Lecture-XI

Modeling Credit Risk-The Basic Framework

1 Introduction

The value of a corporate debt security essentially depends on three items:

1. the required rate of return on riskless debt e.g., treasury and very high grade corporate bonds
2. the various provisions and restrictions contained in the indenture e.g., maturity date, coupon rate, call terms etc.
3. the probability that the firm will be unable to satisfy one or more of the indenture requirements i.e. the probability of default

Until the time Merton published his seminal paper on the pricing of corporate debt, there had been no systematic development of a theory of pricing bonds when there is a significant probability of default, even though there were several theories and empirical studies on the term-structure of interest rates. Merton preferred to call the theory "Risk Structure of Interest Rates". However, the implications of the word **risk** is very specific. It was attributed to the possible gains or losses by bondholders as a result of unanticipated changes in the probability of default and **not** on unanticipated changes in interest rates in general. The development of the theory is based on an assumed term structure and hence the price differentials among bonds will be solely caused by the differences in the probability of default.

Black and Scholes in 1973 presented a complete general equilibrium theory of option pricing which is very attractive because the final formula is a function of observable variables. The same approach is applied here in developing a pricing theory for corporate liabilities in general.

2 The Pricing of Corporate Liabilities

We will now develop a Black-Scholes-type pricing model under the following assumptions.

1. there are no transactions costs, taxes or problems with indivisibilities of assets
2. there are a sufficient number of investors with comparable wealth levels so that each investor believes that he can buy and sell as much of an asset as he wants at the market price
3. there exists an exchange market for borrowing and lending at the same rate of interest
4. short sales of all assets with full use of the proceeds is allowed
5. trading in assets takes place continuously in time

6. the Miller-Modigliani theorem that the value of the firm is invariant to its capital structure obtains

7. the term-structure is flat and known with certainty. i.e. the price of a riskless discount bond which promises a payment of one dollar at time τ in the future is $P(\tau) = \exp(-r\tau)$ where r is the instantaneous riskless rate of interest, the same for all time

8. the dynamics for the value of the firm V through time can be described by a diffusion-type stochastic process with stochastic differential equation

$$dV = (\alpha V - C)dt + \sigma V dz \tag{1}$$

where α is the instantaneous expected rate of return on the firm per unit time, C is the total dollar payouts by the firm per unit time to either its shareholders or liability-holders if positive, and it is the net dollars received by the firm from new financing if negative, σ^2 is the instantaneous variance of the return on the firm per unit time and dz is a standard Gauss-Wiener process.

Many of these assumptions are not necessary for the model to obtain but have been chosen for expositional convenience. For example the first four "perfect market" assumptions can be relaxed. However 5 and 8 are critical and 6 will be proven later. 7 has been chosen to clearly distinguish between risk structure and term structure effects of pricing.

We now turn to pricing securities. Let us consider a security whose market value Y at any point in time can be written as a function of the value of the firm and time i.e.,

$$Y = F(V, t)$$

The dynamics of Y is again a diffusion-type stochastic process with stochastic differential equation

$$dY = (\alpha_y Y - C_y)dt + \sigma_y Y dz \tag{2}$$

where α_y is the instantaneous expected rate of return on this security, C_y is the dollar payout per unit time to this security, σ_y^2 is the instantaneous variance of the return per unit time and dz is a standard Gauss-Wiener process. However, since Y is a function of the value of the firm and time there is an explicit functional relationship between α_y and σ_y in equation 2 and the corresponding variables in equation 1. Using Ito's lemma we can explicitly characterize the relationship:

$$\begin{aligned}
Y &= F(V,t) \\
dY &= \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial V}dV + \frac{1}{2}\frac{\partial^2 F}{\partial V^2}(dV)^2 \qquad \text{applying Ito's lemma} \\
&= F_t dt + F_v dV + \frac{1}{2}F_{vv}(dV)^2 \qquad \text{substitute } dV \text{ from 1} \\
&= F_t dt + F_v((\alpha V - C)dt + \sigma V dz) + \frac{1}{2}F_{vv}((\alpha V - C)dt + \sigma V dz)^2 \\
&= F_t dt + F_v(\alpha V - C)dt + F_v \sigma V dz + \frac{1}{2}F_{vv}\sigma^2 V^2 dt \qquad \text{since} \quad (dt)^2 = dt \cdot dz = 0 \\
&= \left(F_t + F_v(\alpha V - C) + \frac{1}{2}F_{vv}\sigma^2 V^2\right)dt + F_v \sigma V dz
\end{aligned}$$

