# Lecture-XII

## The Black-Cox Framework

In Lecture 11, we saw how Merton applied the Black and Scholes framework to pricing risky bonds. However, one of the chief limitations of Merton's framework is that it considers a firm to default only when it exhausts all its assets. This is clearly unrealistic as firms default long before it exhausts all its assets. In 1976, Black and Cox in a significant development formulated a pricing framework in which default occurs at a prespecified lower boundary. This pricing framework is the agenda for this lecture.

## 1 The Valuation of Corporate Securities

In their development of the pricing framework, Black and Cox made the following assumptions.

1. Individuals buy and sell securities according to their wishes, without affecting the market price.
2. A riskless asset paying a known constant interest rate $r$, exists.
3. Individuals may short-sell assets including the riskless security and reinvest the proceeds.
4. Trading takes place continuously, in time.

5. There are no agency costs, bankruptcy costs, indivisibilities of assets, taxes or transaction costs.

6. The value of the firm can be modeled as a diffusion process with instantaneous variance proportional to the square of the value.

These assumptions are familiar from our study of Merton's model. Although most of the early pricing frameworks assume a diffusion-type process there are frameworks where jump processes have been used as well. If the instantaneous variance is considered to be some function of firm value and time then alternative stochastic processes may be more appropriate. We had derived the following valuation equation to be satisfied by any security, when we studied Merton's model.

$$\frac{1}{2}\sigma^2 V^2 f_{vv} + (rV - p(V,t))f_v - rf + f_t + p'(V,t) = 0 \tag{1}$$

where $f$ represents any of the firm's securities, $V$ is the value of the firm, $t$ denotes time, $\sigma^2$ is the instantaneous variance of the return on the firm, $p(V,t)$ is the net total payout made or inflow received by the firm and $p'(V,t)$ by the specific security $f$.

Let us examine a simple situation. Consider a firm to have only a single debt issue outstanding along with equity. Let the debt issue have a final promised payment $P$ at maturity date $T$. If the value of the firm exceeds $P$ at date $T$, then the shareholders will be able to pay off the bondholders otherwise the ownership of the firm will pass on to the bondholders. Hence at date $T$ the bonds will have a value of $\min(V, P)$ and equity will have a value of $\max(V - P, 0)$.

There are several assumptions about the bond indenture that are implicitly contained in this valuation. $\sigma^2$, $p(V, t)$, $p'(V, t)$, and $P$ are assumed to be known and finite. Hence we have placed certain restrictions on the firm's investment, payout and further financing policies. Up until this point we have generally followed Merton's analysis. We will now derive values for securities issued by the firm without one major restriction. Merton considered the firm to default only when the firm had exhausted all its assets. Here we will consider two boundaries for the firm's value. The first **lower** boundary, where the bondholders will force a reorganization. The second **upper** boundary could be a call provision on the bond. It should also be noted that the final payment at the maturity date could be an arbitrary function of the value of the firm at that time $T$. Ofcourse, bondholders would have to include these conditions in the indenture.

Let us turn now to the valuation equation 1. It does not involve preferences and hence a solution derived for any set of preferences must hold in general. Also the relative value of contingent claims which is a function of the value of the underlying assets must be consistent with risk neutrality.

In a risk-neutral world, if the distribution of the underlying assets is known, we can easily solve many valuation problems. In this case we will consider each security of the firm to have four sources of value: the value if the firm is not reorganized by the maturity date, the value if the firm is reorganized at the lower boundary, the value if the firm is reorganized at the upper boundary and the value of the payouts it will potentially receive. The first three are mutually exclusive, but they each contribute to the present value.

For any claim $f$, let $h_i(V(t), t)$, $i = 1, \ldots, 4$ denote the value of the four components referred to above respectively. Let $g_1(\tau)(g_2(\tau))$ be the value of $f$ as given in the contract if the firm is reorganized at the lower(upper) boundary $C_1(\tau)(C_2(\tau))$

at time $\tau$. Let the distribution of the value of the firm in a risk-neutral world at time $\tau$, $V(\tau)$ conditional on its value at current time $t$, $V(t)$, $C_1(t) < V(t) < C_2(t)$ be $\Phi(V(\tau), \tau | V(t), t)$. Then the first and the fourth component will be given by

$$h_1(V(t), t) = e^{-r(T-t)} \int_{\kappa(T)} \xi(V(T)) d\Phi(V(T), T | V(t), t)$$
$$h_4(V(t), t) = \int_t^T e^{-r(s-t)} \left[ \int_{\kappa(s)} p'(V(s), s) d\Phi(V(s), s | V(t), t) \right] ds$$

where $\kappa(\cdot)$ denotes the interval $(C_1(\cdot), C_2(\cdot))$.

The characterization of the potential values at the reorganization boundaries is done differently. Recall that at the time of receipt we would know the potential value to be received and not the actual value received. In this case, the amount to be received at each boundary is known and is specified in the contract but the time of receipt is a random variable. Its distribution however, is the same as that of the first passage time at the boundary and the approach taken by Cox and Ross can still be applied. Let $\Psi_1(t^*|V(t),t)$ be the distribution of the first passage $t^*$ to the lower boundary and $\Psi_2(t^*|V(t),t)$ be the distribution of the first passage time to the upper boundary. Then

$$h_{i+1}(V(t),t) = \int_t^T e^{-r(t^*-t)} g_i(t^*) d\Psi(t^*|V(t),t) \qquad i = 1,2$$

