Lecture-VI

Bond Pricing Using the P.D.E Approach and the Expectations Approach

1 Three Useful Results

Before we study Merton's, Vasicek and Hull and White models we need three useful results.

1.1 Result 1:

Let $W(t)$ be a Brownian Motion and $\delta(t)$ be a non-random function. Then

$$X(t) = \int_0^t \delta(u)dW(u)$$

is a Gaussian Process with $E[X(t)] = 0$ and $V[X(t)] = \int_0^t \delta^2(u)du$.

1.2 Result 2:

Let $W(t)$ be a Brownian Motion and $\delta(t)$ and $h(t)$ be a non-random functions. Define

$$X(t) = \int_0^t \delta(u)dW(u)$$

$$Y(t) = \int_0^t h(u)X(u)du$$

Then $Y(t)$ is a Gaussian process with $E[Y(t)] = 0$ and Variance

$$V[Y(t)] = \int_0^t \delta^2(v)\left(\int_v^t h(y)dy\right)^2 dv$$

1.3 Result 3:

Let X be normally distributed with mean μ and variance σ^2.Then the Laplace Transform(mgf) of X is given by

$$E(e^{\lambda x}) = \exp\left(\lambda\mu + \frac{1}{2}\lambda^2\sigma^2\right)$$

$$E(e^{\lambda x}) = \exp\left(\lambda E[X] + \frac{1}{2}\lambda^2 V[X]\right)$$

Proof:

$$E[e^{\lambda x}] = \int_{-\infty}^{\infty} e^{\lambda x} f(x)dx$$

where $f(x)$ is the probability density function of the normal distribution.

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$$

$$E[e^{\lambda x}] = \int_{-\infty}^{\infty} e^{\lambda x} \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] dx$$

Let $z = \frac{x-\mu}{\sigma}$. Then $x = \mu + \sigma z$ and $dx = \sigma dz$

$$E[e^{\lambda x}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[\lambda(\mu + \sigma z)\right] \exp\left(-z^2/2\right) dz$$

$$E[e^{\lambda x}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[\lambda \mu\right] \exp\left[\lambda \sigma z)\right] \exp\left(-z^2/2\right) dz$$

$$E[e^{\lambda x}] = e^{\mu\lambda} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}(z^2 - 2\lambda z \sigma)\right] dz$$

$$E[e^{\lambda x}] = e^{\mu\lambda} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}(z^2 - 2\lambda z \sigma + \lambda^2\sigma^2 - \lambda^2\sigma^2)\right] dz$$

$$E[e^{\lambda x}] = e^{\mu\lambda + \lambda^2\sigma^2/2} \int_{-\infty}^{\infty} \underbrace{\frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}(z - \sigma\lambda)^2\right]}_{f(x)} dz$$

The expression within the integral is probability density function of a normal random variable with mean $\sigma\lambda$ and variance 1. Hence the integral is equal to 1.

$$E[e^{\lambda x}] = e^{\mu\lambda + \lambda^2\sigma^2/2} \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}(z - \sigma\lambda)^2\right] dz}_{1}$$

$$E[e^{\lambda x}] = e^{\mu\lambda + \lambda^2\sigma^2/2} \tag{1}$$

2 Merton's Model

This is the simplest model which is given by

$$dr(t) = \mu dt + \sigma dZ(t)$$

the solution of which is given by

$$r(t) = r(0) + \int_0^t \mu du + \int_0^t \sigma dZ(u)$$

The mean and variance of $r(t)$ are given by:

$\Rightarrow E(r(t)) = r(0) + \int_0^t \mu du = r(0) + \mu t$

$\Rightarrow V(r(t)) = \int_0^t \sigma^2 du = \sigma^2 t$

We will now use result 3 to derive the bond price for the Merton's model. The bond price is given by

$$P(t,T) = E\left[\exp\left(-\int_t^T r(v)dv\right)\right]$$

When we calculate the bond price we assume $\lambda = -1$ in equation 1 and proceed as usual. We also consider the time interval to run from 0 to T. Hence

$$P(0,T) = E\left[\exp\left(-\int_0^T r(t)dt\right)\right]$$

$$\Rightarrow P(0,T) = \exp\left[E\left(-\int_0^T r(t)dt\right) + \tfrac{1}{2}V\left(-\int_0^T r(t)dt\right)\right]$$

$$\Rightarrow P(0,T) = \exp\left[-E\left(\int_0^T r(t)dt\right) + \tfrac{1}{2}V\left(\int_0^T r(t)dt\right)\right]$$

