Lecture-VII

The Cox Ingersol and Ross Term Structure Model

1 Bond Prices in the CIR model

The CIR interest rate process is given by:

$$dr(t) = (\alpha - \beta r(t))dt + \sigma\sqrt{r(t)}dW(t) \tag{1}$$

where $r(0)$ is given. In Vasicek's paper it was shown that if the spot rate follows a stochastic differential equation of the form,

$$dr = f(r,t)dt + \rho dz$$

then the bond price satisfies the stochastic differential equation

$$dB = B\mu(t,s)dt - B\sigma(t,s)dz$$

where B is the price of a discount bond. Further the pricing of the bond can be done by using the corresponding partial differential equation which was derived for the Vasicek's model and called the TERM STRUCTURE EQUATION. We restate the TSE here and then use it to price a bond in the CIR's model. The TSE is given by

$$\frac{\partial B}{\partial t} + (f + \rho q)\frac{\partial B}{\partial r} + \frac{1}{2}\rho^2\frac{\partial^2 B}{\partial r^2} - rB = 0$$

where q is the market price of risk and hence in this case equal to 0. If we examine our model we see that $f = \alpha - \beta r$ and $\rho = \sigma\sqrt{r}$. Hence substituting these parameters in our model we get the partial differential equation for the bond price in the CIR model.

$$\frac{\partial B}{\partial t} + ((\alpha - \beta r) + (\sigma\sqrt{r})0)\frac{\partial B}{\partial r} + \frac{1}{2}(\sigma\sqrt{r})^2\frac{\partial^2 B}{\partial r^2} - rB = 0$$

This is exactly equation 4.1 page 311 in shreve's notes with the conditions $0 \leq t < T$ and $r \geq 0$. We will now change to their notation and the same equation is written as

$$-rB(r,t,T) + B_t(r,t,T) + (\alpha - \beta r)B_r(r,t,T) + \frac{1}{2}\sigma^2 rB_{rr}(r,t,T) = 0 \tag{2}$$

where $0 \leq t < T$, and $r \geq 0$. The terminal condition is $B(r,T,T) = 1$, $r \geq 0$.
This equation has a closed form solution. We will look for a solution of the form

$$B(r,t,T) = e^{-rC(t,T) - A(t,T)}$$

where $C(T,T) = 0$, $A(T,T) = 0$. Differentiating B with respect to r and t and differentiating B_r with respect to r we get the following:

- $B_r = -C(t,T)e^{-rC(t,T)-A(t,T)} = -CB$
- $B_{rr} = (-C(t,T))(-C(t,T))e^{-rC(t,T)-A(t,T)} = C^2(t,T)B$
- $B_t = B(-rC_t - A_t)$

Substituting these expressions in equation 2 above the partial differential equation becomes

$$\begin{aligned}
-rB(r,t,T) + B_t(r,t,T) + (\alpha - \beta r)B_r(r,t,T) + \frac{1}{2}\sigma^2 r B_{rr}(r,t,T) &= 0 \\
-rB(r,t,T) + B(-rC_t - A_t) + (\alpha - \beta r)(-CB) + \frac{1}{2}\sigma^2 r C^2(t,T)B &= 0 \\
-rB - rBC_t + rBC\beta + \frac{1}{2}rB\sigma^2 C^2 - BA_T - BC\alpha &= 0 \\
rB\underbrace{\left(-1 - C_t + C\beta + \frac{1}{2}C^2\sigma^2\right)}_{Riccati} - B(A_t + C\alpha) &= 0
\end{aligned}$$

A careful inspection of the above reveals that the expression in underbraces is the well known Riccati equation. We will now solve this equation. After solving this equation we need to set:

$$A(t,T) = \alpha \int_t^T C(u,T)du$$

Since $A(T,T) = 0$ we have $A_t(t,T) = -\alpha C(t,T)$. Although the time starts at t and ends at T we will consider it to start at 0 for the purposes of derivation and end at t. Hence when we refer to t from this point onwards it is the amount of time that has elapsed. The Riccati equation can be rewritten as

$$C_t = -1 + \beta C + \frac{1}{2}\sigma^2 C^2 \tag{3}$$

We introduce another dependent variable u such that

$$C = \frac{-u_1}{\frac{1}{2}\sigma^2 u} = \frac{-2u_1}{\sigma^2 u} = (-2u_1)(\sigma^2 u)^{-1}$$

where u_1 is the derivative of u with respect to t. Differentiating the above expression with respect to t we get

$$\begin{aligned}
C_t &= (\sigma^2 u)^{-1}(-2u_2) + (-2u_1)(-1)(\sigma^2 u)^{-2}(\sigma^2 u_1) \\
C_t &= (\sigma^2 u)^{-1}(-2u_2) + (2u_1)(\sigma^2 u)^{-2}(\sigma^2 u_1)
\end{aligned}$$