Hence we get

$$dY = \left(F_t + F_v(\alpha V - C) + \frac{1}{2}F_{vv}\sigma^2 V^2\right) dt + F_v \sigma V dz \tag{3}$$

Comparing terms in 3 and 2 we get

$$\begin{aligned}
\alpha_y Y &= \alpha_y F \equiv F_t + F_v(\alpha V - C) + \frac{1}{2}F_{vv}\sigma^2 V^2 + C_y \\
\sigma_y Y &= \sigma_y F \equiv \sigma V F_v
\end{aligned}$$

Now consider forming a three security portfolio containing the firm, the particular security and riskless debt such that the aggregate investment in the portfolio is zero. This is achieved by using the proceeds of short sales and borrowings to finance the long positions. Let W_1 be the instantaneous number of dollars invested in the firm, W_2 be the number of dollars invested in the security and $W_3 = -(W_1 + W_2)$ be the number of dollars invested in riskless debt. If dx is the instantaneous dollar return on the portfolio, then

$$\begin{aligned}
dx &= W_1[\text{Return from Firm}] + W_2[\text{Return from Security}] + W_3[\text{Return from Riskless Debt}] \\
&= W_1\frac{(dV + Cdt)}{V} + W_2\frac{(dY + C_y dt)}{Y} + W_3 r dt \\
&= W_1\frac{((\alpha V - C)dt + \sigma V dz + Cdt)}{V} + W_2\frac{((\alpha_y Y - C_y)dt + \sigma_y Y dz_y + C_y dt)}{Y} + [-(W_1 + W_2)]rdt \\
&= W_1(\alpha + \sigma dz) + W_2(\alpha_y + \sigma dz_y) + [-(W_1 + W_2)]rdt \\
&= [W_1(\alpha - r) + W_2(\alpha_y - r)]dt + W_1\sigma dz + W_2\sigma_y dz_y \\
&= [W_1(\alpha - r) + W_2(\alpha_y - r)]dt + [W_1\sigma + W_2\sigma_y]dz \quad \text{since } dz \text{ and } dz_y \text{ are correlated}
\end{aligned}$$

Now consider the final expression for dx.

$$dx = [W_1(\alpha - r) + W_2(\alpha_y - r)]dt + [W_1\sigma + W_2\sigma_y]dz$$

We can impose restrictions on the coefficients to avoid risk and arbitrage. In particular,

$$\begin{aligned}
W_1\sigma + W_2\sigma_y &= 0 \quad \text{no risk} \\
W_1(\alpha - r) + W_2(\alpha_y - r) &= 0 \quad \text{no arbitrage}
\end{aligned}$$

A non-trivial solution to the system above will exist iff

$$\left(\frac{\alpha - r}{\sigma}\right) = \left(\frac{\alpha_y - r}{\sigma_y}\right)$$

However we have derived expressions for α_y and σ_y above, we substitute them here

$$\begin{aligned}
\left(\frac{\alpha - r}{\sigma}\right) &= \left(\frac{\alpha_y F - rF}{\sigma_y F}\right) \\
\left(\frac{\alpha - r}{\sigma}\right) &= \frac{\left(F_t + F_v(\alpha V - C) + \frac{1}{2}F_{vv}\sigma^2 V^2 + C_y - rF\right)}{\sigma V F_v} \\
(\alpha - r)VF_v &= F_t + F_v(\alpha V - C) + \frac{1}{2}F_{vv}\sigma^2 V^2 + C_y - rF \\
\alpha V F_v - rVF_v &= F_t + F_v\alpha V - F_v C + \frac{1}{2}F_{vv}\sigma^2 V^2 + C_y - rF \\
0 &= F_t + rVF_v - F_v C + \frac{1}{2}F_{vv}\sigma^2 V^2 + C_y - rF \\
0 &= F_t + F_v(rV - C) + \frac{1}{2}F_{vv}\sigma^2 V^2 + C_y - rF
\end{aligned}$$

$$0 = F_t + F_v(rV - C) + \frac{1}{2}F_{vv}\sigma^2 V^2 + C_y - rF \quad (4)$$

Hence we have derived a parabolic partial differential equation for F, which must be satisfied by any security whose value can be written as a function of the value of the firm and time. However to solve the equation we need boundary conditions. It is precisely these boundary conditions which distinguish one security from another(i.e. the debt of a firm from its equity.)