We now need to study $\int_0^T r(t)dt$. Define $X(t) = \int_0^t \sigma dZ(v)$ and $Y(T) = \int_0^T X(t)dt$. Now

$$\Rightarrow r(t) = r(0) + \int_0^t \mu dv + X(t)$$

$$\Rightarrow \int_0^T r(t)dt = \int_0^T \left[r(0) + \int_0^t \mu dv\right] dt + \underbrace{\int_0^T X(t)dt}_{Y(T)}$$

Now

$$\Rightarrow E\left[\int_0^T r(t)dt\right] = r(0)T + \tfrac{\mu}{2}T^2$$

We now need to apply result 2 to determine the variance in the above equation.

$$V\left(-\int_0^T r(t)dt\right) = V\left(\int_0^T r(t)dt\right)$$

$$V\left(\int_0^T r(t)dt\right) = V[Y(T)]$$

Since $\delta(v) = \sigma$ and $h(v) = 1$ using result 2 we have

$$V(Y(T)) = \int_0^T \sigma^2 \left(\int_v^T dy\right)^2 dv$$

$$V(Y(T)) = \int_0^T \sigma^2 (T-v)^2 dv$$

$$V(Y(T)) = -\sigma^2 \frac{(T-v)^3}{3}|_0^T$$

$$V(Y(T)) = \sigma^2 \frac{T^3}{3}$$

Hence the bond price solution for the Merton model is given by

$$P(0,T) = \exp\left[-\left(r(0)T + \mu\frac{T^2}{2}\right) + \sigma^2\frac{T^3}{6}\right] \tag{2}$$

3 General PDE for bond pricing

$$\frac{\partial P(t,T)}{\partial t} + [\mu_r + \sigma_r q(t,r)] \frac{\partial P(t,T)}{\partial r} + \frac{1}{2}\sigma_r^2 \frac{\partial^2 P}{\partial r^2} - rP(t,T) = 0$$

where $P(T,T) = 1$.
Try a solution of the form:

$$P(t,T) = e^{-rC(t,T)-A(t,T)}$$

where $C(T,T) = A(T,T) = 0$.
Now

$$\frac{\partial P(t,T)}{\partial t} = \left[-r\frac{\partial C(t,T)}{\partial t} - \frac{\partial A(t,T)}{\partial t}\right] P(t,T)$$

$$\frac{\partial P(t,T)}{\partial r} = -C(t,T)P(t,T)$$

$$\frac{\partial^2 P(t,T)}{\partial r^2} = C^2(t,T)P(t,T)$$

Substituting these in the PDE we get

$$\begin{aligned} 0 &= -rP(t,T) + \left[-r\frac{\partial C(t,T)}{\partial t} - \frac{\partial A(t,T)}{\partial t}\right] P(t,T) + \mu_r[-C(t,T)P(t,T)] \\ &+\sigma_r q(t,r)[-C(t,T)P(t,T)] + \frac{1}{2}\sigma_r^2[C^2(t,T)P(t,T)] \end{aligned}$$

4 Merton's Model

Let $\mu_r = \mu$ and $\sigma_r = \sigma$.

$$rP(t,T)\left[-1 - \tfrac{\partial C(t,T)}{\partial t}\right] - P(t,T)\tfrac{\partial A(t,T)}{\partial t} - P(t,T)[\mu + \sigma q]C(t,T) + P(t,T)\tfrac{1}{2}\sigma^2 C^2(t,T) = 0$$

Set $-1 - \frac{\partial C(t,T)}{\partial t} = 0$. such that $C(T,T) = 0$ and

$$\frac{\partial A(t,T)}{\partial t} + [\mu + \sigma q]C(t,T) - \frac{1}{2}\sigma^2 C^2(t,T) = 0$$

such that $A(T,T) = 0$ Hence $C(t,T) = T - t$ Note that

$$A(t,T) = -\int_t^T \left[-[\mu + \sigma q](T - v) + \frac{1}{2}\sigma^2(T - v)^2\right] dv$$