Substituting C and C_t in 3 we get

$$\begin{aligned}
(\sigma^2 u)^{-1}(-2u_2) + (2u_1)(\sigma^2 u)^{-2}(\sigma^2 u_1) &= -1 - \beta\left(\frac{2u_1}{\sigma^2 u}\right) + \frac{1}{2}\sigma^2\left(\frac{2u_1}{\sigma^2 u}\right)^2 \\
\frac{-2u_2}{\sigma^2 u} + \frac{2u_1}{(\sigma^2 u)^2}\sigma^2 u_1 &= -1 - \beta\left(\frac{2u_1}{\sigma^2 u}\right) + \frac{1}{2}\sigma^2\left(\frac{2u_1}{\sigma^2 u}\right)^2
\end{aligned}$$

Multiplying both sides of the above expression by $\sigma^4 u^2$ we get

$$\begin{aligned}(-2u_2)(\sigma^2 u) + (2u_1)(\sigma^2 u_1) &= -(\sigma^4 u^2) - \beta(2u_1)(\sigma^2 u) + \frac{\sigma^2}{2}(2u_1)^2 \\ (-2u_2)(\sigma^2 u) &= -(\sigma^4 u^2) - \beta(2u_1)(\sigma^2 u)\end{aligned}$$

Multiplying both sides of the above expression by $\frac{1}{u\sigma^2}$ we get

$$2u_2 - 2\beta u_1 - \sigma^2 u = 0$$

The auxiliary equation is given by

$$(2D^2 - 2\beta D - \sigma^2)u = 0 \quad \text{where} \quad D = \frac{d}{dt}$$

The solution to this quadratic in D is given by

$$D_1 = \frac{\beta}{2} + \frac{1}{2}\sqrt{\beta^2 + 2\sigma^2} \quad \text{and} \quad D_2 = \frac{\beta}{2} - \frac{1}{2}\sqrt{\beta^2 + 2\sigma^2}$$

The general solution to the differential equation above will be given by

$$u = Ae^{D_1 t} + Be^{D_2 t}$$

where D_1 and D_2 are the two values of D. Hence

$$u = Ae^{\left(\frac{\beta}{2}+\frac{1}{2}\sqrt{\beta^2+2\sigma^2}\right)t} + Be^{\left(\frac{\beta}{2}-\frac{1}{2}\sqrt{\beta^2+2\sigma^2}\right)t}$$

Following the notation of Shreve's notes we have

$$\gamma = \frac{1}{2}\sqrt{\beta^2 + 2\sigma^2}$$

Hence the derivative of u i.e.u_1 with respect to t will be given by

$$u_1 = A\left(\frac{1}{2}\beta + \gamma\right)e^{\left(\frac{1}{2}\beta+\gamma\right)t} + B\left(\frac{1}{2}\beta - \gamma\right)e^{\left(\frac{1}{2}\beta-\gamma\right)t}$$

Remember before we expressed C as

$$C = \frac{-2u_1}{\sigma^2 u}$$

Substituting the values of u and u_1 we just obtained above we get

$$C = \left(\frac{-2}{\sigma^2}\right)\frac{A\left(\frac{1}{2}\beta + \gamma\right)e^{\left(\frac{1}{2}\beta+\gamma\right)t} + B\left(\frac{1}{2}\beta - \gamma\right)e^{\left(\frac{1}{2}\beta-\gamma\right)t}}{Ae^{\left(\frac{\beta}{2}+\frac{1}{2}\sqrt{\beta^2+2\sigma^2}\right)t} + Be^{\left(\frac{\beta}{2}-\frac{1}{2}\sqrt{\beta^2+2\sigma^2}\right)t}}$$

which reduces to

$$C = \frac{(-2/\sigma^2)A(1/2\beta + \gamma)e^{(1/2\beta+\gamma)t} - (-2/\sigma^2)B(\gamma - 1/2\beta)e^{(1/2\beta-\gamma)t}}{Ae^{(\beta/2+\gamma)t} + Be^{(\beta/2-\gamma)t}}$$

Choosing appropriate constants A and B the above expression can be written as

$$C = \frac{e^{\gamma t} - e^{-\gamma t}}{(\gamma + 1/2\beta)e^{\gamma t} + (\gamma - 1/2\beta)e^{-\gamma t}}$$

which is our required value of C. Now set

$$A(t,T) = \alpha \int_t^T C(u,T)du$$

Since we are considering 0 to t instead of t to T this integral becomes:

$$A(0,t) = \alpha \int_0^t C(u)du$$

which gives us

$$A(0,t) = -\frac{2\alpha}{\sigma^2} \log \left[\frac{\gamma e^{1/2\beta t}}{(\gamma + 1/2\beta)e^{\gamma t} + (\gamma - 1/2\beta)e^{-\gamma t}} \right]$$