unique), this is the *only* possible value.

Two assets plus a derivative

Market has bond + stock

initial prices $B_0 = e^{-}rT$, S_0

S_0 is as quoted in the market right now

World has *exactly two* states: stock moves to S_1 or S_2 ($S_1 < S_2$)

We choose S_1 and S_2 however we like

$$\bar{D} = \begin{pmatrix} -e^{-rT} & -S_0 \\ 1 & S_1 \\ 1 & S_2 \end{pmatrix}$$

$S_1 \neq S_2 \implies$ market is complete $\implies$ q is unique if it exists

Set $q_2 = q$, $q_1 = 1 - q$, solve

$$S_0 \;=\; e^{-rT}\Big(qS_2 + (1-q)S_1\Big)$$

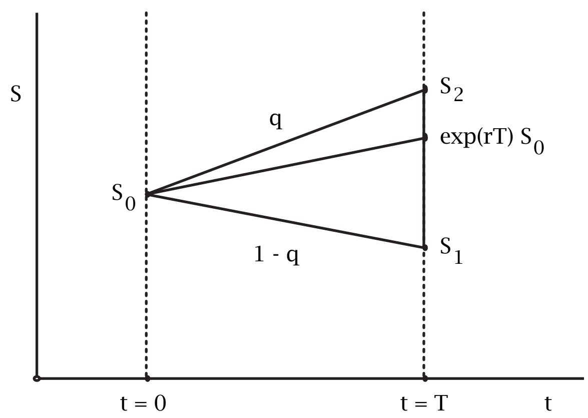
$$q \;=\; \frac{e^{rT}S_0 - S_1}{S_2 - S_1}, \qquad 1 - q \;=\; \frac{S_2 - e^{rT}S_0}{S_2 - S_1}.$$

For $0 < q < 1$, need

$$S_1 \;<\; e^{rT}S_0 \;<\; S_2$$

$S_1 \geq e^{rT}S_0 \quad \Longrightarrow \quad$ Borrow money to buy the stock

$e^{rT}S_0 \geq S_2 \quad \Longrightarrow \quad$ Short the stock and invest the proceeds


S
q
S2
exp(rT) S0
S0
S1
1 - q
t = 0
t = T
t

Add a security to the market

General new security $V \quad \Longrightarrow \quad$ more states
(e.g. a different stock)

We add a *derivative security*
Value of V at time T determined by value of S at T
World still has the same two future states

Example: call option with strike K
Right to buy the stock for K at time T

$$V \text{ at time } T \;=\; \begin{cases} S - K, & \text{if } S \geq K \text{ (buy for } K\text{, sell in market for } S\text{)} \\ 0, & \text{if } S \leq K \text{ (not worth using the option)} \end{cases}$$

$$V_j = \max\{S_j - K, 0\} \quad (k = 1, 2)$$

Market is now

$$\bar{D} = \begin{pmatrix} -e^{-rT} & -S_0 & -V_0 \\ 1 & S_1 & V_1 \\ 1 & S_2 & V_2 \end{pmatrix}$$

Must have rank $\leq 2 \quad \Longrightarrow \quad$ determines V_0

Risk-neutral expectation formula:

$$V_0 = e^{-rT}\Big(qV_2 + (1-q)V_1 \Big)$$
$$= \frac{S_0 - e^{-rT}S_1}{S_2 - S_1}V_2 + \frac{e^{-rT}S_2 - S_0}{S_2 - S_1}V_1.$$

Formula for V_0 in terms of values at *later* times.

Hedging

You are the bank. You wrote option to a customer.

You have risk since you have to pay V_1 or V_2.

You *hedge* your risk by investing in stock and bond.

Choose investments to *replicate* option payoff.

Heding portfolio Π has b units of bond bond, Δ shares of stock

Initial value: $$\Pi_0 = b\, e^{-rT} + \Delta S_0$$

Final value: $$\begin{cases} \Pi_1 = b + \Delta S_1, & \text{if stock moves to } S_1 \\ \Pi_2 = b + \Delta S_2, & \text{if stock moves to } S_2 \end{cases}$$

Choose b, Δ so

$$\Pi_1 = V_1 \qquad \Pi_2 = V_2$$

No matter what happens, you have exactly enough money to cover your obligation.
Then it must be that

$$V_0 = \Pi_0$$

Solution (two linear equations in two variables):

$$\Delta = \frac{V_2 - V_1}{S_2 - S_1}, \qquad b = \frac{S_2 V_1 - S_1 V_2}{S_2 - S_1}.$$

Δ = *hedge ratio:* shares of stock held per option sold
Hold this amount stock, loan/borrow cash difference

Review of why this has to be the price:

- Suppose option were being bought in the market for $V_0' > V_0$ (V_0 determined by no-arbitrage). You sell the option and buy the hedging portfolio $\implies$ guaranteed profit $V_0' - V_0$.
- If the option were being sold for a price less than V_0 you do the reverse. (Remember there is only *one* price for any security, including the option.)