Finally to conclude the above analysis, it is important to note the following

- F depends on
    1. the value of the firm
    2. the time
    3. the interest rate
    4. the volatility of the firm's value or its business risk as measured by the variance
    5. the payout policy of the firm
    6. the promised payout policy to the holders of the security
- F does not depend on
    1. expected rate of return on the firm
    2. risk preferences of investors
    3. characteristics of other assets available to investors beyond the three mentioned
- two investors with very different utility functions and different expectations for the company's future but who agree on the volatility of the firm's value will for a given interest rate and current firm value agree on the value of the particular security$F$.
- all parameters and variables except the variance are directly observable and the variance can be reasonably estimated from time series data.

3 The Pricing of Risky Discount Bonds

We will now apply what we have learned in the previous section in the simplest case of corporate debt pricing. Consider a corporation with two classes of claims

1. a single homogenous class of debt
2. the residual claim equity

Also assume that the indenture of the bond issue contains the following provisions and restrictions.

1. the firm promises to pay a total of B dollars to the bondholders on the specified calendar date T
2. in the event this payment is not met, the bondholders immediately take over the company and the shareholders receive nothing

3. the firm cannot issue any new senior(or of equivalent rank) claims on the firm nor can it pay cash dividends or repurchase shares prior to the maturity of the debt. If F is the value of the debt issue we can write equation 4 as

$$0 = -F_\tau + F_v rV + \frac{1}{2}F_{vv}\sigma^2 V^2 + -rF \tag{5}$$

where $C_y = 0$ because of the zero-coupon condition

4. $\tau = T - t$ is length of time until maturity so that $F_t = -F_\tau$

To solve equation 5 for the value of the debt we need to specify boundary conditions. These boundary conditions are derived from the provisions in the indenture and the limited liability of claims. By definition, we have

$$V \equiv F(v,\tau) + f(V,\tau)$$

where f is the value of the equity. Since both F and f can only take on non-negative values we have

$$F(0,\tau) = f(0,\tau) = 0 \tag{5a}$$

Further $F(V,\tau) \leq V$ which implies the regularity condition

$$\frac{F(V,\tau)}{V} = 1 \tag{5b}$$

which is the other boundary condition in this semi-infinite boundary problem where $0 \leq V \leq \infty$. The initial condition follows from indenture conditions 1 and 2 that management is elected by equity owners and must act in their best interests. On the maturity date T(i.e.$\tau = 0$) the firm must pay the promised amount of B to the debtholders or else the current equity will be valueless. Clearly if at time T, $V(T) > B$ the firm will pay the bondholders B and the value of equity will be $V(T) - B$ but if they do not the value of equity will be zero. If however, $V(T) < B$ then the firm will not be able to make payments to the bondholders on the maturity date and default the firm to the bondholders. The other choice is to pay the bondholders by equity holders contributing additional money which is unwise and limited liability shields the equity holders from any such payments. Thus the initial condition for the debt at $\tau = 0$ is

$$F(V,0) = \min\left[V, B\right]$$

We can now solve equation 5 as we have the boundary conditions. There are several methods by which we can solve 5, standard methods of fourier transforms or separation of variables to name a few. However, note that this is exactly like the Black and Scholes partial differential equation and the solution is available in the literature. To determine the value of equity $f(V,\tau)$ we note that $f(V,\tau) = V - F(V,\tau)$ and substitute for F in equation 5 to deduce the partial differential equation for f, which is

$$0 = -f_\tau + f_v rV + \frac{1}{2}f_{vv}\sigma^2 V^2 - rf \tag{6}$$

subject to

$$f(V,0) = \max\left[0, V - B\right] \tag{6a}$$

and boundary conditions 5a and 5b above. At this stage let us explicitly state the similarities between this and the Black and Scholes equation. Equations 6 and 6a are identical to the equations for a European call option for a non-dividend paying stock where firm value in 6 and 6a corresponds to the stock price and B to the exercise price. In the case of Black and Scholes if the stock price is greater than the exercise price, the option will be exercised. In this case, if the value of the firm is greater than the payments promised, the firm can meet its promises. However, if the stock price is lower than the exercise price then the call option holder will not exercise the option, which will be valueless. In this case if the firm defaults then the bondholders take over the firm and the shareholders get nothing.

