Lecture-V

Stochastic Processes and the Basic Term-Structure Equation

1 Stochastic Processes

Any variable whose value changes over time in an uncertain way is called a Stochastic Process. Stochastic Processes can be classified as DISCRETE time or CONTINUOUS time. A type of process that is of extensive interest in Finance is the Markov Process. An example of the Markov process is the well known Brownian Motion also called the Wiener Process.

1.1 Markov Process

A Markov Process is a particular type of stochastic process where only the present value of a variable is relevant for predicting the future. The past history of the variable and the way in which the present has emerged from the past are irrelevant.

1.2 Wiener Process

A Wiener process is a particular type of Markov process. The behavior of a variable Z which follows a Wiener process, can be understood by considering the changes in its value in small intervals of time. Consider a small interval of time Δt and define ΔZ as the change in Z during Δt. There are two basic properties ΔZ must have for Z to be following a Wiener process.

1. ΔZ is related to Δt by the equation $\Delta Z = \epsilon\sqrt{\Delta t}$ where $\epsilon \sim N(0,1)$
2. The values of ΔZ for any two different short intervals of time Δt are independent. Property 2 implies that Z is a Markov Process. Now
   - $E(\Delta Z) = E\left(\epsilon\sqrt{\Delta t}\right) = \sqrt{\Delta t}E(\epsilon) = 0$
   - $V(\Delta Z) = V\left(\epsilon\sqrt{\Delta t}\right) = \Delta t V(\epsilon) = \Delta t$

Property 2 implies that Z is a Markov Process. Let us now consider the change in Z for a relatively longer period of time. This could be broken up into smaller intervals of Δt. Let there be N intervals. Then $N = \frac{T}{\Delta t}$. Thus

$$Z(T) - Z(0) = \sum_{i=1}^{N} \epsilon_i\sqrt{\Delta t}$$

and

$$E[Z(T) - Z(0)] = \sum_{i=1}^{N} \sqrt{\Delta t} E(\epsilon_i) = 0$$

$$V[Z(T) - Z(0)] = \sum_{i=1}^{N} \Delta t V(\epsilon_i) = N\Delta t = T$$

A function is

- $o(\Delta t)$ if $\lim_{\Delta t \to 0} \frac{f(\Delta t)}{\Delta t} \longrightarrow 0$
- $O(\Delta t)$ if $\lim_{\Delta t \to 0} \frac{f(\Delta t)}{\Delta t} \longrightarrow$ constant

$$\Rightarrow V[(\Delta Z)^2] = V[\epsilon^2 \Delta t] = \underbrace{[\Delta t]^2}_{o(dt)} \overbrace{V[\epsilon^2]}^{c}$$

where c is a constant.

1.3 Generalised Wiener Process

A generalized Wiener process for a variable x can be defined in terms of dZ as follows:

$$dx = adt + bdZ$$

where a and b are constants.

1.4 Ito's Process

This is a generalized Wiener process where the parameters a and b are functions of the value of the underlying variable x and time t. Algebraically it can be written as

$$dx = a(t,x)dt + b(t,x)dZ$$

where $a(t,x)$ is called the drift rate and $b^2(t,x)$ is called the variance rate.

1.5 Ito's Lemma

Consider the Ito's process

$$dx = a(t,x)dt + b(t,x)dZ$$

The simplest form of Ito's lemma is given by:

$$dG = \frac{\partial G}{\partial x}dx + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}(dx)^2 + \frac{\partial G}{\partial t}dt$$

Ito's Lemma shows that a function G of t and x follows the process

$$dG = \left(\frac{\partial G}{\partial t} + a\frac{\partial G}{\partial x} + \frac{1}{2}b^2\frac{\partial^2 G}{\partial x^2}\right)dt + \frac{\partial G}{\partial x}bdZ$$

where dZ is the same Wiener process and G also follows an Ito process with the given drift and variance rates.

1.6 Some examples of Stochastic Integration

1. If $dX(t) = \mu(X(t),t)dt + \sigma(X(t),t)dZ(t)$ then

$$\Rightarrow \int_0^t dX(v) = \int_0^t \mu(X(v),v)dv + \int_0^t \sigma(X(v),v)dZ(v)$$

$$\Rightarrow X(t) = X(0) + \int_0^t \mu(X(v),v)dv + \int_0^t \sigma(X(v),v)dZ(v)$$

2. If

$$\Rightarrow X(t) = X(0) + \int_0^t \mu(X(v),v)dv + \int_0^t \sigma(X(v),v)dZ(v)$$

There is no t contained in μ and σ. Then

$$\Rightarrow dX(t) = \mu(X(t),t)dt + \sigma(X(t),t)dZ(t)$$

3. If

$$X(t) = X(0) + \int_0^t \mu(X(v),v,t)dv + \int_0^t \sigma(X(v),v,t)dZ(v)$$

Then

$$\Rightarrow dX(t) = \mu(X(t),t,t)dt + \sigma(X(t),t,t)dZ(t) + \left[\int_0^t \frac{\partial\mu(X(v),v,t)}{\partial t}dv + \int_0^t \frac{\partial\sigma(X(v),v,t)}{\partial t}dZ(v)\right] dt$$