Hence,

$$A(t,T) = -\left[-[\mu + \sigma q]\frac{(T-t)^2}{2} + \frac{\sigma^2(T-t)^3}{6}\right]$$

$$A(t,T) = [\mu + \sigma q]\frac{(T-t)^2}{2} - \frac{\sigma^2(T-t)^3}{6}$$

Having obtained $C(t,T)$ and $A(t,T)$ we need to substitute these values in the following equation which gives the bond price solution.

$$P(t,T) = e^{-rC(t,T)-A(t,T)}$$

5 Vasicek's Model

The model is given by

$$dr = \alpha(m - r(t))dt + \sigma dZ(t)$$

5.1 Properties

1. The long term mean is given by m.
2. The speed of mean reversion is given by $\alpha > 0$.
3. If $r(t) > m$ then the drift is negative and if $r(t) < m$ then the drift is positive.
4. The above process is Gaussian so interest rates can be negative.
5. $$\int_0^t dr(v) = \int_0^t \alpha(m - r(v))dv + \int_0^t \sigma dZ(v)$$

   This expression is not easy to integrate as $r(v)$ appears in the integrals on the right hand side.

5.2 A Transformation

Set $\alpha m = \beta$. The drift can be rewritten as $\beta - \alpha r(v) = \alpha(m - r(v))$. Then

$$dr = (\beta - \alpha r(t))dt + \sigma dZ$$

Consider $Y(t) = Y(r(t), t) = e^{\alpha t} r(t)$. We then have

$$\begin{aligned}
dY(t) &= \frac{\partial f}{\partial r} dr + \frac{1}{2}\frac{\partial^2 f}{\partial r^2} dr^2 + \frac{\partial f}{\partial t} dt \\
&= e^{\alpha t} dr + \frac{1}{2}.0.dr^2 + \alpha e^{\alpha t} r(t) dt \\
&= e^{\alpha t} dr + \alpha e^{\alpha t} r(t) dt \\
&= e^{\alpha t}[(\beta - \alpha r(t))dt + \sigma dZ] + \alpha e^{\alpha t} r(t) dt \\
&= e^{\alpha t}[\beta dt - \alpha r(t) dt + \sigma dZ + \alpha r(t) dt] \\
&= e^{\alpha t}[\beta dt + \sigma dZ] \\
\int_0^t dY(u) &= \int_0^t e^{\alpha u} \beta du + \int_0^t e^{\alpha u} \sigma dZ(u) \\
Y(t) - Y(0) &= \int_0^t e^{\alpha u} \beta du + \int_0^t e^{\alpha u} \sigma dZ(u) \\
Y(t) &= Y(0) + \int_0^t e^{\alpha u} \beta du + \int_0^t e^{\alpha u} \sigma dZ(u) \\
e^{\alpha t} r(t) &= r(0) + \int_0^t e^{\alpha u} \beta du + \int_0^t e^{\alpha u} \sigma dZ(u) \\
r(t) &= e^{-\alpha t}\left[r(0) + \int_0^t e^{\alpha u} \beta du + \int_0^t e^{\alpha u} \sigma dZ(u)\right]
\end{aligned}$$

The mean and variance of this process respectively are given by:

$$E(r(t)) = e^{-\alpha t}\left[r(0) + \int_0^t e^{\alpha u}\beta du\right]$$

Using result 1 we get

$$V(r(t)) = e^{-2\alpha t}\int_0^t e^{2\alpha u}\sigma^2 (dZ(u))^2$$

But

$$(dZ(u))^2 = du$$

Hence

$$V(r(t)) = e^{-2\alpha t}\int_0^t e^{2\alpha u}\sigma^2 du$$

We now need to study $\int_0^T r(t)dt$. We will use results 1 and 2. Define $X(t) = \int_0^t e^{\alpha u}\sigma dZ(u)$ and $Y(T) = \int_0^T e^{-\alpha t}X(t)dt$. Now