This isomorphic price-relationship between the levered equity of a firm and the call option not only allows us to write the solution to 6 and 6a but also allows us to apply the comparative statics results in the Black and Scholes analysis to the equity and debt cases. We will examine those in detail a little later. Hence we have

$$f(V,\tau) = VN(x_1) - Be^{-r\tau}N(x_2) \tag{7}$$

where

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left[\frac{1}{2}z^2\right] dz$$

and

$$\begin{aligned} x_1 &= \frac{\log\left[\frac{V}{B}\right] + \left(r + \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}} \\ x_2 &= x_1 - \sigma\sqrt{\tau} \end{aligned}$$

Using 7 and the fact that $F = V - f$ we can write the value of the debt issue as

$$F(V,\tau) = Be^{-r\tau}\left[N[h_2(d,\sigma^2\tau)] + \frac{1}{d}N[h_1(d,\sigma^2\tau)]\right] \tag{8}$$

where

$$\begin{aligned} d &= \frac{Be^{-r\tau}}{V} \\ h_1(d,\sigma^2\tau) &= -\frac{\left[\frac{1}{2}\sigma^2\tau - \log(d)\right]}{\sigma\sqrt{\tau}} \\ h_2(d,\sigma^2\tau) &= -\frac{\left[\frac{1}{2}\sigma^2\tau + \log(d)\right]}{\sigma\sqrt{\tau}} \end{aligned}$$

Since in most discussions of bond pricing yields are used more than prices we can rewrite 8 in terms of yields as

$$R(\tau) - r = \frac{-1}{\tau}\log\left[N[h_2(d,\sigma^2\tau)] + \frac{1}{d}N[h_1(d,\sigma^2\tau)]\right] \tag{9}$$

where

$$\exp\left[-R(\tau)\tau\right] \equiv \frac{F(V,\tau)}{B}$$

and $R(\tau)$ is the yield-to-maturity on the risky debt provided that the firm does not default. Merton calls $R(\tau) - r$ a risk premium and equation 9 a definition for the risk structure of interest rates. Finally it is important to note the following

- for a given maturity the risk premium is a function of only two variables
  1. the volatility of the firm's operations σ^2

2. the ratio of the present value(at the riskless rate) of the promised payment to the current value of the firm d

- because d is the debt-to-firm value ratio where debt is valued at the riskless rate, it is a biased upward estimate of the actual(market-value) debt-to-firm value ratio.
- Merton in a 1973 paper has solved the option pricing problem when the term-structure is not flat and is stochastic(by again using isomorphic correspondence between options and levered equity). Using that approach the risk-structure can be deduced under a stochastic term-structure.

4 A Comparative Statics Analysis of the Risk Structure

A careful examination of equation 8 shows that the value of debt can be written showing its full functional dependence as $F(V, \tau, B, \sigma^2, r)$. Because of the isomorphic relationship between levered equity and a European call option we can use analytical results presented in Merton[1973] to show that F is a first-degree homogenous concave function of V and B. We also have the following:

1. $F_V = 1 - f_V \geq 0$
2. $F_B = -f_B > 0$
3. $F_\tau = -f_\tau < 0$
4. $F_{\sigma^2} = -f_{\sigma^2} < 0$
5. $F_r = -f_r < 0$

where the subscripts denote partial derivatives. The results presented above are as one would have expected for a discount bond i.e. the value of debt is an increasing function of the current market-value of the firm and the promised payment at maturity and a decreasing function of the time to maturity, the business risk of the firm and the riskless rate of interest. Since we are interested in the risk structure of interest rates, which is a cross-section of bond prices at a point in time, it will shed more light on the characteristics of this structure to work with the price ratio

$$P = \frac{F(V, \tau)}{B} = \exp\left[-rt\right]$$

rather than the absolute price level F. P is the price today of a risky dollar promised at time τ in the future in terms of a dollar delivered at that date with certainty and it is always less than or equal to one. From equation 8 we have

$$P[d, T] = N(h_2(d, T)) + \frac{1}{d} N(h_1(d, T)) \tag{10}$$

where $T = \sigma_2 \tau$. Note that unlike F, P is completely determined by d, the "quasi" debt-to-firm value ratio and T which is a measure of the volatility of the firm's value over the life of the bond and it is a decreasing function of both i.e.

$$P_d = -\frac{N(h_1)}{d^2} < 0 \tag{11}$$

$$P_T = -\frac{N'(h_1)}{2d\sqrt{T}} < 0 \tag{12}$$

where $N(x) = \frac{\exp\left(-\frac{x^2}{2}\right)}{\sqrt{2\pi}}$ is the standard normal density function.