In either case, a lot of people would quickly notice opportunity; price would respond and move back to the "correct" level.
(Le Chatelier's principle)

Assumptions:

- *Buy/sell at same price*
  No transaction costs, bid/ask spread, margin requirements
- *Arbitrary positive/negative amounts*
  No impact on market, underlying stock can always be traded
  (The price is determined *only* if you can trade stock.)
- *Stock can move to only two specific values*
  In continuous limit, you must guess *amplitude*
  (not direction!) of future price changes. ("volatility")
  In practice this is not unambiguously determined.

Multiple periods

Divide time T into N subintervals

$$t_0 = 0, \quad t_1, \quad t_2, \quad \cdots, \quad t_{N-1}, \quad t_N = T$$

Bond price known at each time (constant interest rate r)

$$B(t_j) \;=\; e^{-r(T-t_j)}$$

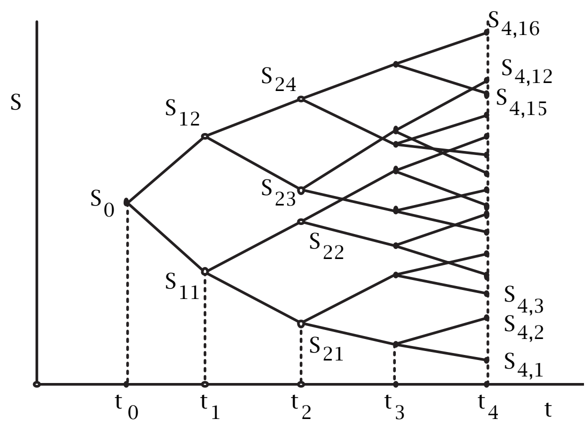
Stock price can move to two different values at t_{j+1} from its value at t_j.

N levels $\quad\Longrightarrow\quad$ 2^N possible final states

Make a list of all possible prices at all nodes:

$$S_{0,1}, \quad S_{1,1}, S_{1,2}, \quad S_{2,1}, S_{2,2}, S_{2,3}, S_{2,4}, \quad \cdots \quad S_{N,1}, \ldots, S_{N,2^N}$$

A *binomial tree* model (nonrecombining)


S
S0
S12
S11
S24
S23
S22
S21
S4,16
S4,12
S4,15
S4,3
S4,2
S4,1
t0
t1
t2
t3
t4
t

Add an option V to the market.
We want to determine $V_{0,1} \;=\;$ option value right now
(We are about to buy it, or quote its price to a customer.)

Suppose value of V is known in terms of S at time T.
(example: call option has $V = \max\{S-K,0\}$).

Option pricing procedure:

1. Determine prices $V_{N,1},\dots,V_{N,2^N}$ using known formula.
2. Work back up the tree applying our formula

$$V_0 \;=\; e^{-rT}\Big(\, qV_2 \;+\; (1-q)V_1 \,\Big)$$

   (and the formula for q in terms of the S) at each node
   V_0 is any node, V_1 and V_2 are its children.

Why is this true?

If the formula were violated at even one node (i_*, j_*),

there would be an arbitrage:

Wait until time i.

It may be that the stock price happens to be equal to S_{i_*,j_*}, that is, it happened to take all the right jumps at all times up to i.

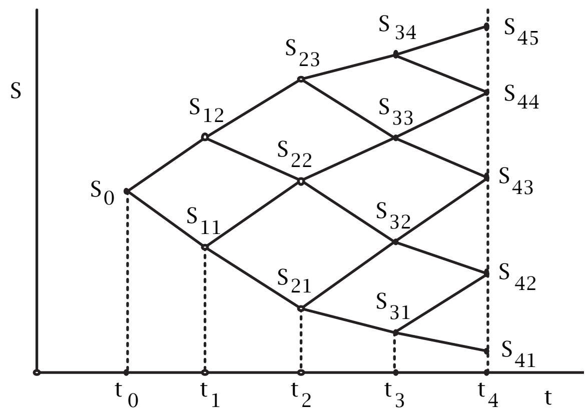
If this node doesn't satisfy the pricing relation, then the option price V at that time will not equal its no-arbitrage price.

If the stock price is S_{i_*,j_*}, then buy or sell the option and the opposite hedging portfolio at that time. If the stock price happens to be something else, then do nothing.

With this strategy, you have a positive probability of getting something for nothing.

Recombining trees: $\mathcal{O}(N^2)$ elements much more practical.

Price moves up and down on a mesh.



Dynamic Hedging:
If you sell the option at $t = 0$ for price V_0:
Purchase Δ_0 shares of stock at same time
Borrow/lend difference to risk-free account

As time evolves, price moves up and down on tree.
Continually adjust stock holdings to maintain $\Delta_{i,j}$
(Completely deterministic in terms of observed motions)
Cash difference goes in/out of risk-free account.

At $t = T$ you are *guaranteed* to have exactly
right amount of stock and cash to cover option.
(For profit, charge a little more than V_0 at beginning)

One more assumption:

- You can *rebalance* portfolio as often as prices move.
  Continuous-time limit $\implies$ continuous trading.