2 Vasicek

A general form of term structure models was proposed by Vasicek in 1977. He made three assumptions which are:

1. the spot rate follows a continuous Markov process
2. the price $P(t,s)$ of a discount bond is determined by the assessment at time t, of the segment $r(\tau), t \leq \tau \leq s$ of the spot rate process over the term of the bond
3. the market is efficient; that is, there are no transactions costs, information is available to all investors simultaneously andevery investor acts rationally.

The Markov property implies that the spot rate process is characterized by a single state variable namely its current value. Processes that are Markov and Continuous are called diffusion processes.We now formally present the model as

$$dr = \mu_r(t,r)dt + \sigma_r(t,r)dZ \tag{1}$$

where $Z(t)$ is a Wiener process with incremental variance dt. The functions μ_r and σ_r are the instantaneous drift and variance respectively of the process r. All the models we study in this section of the course are variations of the above model. In the Merton as is the case here σ_r is constant. In the CIR and Courtadon however, σ_r may depend on $r(t)$.

2.1 The Term Structure Equation

Assumption (2) implies that $P = P(t, r)$. Using Ito's lemma to differentiate this expression we get

$$dP = \left(\frac{\partial P}{\partial t} + \mu_r \frac{\partial P}{\partial r} + \frac{1}{2}\sigma_r^2 \frac{\partial^2 P}{\partial r^2}\right) dt + \frac{\partial P}{\partial r}\sigma_r dZ$$
$$dP = P\mu_p(t, r) - P\sigma_p(t, r)dZ$$

where μ_p and σ_p are the expressions given in the equation above. Hence μ_p and σ_p^2 are the mean and variance respectively of the instantaneous rate of return t on a bond with maturity date s, given that the current spot rate is r. Now consider an investor who at time t issues an amount W_1 of a bond with maturity date s_1 and simultaneously buys an amount W_2 maturing at time s_2. The total worth $W = W_2 - W_1$ of the portfolio thus constructed changes over time according to the accumulation equation

$$dW = (W_2\mu_p(t, s_2) - W_1\mu_p(t, s_1))dt - (W_2\sigma_p(t, s_2) - W_1\sigma_p(t, s_1))dZ$$

Let

$$W_1 = \frac{W\sigma_p(t, s_2)}{\sigma_p(t, s_1) - \sigma_p(t, s_2)}$$
$$W_2 = \frac{W\sigma_p(t, s_1)}{\sigma_p(t, s_1) - \sigma_p(t, s_2)}$$

substituting these values in the equation above we get

$$dW = \frac{W[\mu_p(t, s_2)\sigma_p(t, s_1) - \mu_p(t, s_1)\sigma_p(t, s_2)]}{\sigma_p(t, s_1) - \sigma_p(t, s_2)}dt$$

Since the above equation does not have a random term, the investor's portfolio is riskless. Hence the portfolio should provide a riskless return. Hence

$$dW = Wr(t)dt$$

$$\frac{\mu_p(t, s_2)\sigma_p(t, s_1) - \mu_p(t, s_1)\sigma_p(t, s_2)}{\sigma_p(t, s_1) - \sigma_p(t, s_2)} = r(t)$$
$$\mu_p(t, s_2)\sigma_p(t, s_1) - \mu_p(t, s_1)\sigma_p(t, s_2) = r(t)[\sigma_p(t, s_1) - \sigma_p(t, s_2)]$$
$$\mu_p(t, s_2)\sigma_p(t, s_1) - r(t)\sigma_p(t, s_1) = \mu_p(t, s_1)\sigma_p(t, s_2) - r(t)\sigma_p(t, s_2)$$
$$\frac{\mu_p(t, s_2) - r(t)}{\sigma_p(t, s_2)} = \frac{\mu_p(t, s_1) - r(t)}{\sigma_p(t, s_1)}$$

Excess return divided by standard deviation should be equal across all maturities. Since the above equation is valid for arbitrary maturity dates s_1, s_2 it follows that the ratio $\frac{\mu_p(t,s) - r(t)}{\sigma_p(t,s)}$ is independent of s. Let $q(t, r)$ denote the common value of such ratio for a bond of a maturity date given that the current spot rate is $r(t) = r$. Then

$$q(t, r) = \frac{\mu_p(t, s, r) - r(t)}{\sigma_p(t, s, r)}$$

where $s \geq t$. The quantity $q(t,r)$ is called the market price of risk as it specifies the increase in expected instantaneous rate of return on a bond per an additional unit of risk. We will now use the above equation for the price of a discount bond.