We now define another ratio which is of critical importance in analyzing the risk-structure

$$g = \frac{\sigma_y}{\sigma}$$

where σ_y is the instantaneous standard deviation of the return of the bond and σ is the instantaneous standard deviation of the return of the firm. Because these two returns are instantaneously perfectly correlated, g is a measure of the relative riskiness of the bond in terms of the riskiness of the firm at a given point in time. We can deduce the formula for g to be

$$\begin{aligned} \frac{\sigma_y}{\sigma} &= \frac{VF_V}{F} \\ &= \frac{N[h_1(d,T)]}{(P[d,T]d)} \\ &= g[d,T] \end{aligned}$$

The characteristics of g are examined in detail later. However we note here that g is a function of d and T only and that from the no-arbitrage condition we have

$$\frac{\alpha_y - r}{\alpha - r} = g[d,T]$$

where $\alpha_y - r$ is the expected excess return on the debt and $\alpha - r$ is the expected excess return on the firm as a whole. We can rewrite equations 11 and 12 in elasticity form in terms of g to be

$$\frac{dP_d}{P} = -g[d,T] \tag{13}$$

and

$$\frac{TP_T}{P} = \frac{-g[d,T]\sqrt{T}N'(h_1)}{2N(h_1)} \tag{14}$$

Since it is common to use yield-to-maturity in excess of the riskless debt as a measure of the risk premium on debt, we will get expressions in terms of g. If we define $R(\tau) - r \equiv H(d,\tau,\sigma^2)$ then we get the following:

$$\begin{aligned} H_d &= \frac{1}{\tau d} g[d,T] > 0 \\ H_{\sigma^2} &= \frac{g[d,T]N'(h_1)}{2\sqrt{T}N(h_1)} > 0 \\ H_\tau &= \frac{\log[P] + \frac{\sqrt{T}N'(h_1)g[d,T]}{2N(h_1)}}{\tau^2} \end{aligned}$$

It is important to note the following:

- the term premium is an increasing function of both d and σ^2
- From the expression for H_τ, it is important to note that a change in the premium with respect to a change in maturity can be either sign, however for $d \geq 1$ it will be negative.
- the term premium is a decreasing function of the riskless rate of interest i.e.,

$$\begin{aligned} \frac{dH}{dr} &= H_d \frac{\partial d}{\partial r} \\ &= -g[d,T] < 0 \end{aligned}$$

- it still remains to be determined whether $R-r$ is a valid measure of the riskiness of the bond i.e. can one assert that if $R-r$ is larger for one bond than another then the former is riskier than the latter

- to answer the question in the previous item one must establish and appropriate definition of "riskier"

- since the risk-structure like the corresponding term-structure is a "snap shot" at one point in time, it seems natural to define the riskiness in terms of the uncertainty of the rate of return over the next trading interval

- in the sense of riskier the natural choice as a measure of risk is the instantaneous standard deviation of the return on the bond

$$\sigma_y = \sigma g[d, T] = G(d, \sigma, \tau)$$

- the standard deviation is a sufficient statistic for comparing the relative riskiness of securities in the Rothschild-Stiglitz sense

- however the standard deviation is not sufficient for comparing the riskiness of the debt of different companies in a portfolio sense because the correlations of the returns of the two firms with other assets in the economy may be different, Since $R-r$ is computed for each bond without the knowledge of such correlations it cannot reflect such differences except indirectly through the market value of the firm

- thus atleast a necessary condition for $R-r$ to be a valid measure of risk is that it should move in the same direction as G does in response to changes in the underlying variables.

We also have

$$\begin{aligned}
G_d &= \frac{\sigma g^2}{\sqrt{T}} \frac{N(h_2)}{N(h_1)} \left[\frac{N'(h_2)}{N(h_2)} + \frac{N'(h_1)}{N(h_1)} + h_1 + h_2\right] \\
G_\sigma &= g(N(h_1) - N'(h_1)) \frac{\left[\frac{1}{2}(1-2g) + \frac{\log[d]}{T}\right]}{N(h_1)} \\
G_\tau &= \frac{-\sigma^2 G}{\sqrt{T}} \frac{N'(h_1)}{N(h_1)} \left[\frac{1}{2}(1-2g) + \frac{\log[d]}{T}\right]
\end{aligned}$$

Comparing the partial derivatives of G and H we see that the term premium and the standard deviation change in the same direction in response to a change in the "quasi" debt-to-firm value ratio or the business risk of the firm. However they need not change in the same direction with a change in maturity. Hence while comparing the term premiums on bonds of the same maturity does provide a valid comparison of the riskiness of such bonds one cannot conclude that a higher term premium on bonds of different maturities implies a higher standard deviation.