$$\begin{aligned}
r(t) &= e^{-\alpha t}\left[r(0) + \int_0^t e^{\alpha u}\beta du\right] + e^{-\alpha t}X(t) \\
\int_0^T r(t)dt &= \int_0^T e^{-\alpha t}\left[r(0) + \int_0^t e^{\alpha u}\beta du\right]dt + \underbrace{\int_0^T e^{-\alpha t}X(t)dt}_{Y(T)} \\
E\left[\int_0^T r(t)dt\right] &= \int_0^T e^{-\alpha t}\left[r(0) + \int_0^t e^{\alpha u}\beta du\right]dt \\
E\left[\int_0^T r(t)dt\right] &= \int_0^T e^{-\alpha t}\left[r(0) + \frac{\beta}{\alpha}e^{\alpha u}|_0^t\right]dt \\
E\left[\int_0^T r(t)dt\right] &= \int_0^T e^{-\alpha t}\left[r(0) + \frac{\beta}{\alpha}e^{\alpha t} - \frac{\beta}{\alpha}\right]dt \\
E\left[\int_0^T r(t)dt\right] &= \int_0^T e^{-\alpha t}r(0)dt + \int_0^T \frac{\beta}{\alpha}dt - \int_0^T \frac{\beta}{\alpha}e^{-\alpha t}dt \\
E\left[\int_0^T r(t)dt\right] &= \frac{r(0)}{(-\alpha)}e^{-\alpha t}|_0^T + \frac{\beta}{\alpha}T - \frac{\beta}{\alpha}\frac{1}{(-\alpha)}e^{-\alpha t}|_0^T \\
E\left[\int_0^T r(t)dt\right] &= \frac{r(0)}{(-\alpha)}e^{-\alpha t} + \frac{r(0)}{\alpha} + \frac{\beta}{\alpha}T - \left[\frac{\beta}{(-\alpha^2)}e^{-\alpha T} - \frac{\beta}{(-\alpha^2)}\right] \\
E\left[\int_0^T r(t)dt\right] &= -\frac{r(0)}{(-\alpha)}e^{-\alpha T} + \frac{r(0)}{\alpha} + \frac{\beta}{\alpha}T + \frac{\beta}{\alpha^2}e^{-\alpha T} - \frac{\beta}{\alpha^2} \\
E\left[\int_0^T r(t)dt\right] &= \frac{r(0)}{\alpha}\left[1 - e^{-\alpha T}\right] + \frac{\beta}{\alpha^2}\left[e^{-\alpha T} - 1\right] + \frac{\beta}{\alpha}T \\
E\left[\int_0^T r(t)dt\right] &= \left[1 - e^{-\alpha T}\right]\left[\frac{r(0)}{\alpha} - \frac{\beta}{\alpha^2}\right] + \frac{\beta}{\alpha}T
\end{aligned}$$

This can also be expressed as:

$$E\left[\int_0^T r(t)dt\right] = \left[1 - e^{-\alpha T}\right]\left[\frac{\alpha r(0) - \beta}{\alpha^2}\right] + \frac{\beta}{\alpha}T$$

We now need to calculate the variance of $\int_0^T r(t)dt$. We note that

$$V(Y(T)) = V\left[\int_0^T r(t)dt\right]$$

According to result 2 the variance is given by the following expression:

