Lecture-VIII

Contingent Claims Pricing using the Equivalent Martingale Measure

1 Probability as Measure

We consider a normally distributed random variable z_t at a fixed time t with zero mean and unit variance. Formally $z_t \sim N(0, 1)$. The probability density function of this random variable is given by the well known expression

$$f(z_t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z_t^2}$$

where $-\infty < z_t < \infty$. If we were interested in the probability that z_t falls near a specific value 1 then this probability can be expressed by first choosing a small interval $\Delta > 0$ and next by calculating the integral of the normal density over the region in question:

$$P\left(1 - \frac{1}{2}\Delta < z_t < 1 + \frac{1}{2}\Delta\right) = \int_{1-\frac{1}{2}\Delta}^{1+\frac{1}{2}\Delta} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z_t^2} dz_t$$

Visualizing this way, the probability corresponds to "measure" that z_t falls between $1 - \frac{1}{2}\Delta$ and $1 + \frac{1}{2}\Delta$. Probabilities are called measures because they are mappings from arbitrary sets to nonnegative real numbers R^+. For infinitesimal Δ which we write as dz_t these measures are denoted by the symbol $dP(z_t)$, or simply dP when there is no confusion of the underlying random variable: Thus

$$dP(1) = P\left(1 - \frac{1}{2}\Delta < z_t < 1 + \frac{1}{2}\Delta\right)$$

This can be read as the probability that the random variable z_t will fall within a small interval centered around 1 and of infinitesimal length dz_t. Formally

$$\int_{-\infty}^{\infty} dP(z_t) = 1$$

$$E[z_t] = \int_{-\infty}^{\infty} z_t dP(z_t)$$

and

$$E[z_t - Ez_t]^2 = \int_{-\infty}^{\infty} [z_t - Ez_t]^2 dP(z_t)$$

and in the discrete cases the integral will be replaced by the sum. Some famous continuous probability measures are the Normal probability measure, Chi-square probability measure and the Gamma probability measure and some well known discrete probability measures are the Binomial probability measure, Poisson probability measure and the Hypergeometric probability measure.

2 An Example of Changing Means

Suppose a random variable Z is defined as follows. A die is rolled and the values of Z are set according to the rule

$$Z = \begin{cases} 10 & \text{roll of 1 or 2} \\ -3 & \text{roll of 3 or 4} \\ -1 & \text{roll of 5 or 6} \end{cases}$$

The Mean and Variance of this random variable are calculated as follows:

$E[Z] = \frac{1}{3}[10] + \frac{1}{3}[-3] + \frac{1}{3}[-1] = 2$

$V[Z] = \frac{1}{3}[10-2]^2 + \frac{1}{3}[-3-2]^2 + \frac{1}{3}[-1-2]^2 = \frac{98}{3}$

We will now obtain a new probability measure under which the mean becomes one and the variance remains unchanged. This can be done by solving the system of equations that the probabilities sum to one, the new mean is one and the variance is $\frac{98}{3}$. Hence we obtain

$P(\text{getting 1 or 2}) = \frac{1}{3} \Rightarrow \tilde{P}(\text{getting 1 or 2}) = \frac{122}{429}$

$P(\text{getting 3 or 4}) = \frac{1}{3} \Rightarrow \tilde{P}(\text{getting 3 or 4}) = \frac{22}{39}$

$P(\text{getting 5 or 6}) = \frac{1}{3} \Rightarrow \tilde{P}(\text{getting 5 or 6}) = \frac{5}{33}$

Now the new probabilities are designated by $\tilde{P}$. The mean of Z under$\tilde{P}$ is equal to one. This was a simple discrete case where the variable assumed a finite number of values and the state space was finite. We will now move to the continuous case which is more complicated and the transformations are extended to continuous time stochastic processes.

3 Equivalent Probabilities in the continuous setting

Consider a normally distributed random variable Z_t, $Z_t \sim N(\mu, 1)$. Denote the density function of Z_t by $f(Z_t)$ and the implied probability measure by P such that

$$dP(Z_t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(Z_t-\mu)^2}$$

In this example the state space is continuous although we are still working with a random variable, instead of a random process. Now define a function

$$\xi(Z_t) = e^{-Z_t\mu + \frac{1}{2}\mu^2}$$

Multiplying $\xi(Z_t)$ by $dP(Z_t)$ we obtain a new probability measure $\tilde{P}$.

$$dP(Z_t)\xi(Z_t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z_t^2}$$

$$d\tilde{P}(Z_t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z_t^2}$$

Hence $d\tilde{P}(Z_t)$ ia a new probability measure defined by

$$d\tilde{P}(Z_t) = dP(Z_t)\xi(Z_t)$$

This is a normal probability measure with mean zero and variance one. The transformation leaves the variance unchanged and is unique given the mean and variance. We can also accomplish this for a sequence of normally distributed random variables Given $\xi(Z_t)$ you may wonder if we could attach a deeper interpretation of what $\xi(Z_t)$ really represents. We will discuss that in the next section.

4 Radon-Nikodym Derivative

Let us look at the function $\xi(Z_t)$ more carefully. Consider the previous example of the Normal probability measure with $\sigma = 1$.

$$\xi(Z_t) = e^{-Z_t\mu+\frac{1}{2}\mu^2}$$

We used the above expression to obtain the new probability measure from $dP(Z_t)$:

$$d\tilde{P}(Z_t) = dP(Z_t)\xi(Z_t)$$

On dividing both sides by $dP(Z_t)$:

$$\frac{d\tilde{P}(Z_t)}{dP(Z_t)} = \xi(Z_t)$$

This expression is called the derivative of the measure $\tilde{P}$ with respect to P and is given by $\xi(Z_t)$. Such derivatives are called Radon-Nikodym derivatives. Hence if the Radon-Nikodym derivative of $\tilde{P}$ with respect to P exists then we can use the expression $\xi(Z_t)$ to transform the mean of Z_t by leaving its variance structure unchanged. This transformation is very useful in the Heath-Jarrow-Morton model because the market price of risk can be eliminated.

5 Equivalent Measures

We also need to determine carefully conditions under which the Radon-Nikodym derivative will exist. When can transformations such as

$$d\tilde{P}(Z_t) = dP(Z_t)\xi(Z_t)$$

be performed? The answer is that the Radon-Nikodym derivative will exist when for every non-zero probability assignment of $\tilde{P}$, P also assigns a non-zero probability and vice-versa. Finally if $\tilde{P}$ assigns a zero probability P also must assign a zero probability to that event. If the above holds then the two measures are equivalent measures and are called equivalent probability measures.

6 Girsanov Theorem for a Random Process

The setting of the Girsanov theorem is the following: We are given a family of information sets I_t over a period $[0, T]$. T is finite. In this case we define a random process ξ_t such that

$$\xi_t = e^{\left(\int_0^t X_u dW_u - \frac{1}{2}\int_0^t X_u^2 du\right)}$$

where $t \in [0, T]$ and X_t is an $I_t -$ measurable process. That is given the information set I_t, the value of X_t is known exactly. The W_t is a Wiener process with probability distribution P. If the process above is a martingale with respect to the information sets I_t, then $\tilde{W_t}$ defined by

$$\tilde{W_t} = W_t - \int_0^t X_u du$$

where $t \in [0, T]$ is a Wiener process with respect to I_t and the probability measure $\tilde{P}$ given by

$$\tilde{P_T}(A) = E^P[1_A \xi_T]$$

with A being an event in I_T and 1_A being the indicator function of that event.