To complete the comparison between $R-r$ and G the standard deviation is a decreasing function of the riskless rate of interest as was the case for the term premium i.e.,

$$\begin{aligned}
\frac{dG}{dr} &= G_d \frac{\partial d}{\partial r} \\
&= -\tau d G_d < 0
\end{aligned}$$

5 The Modigliani-Miller Theorem with Bankruptcy

In the derivation of equation 4 the fundamental equation for the pricing of corporate liabilities, is was assumed that the Modigliani-Miller theorem held so that the value of the firm could be treated as exogenous. Under conditions of bankruptcy and corporate taxes i.e. the M-M theorem not obtaining, the value of the firm would depend on the debt-equity ratio but the formal analysis is still valid. However the linear property of equation 4 would be lost and instead a non-linear simultaneous solution $F = F[V(F), \tau]$ would be required.

In the previous section a cross-section of bonds across firms at a point in time were analyzed to describe the risk structure of interest rates. We now examine a debt issue of a single firm. In this context, we are interested in measuring the risk of the debt relative to the risk of a firm. As discussed before, the correct measure of this relative riskiness is $g[d,T] = \sigma_y/\sigma$. We also have the following relationship.

$$\frac{1}{g} = 1 + \frac{dN(h_2)}{N(h_1)}$$

From the equation above we have $0 \leq g \leq 1$, i.e. the debt of the firm can never be more risky than the firm as a whole and as a corollary the equity of a levered firm must always be atleast as risky as the firm. Thus as the ratio of the present value of the promised payment to the current value of the firm becomes large and therefore the probability of eventual default becomes large, the market value of the debt approaches that of the firm and the risk characteristics of the debt approaches that of unlevered equity. As $d \to 0$ the probability of default approaches zero and $F[V,\tau] \to B\exp[-r\tau]$ the value of a riskless bond and $g \to 0$. In this case the characteristics of debt become the same as riskless debt. Between these two extremes, the debt will behave like a combination of riskless debt and equity and will change in a continuous fashion. Now

$$g_d = \frac{g}{d}\left[-(1-g) + \frac{1}{\sqrt{T}}\frac{N'(h_1)}{N(h_1)}\right] > 0$$

i.e., the relative riskiness of the debt is an increasing function of d, and

$$g_T = \frac{-gN'(h_1)}{2\sqrt{T}N(h_1)}\left[\frac{1}{2}(1-2g) + \frac{\log[d]}{T}\right] \tag{15}$$

i.e.,

$$\begin{aligned} g_T &> 0 \quad \text{as} \quad d < 1 \\ g_T &= 0 \quad \text{as} \quad d = 1 \\ g_T &< 0 \quad \text{as} \quad d > 1 \end{aligned}$$

Further we have that

$$\begin{aligned} g[1,T] &= \frac{1}{2}, \quad \text{as} \quad T > 0 \\ \lim_{T\to\infty} g[d,T] &= \frac{1}{2} \quad \text{as} \quad 0 < d < \infty \end{aligned}$$

Hence for $d = 1$ independent of the business risk of the firm or the length of time until maturity, the standard deviation of the return on debt equals half the standard deviation of the return on the whole firm. As the business risk of the firm or the time to maturity gets large $\sigma_y \to \sigma/2$ for all d.

A somewhat surprising result is that the relative riskiness of the debt can decline as either the business risk of the firm or the time until maturity increases. Examining equation 15 carefully reveals that this is the case if $d > 1$(i.e. the present value of the promised payment is less than the current value of the firm). This is not unreasonable, considering the following: for small T(i.e. σ^2 or τ small), the chances that the debt will become equity through default are large and this will be reflected in the risk characteristics of the debt through a large g. By increasing T(either through an increase in σ^2 or τ) the chances are better that the firm value will increase enough to meet the promised payment. It is also true that the chances that the firm value will be lower are increased. However remember that g is a measure of how much the risky debt behaves like equity versus debt. Since for large g, the debt is already more aptly described by equity than riskless debt. Thus the increased probability of meeting the promised payment dominates, and g declines.