$$q(t,r) = \frac{\mu_p(t,s,r) - r(t)}{\sigma_p(t,s,r)}$$

$$\mu_p(t,s,r) - r(t) = \sigma_p(t,s,r) q(t,r)$$

Substituting μ_p and σ_p from the term structure equation we get

$$\frac{1}{P(t,s,r)}\left[\frac{\partial}{\partial t} + \mu_r \frac{\partial}{\partial r} + \frac{1}{2}\sigma_r^2 \frac{\partial^2}{\partial r^2}\right] P - r(t) = -\frac{1}{P}\sigma_r \frac{\partial P}{\partial r} q(t,r)$$

$$\frac{\partial P}{\partial t} + \mu_r \frac{\partial P}{\partial r} + \frac{1}{2}\sigma_r^2 \frac{\partial^2 P}{\partial r^2} - rP = -\sigma_r \frac{\partial P}{\partial r} q(t,r)$$

$$\frac{\partial P}{\partial t} + \mu_r \frac{\partial P}{\partial r} + \frac{1}{2}\sigma_r^2 \frac{\partial^2 P}{\partial r^2} + \sigma_r \frac{\partial P}{\partial r} q(t,r) - rP = 0$$

$$\frac{\partial P}{\partial t} + \left[\mu_r + \sigma_r q(t,r)\right] \frac{\partial P}{\partial r} + \frac{1}{2}\sigma_r^2 \frac{\partial^2 P}{\partial r^2} - rP = 0 \tag{2}$$

for $t \leq s$

This is the basic equation for pricing of discount bonds in a market characterised by our three assumptions. This equation is called the TERM STRUCTURE EQUATION(TSE). All single factor term structure models are special cases of equation (2). The TSE is a partial differential equation for $P(t,s,r)$. Once the character of the spot rate process $r(t)$ is described and the market price of risk $q(t,r)$ specified, the bond prices are obtained by solving the TSE subject to the boundary condition $P(s,s,r) = 1$. The term structure $R(t,T)$ of interest rates is then readily evaluated from the equation

$$R(t,T) = -\frac{1}{T}\log P(t, t+T, r(t))$$

2.2 Stochastic Representation Of The Bond Price

Solutions of stochastic differential equations of the type such as the TSE can be represented in an integral form in terms of the underlying stochastic process. Such representation for the bond price as a solution to the TSE and its boundary condition is as follows:

$$P(t,s) = E_t \exp\left(-\int_t^s r(\tau)d\tau - \frac{1}{2}\int_t^s q^2(\tau, r(\tau))d\tau + \int_t^s q(\tau, r(\tau))dZ(\tau)\right)$$

for $t \leq s$ To prove the above define

$$V(u) = \exp\left(-\int_t^u r(\tau)d\tau - \frac{1}{2}\int_t^u q^2(\tau, r(\tau))d\tau + \int_t^u q(\tau, r(\tau))dZ(\tau)\right)$$

Now let us Ito differentiate the process $P(u,s)V(u)$. Let $f = PV$ and then $df = d(PV)$

$$d(PV) = \frac{\partial(PV)}{\partial P}dP + \frac{1}{2}\frac{\partial^2(PV)}{\partial P^2}dP^2 + \frac{\partial(PV)}{\partial V}dV + \frac{1}{2}\frac{\partial^2(PV)}{\partial V^2}dV^2 + \frac{\partial^2(PV)}{\partial P \partial V}dPdV$$

Now

$$\frac{1}{2}\frac{\partial^2(PV)}{\partial V^2}dV^2 = 0$$

and

$$\frac{1}{2}\frac{\partial^2(PV)}{\partial P^2}dP^2 = 0$$

Hence

$d(PV) = VdP + PdV + dPdV$

$\Rightarrow d(PV) = V\left(\frac{\partial P}{\partial t} + \mu_r \frac{\partial P}{\partial r} + \frac{1}{2}\sigma_r \frac{\partial^2 P}{\partial r^2}\right) du + V\frac{\partial P}{\partial r}\sigma_2 dZ + PV\left(-r - \frac{1}{2}q^2\right) du + PVqdZ + \frac{1}{2}PVq^2 du + V\frac{\partial P}{\partial r}\sigma_p q du$

$\Rightarrow d(PV) = V\left(\frac{\partial P}{\partial t} + (\mu_r + \sigma_r q)\frac{\partial P}{\partial r} + \frac{1}{2}\sigma_r^2 \frac{\partial^2 P}{\partial r^2} - rP\right) du + PVqdZ + V\frac{\partial P}{\partial r}\sigma_r dZ$

$$d(PV) = PVqdZ + V\frac{\partial P}{\partial r}\sigma_r dZ$$

by virtue of the TSE. Integrating from t to s and taking expectation yields

$$E_t(P(s,s)V(s) - P(t,s)V(t)) = 0$$

because $P(s,s) = 1, V(t) = 1$ and $P(t,s) = E[V(S)]$. Hence proved.