## 2 Bonds with Safety Covenants

We will now study the impact that safety covenants have on a firm's securities. Provisions in the contract which give debtholders the right to force a reorganization if the firm is performing poorly according to some standard are called **safety covenants**. A standard for example could be if the firm stops paying interest on the debt. In that situation if the stockholders were allowed to sell the assets of the firm then the restrictions are not very effective. A safety covenant could be the following: if the value of the firm falls below a certain level, which could change over time the bondholders have the right to force the firm into bankruptcy or take ownership of the assets. Under this agreement, interest payments will not play a prominent role and hence we will assume that the firm has only one class of discount bonds outstanding. However we will assume that the firm pays dividends and stockholders receive a continuous dividend payment $aV$ proportional of the value of the firm. Since, in continuous time it is reasonable to characterize time dependence of the safety covenant as exponential, we will let the bankruptcy level $C_1(t)$ be $Ce^{-\gamma(T-t)}$. The valuation equation for a bond $B$ will be

$$\frac{1}{2}\sigma^2 V^2 B_{vv} + (r-a)VB_v - rB + B_t = 0 \tag{2}$$

with boundary conditions

$$B(V,T) = \min(V,P)$$

$$B(Ce^{-\gamma(T-t)}, t) = Ce^{-\gamma(T-t)}$$

Similarly the value of the stock must satisfy

$$\frac{1}{2}\sigma^2 V^2 S_{vv} + (r-a)VS_v - rS + S_t + aV = 0 \tag{3}$$

with boundary conditions

$$S(V,T) = \min(V-P, 0)$$

$$S(Ce^{-\gamma(T-t)}, t) = 0$$

In order to follow the probabilistic framework of valuation we need the conditional distribution

$$\Phi(V(\tau), \tau | V(t), t)$$

i.e. the value of the firm at time $\tau$ given the value of the firm at the current time $V$, in a risk neutral setting. In this case the distribution will be a lognormal process, with an absorbing barrier at the reorganization boundary $C_1(\tau) = Ce^{-\gamma(T-\tau)}$. The probability that $V(\tau) \geq K$ and has not reached the reorganization boundary in the meantime is given by

$$N\left(\frac{\ln V - \ln K + (r - a - \frac{1}{2}\sigma^2)(\tau - t)}{\sqrt{\sigma^2(\tau - t)}}\right) - \left(\frac{V}{Ce^{-\gamma(T-t)}}\right)^{1-(2(r-a-\gamma)/\sigma^2)} N\left(\frac{2\ln Ce^{-\gamma(T-t)} - \ln V - \ln K + (r - a - \frac{1}{2}\sigma^2)(\tau - t)}{\sqrt{\sigma^2(\tau - t)}}\right)$$

where $N(\cdot)$ is the standard normal distribution function. Setting $K = Ce^{-\gamma(T-\tau)}$ we get the risk neutral probability of the firm not being reorganized before time $\tau$. The intuition comes from first passage time distributions. If $t^*$ is the first passage time to the boundary i.e. the time at which the value of the firm first hits the reorganization level then the probability that $t^* \geq \tau$ is given by the expression above by letting $K = Ce^{-\gamma(T-\tau)}$. The valuation formula for $B$ can be solved as

$$B(V,t) = Pe^{r(T-t)}[N(z_1) - y^{2\theta-2}N(z_2)] + Ve^{a(T-t)}[N(z_3) - y^{2\theta}N(z_4)$$

$$+ y^{\theta+\zeta}e^{a(T-t)}N(z_5) + y^{\theta-\zeta}e^{a(T-t)}N(z_6) - y^{\theta-\eta}N(z_7) - y^{\theta-\eta}N(z_8)]$$

where

$$y = Ce^{\gamma(T-t)}/V$$

$$\theta = (r - a - \gamma + \frac{1}{2}\sigma^2)/\sigma^2$$

$$\delta = (r - a - \gamma - \frac{1}{2}\sigma^2)^2 + 2\sigma^2(r - \gamma)$$

$$\zeta = \sqrt{\delta}/\sigma^2$$

$$\eta = \sqrt{\delta - 2\sigma^2 a}/\sigma^2$$

$$z_1 = [\ln V - \ln P + (r - a - \frac{1}{2}\sigma^2)(T-t)]/\sqrt{\sigma^2(T-t)}$$

$$z_2 = [\ln V - \ln P + 2\ln y + (r - a - \frac{1}{2}\sigma^2)(T-t)]/\sqrt{\sigma^2(T-t)}$$

$$z_3 = [\ln V - \ln P - (r - a + \frac{1}{2}\sigma^2)(T-t)]/\sqrt{\sigma^2(T-t)}$$

$$z_4 = [\ln V - \ln P + 2\ln y + (r - a + \frac{1}{2}\sigma^2)(T-t)]/\sqrt{\sigma^2(T-t)}$$

$$z_5 = [\ln y + \zeta\sigma^2(T-t)]/\sqrt{\sigma^2(T-t)}$$

$$z_6 = [\ln y - \zeta\sigma^2(T-t)]/\sqrt{\sigma^2(T-t)}$$

$$z_7 = [\ln y + \eta\sigma^2(T-t)]/\sqrt{\sigma^2(T-t)}$$

$$z_8 = [\ln y - \eta\sigma^2(T-t)]/\sqrt{\sigma^2(T-t)}$$

This expression holds for all $Ce^{-\gamma(T-t)} \leq Pe^{r(T-t)}$. As noted by Black and Cox, an interesting choice is $Ce^{-\gamma(T-t)} = \rho Pe^{-r(T-t)}$ with $0 \leq \rho \leq 1$ so that the reorganization value specified in the safety covenant is a constant fraction of the present value of the promised final payment.