$$V[Y(t)] = \int_0^t \delta^2(v)\left(\int_v^t h(y)dy\right)^2 dv$$

Here $\delta(v) = e^{\alpha v}\sigma$ and $h(y) = e^{-\alpha y}$: Hence we get

$$\begin{aligned}
V(Y(T)) &= \int_0^T (e^{\alpha u}\sigma)^2 \left[\int_u^T e^{-\alpha y}dy\right]^2 du \\
V(Y(T)) &= \int_0^T (e^{\alpha u}\sigma)^2 \left[\frac{e^{-\alpha y}}{(-\alpha)}|_u^T\right]^2 du \\
V(Y(T)) &= \int_0^T (e^{\alpha u}\sigma)^2 \left[\frac{e^{-\alpha T}}{(-\alpha)} - \frac{e^{-\alpha u}}{(-\alpha)}\right]^2 du \\
V(Y(T)) &= \int_0^T (e^{\alpha u}\sigma)^2 \left[\frac{e^{-\alpha u}}{\alpha} - \frac{e^{-\alpha T}}{\alpha}\right]^2 du \\
V(Y(T)) &= \int_0^T \left((e^{\alpha u}\sigma)\left[\frac{e^{-\alpha u}}{\alpha} - \frac{e^{-\alpha T}}{\alpha}\right]\right)^2 du \\
V(Y(T)) &= \int_0^T \left(\frac{\sigma}{\alpha} - \frac{\sigma}{\alpha}e^{\alpha(u-t)}\right)^2 du \\
V(Y(T)) &= \frac{\sigma^2}{\alpha^2}\int_0^T \left(1 - e^{\alpha(u-t)}\right)^2 du \\
V(Y(T)) &= \frac{\sigma^2}{\alpha^2}\int_0^T \left(1 - 2e^{\alpha(u-t)} + e^{2\alpha(u-T)}\right) du \\
V(Y(T)) &= \frac{\sigma^2}{\alpha^2}T - \frac{2\sigma^2}{\alpha^2}\int_0^T e^{\alpha(u-T)}du + \frac{\sigma^2}{\alpha^2}\int_0^T e^{2\alpha(u-T)}du \\
V(Y(T)) &= \frac{\sigma^2}{\alpha^2}T - \frac{2\sigma^2}{\alpha^2}e^{-\alpha T}\frac{e^{\alpha u}}{\alpha}|_0^T + \frac{\sigma^2}{\alpha^2}e^{-2\alpha T}\frac{e^{2\alpha u}}{2\alpha}|_0^T \\
V(Y(T)) &= \frac{\sigma^2}{\alpha^2}T - \frac{2\sigma^2}{\alpha^2}e^{-\alpha T}\left[\frac{e^{\alpha T}}{\alpha} - \frac{1}{\alpha}\right] + \frac{\sigma^2}{\alpha^2}e^{-2\alpha T}\left[\frac{e^{2\alpha T}}{2\alpha} - \frac{1}{2\alpha}\right] \\
V(Y(T)) &= \frac{\sigma^2}{\alpha^2}T - \frac{2\sigma^2}{\alpha^3}[1 - e^{-\alpha T}] + \frac{\sigma^2}{2\alpha^3}[1 - e^{-2\alpha T}]
\end{aligned}$$

Now having calculated the mean and variance of $\int_0^T r(t)dt$, we can substitute these in the following equation to get the bond price solution. The equation is given by

$$P(0,T) = \exp\left(-E\left[\int_0^T r(t)dt\right] + \frac{1}{2}V\left[\int_0^T r(t)dt\right]\right)$$

Let

$$M = -E\left[\int_0^T r(t)dt\right] + \frac{1}{2}V\left[\int_0^T r(t)dt\right]$$

Then

$$
\begin{aligned}
M &= -[1-e^{-\alpha T}]\left(\frac{\alpha r(0)-\beta}{\alpha^2}\right) - \frac{\beta}{\alpha}T + \frac{1}{2}\frac{\sigma^2}{\alpha^2}T - \frac{\sigma^2}{\alpha^3}[1-e^{-\alpha T}] + \frac{\sigma^2}{4\alpha^3}[1-e^{-2\alpha T}] \\
&= \frac{1}{\alpha}[1-e^{-\alpha T}](\beta/\alpha - r(0)) - \frac{\beta}{\alpha}T + \frac{1}{2}\frac{\sigma^2}{\alpha^2}T - \frac{1}{2}\frac{\sigma^2}{\alpha^3}[1-e^{-\alpha T}] - \frac{1}{2}\frac{\sigma^2}{\alpha^3}[1-e^{-\alpha T}] \\
&\quad + \frac{\sigma^2}{4\alpha^3}[1-e^{-2\alpha T}] \\
&= \frac{1}{\alpha}[1-e^{-\alpha T}]\left(\frac{\beta}{\alpha} - \frac{1}{2}\frac{\sigma^2}{\alpha^2} - r(0)\right) - T[\beta/\alpha - \frac{1}{2}\frac{\sigma^2}{\alpha^2}] - \frac{\sigma^2}{4\alpha^3}\left[e^{-2\alpha T} - 1 + 2 - 2e^{-\alpha T}\right] \\
&= \frac{1}{\alpha}[1-e^{-\alpha T}]\left(\frac{\beta}{\alpha} - \frac{1}{2}\frac{\sigma^2}{\alpha^2} - r(0)\right) - T\left[\beta/\alpha - \frac{1}{2}\frac{\sigma^2}{\alpha^2}\right] - \frac{\sigma^2}{4\alpha^3}[1-e^{-\alpha T}]^2
\end{aligned}
$$

$$P(0,T) = \exp M$$

$$P(0,T) = \exp(-r(0)C(T) - A(T))$$

for homework figure out A(T) and C(T)