Chapter 8

Some notes on term structure modelling

8.1   Introduction

After the brief encounter with continuous time modelling in Chapter 7 we now return to the discrete time, finite state space models of Chapter 5. They still have a great deal to offer.

One of the most widespread applications of arbitrage pricing in the multiperiod finite state space model is in the area of term structure modelling. We saw in Chapter 3 how the term structure could be defined in several equivalent ways through the discount function, the yields of zero coupon bonds and by looking at forward rates. In this chapter we will think of the term structure as the yield of zero coupon bonds as a function of time to maturity. In Chapter 3 we considered the term structure at a fixed point in time. In this chapter our goal is to look at dynamic modelling of the evolution of the term structure. This topic could easily occupy a whole course in itself so here we focus merely on explaining a fundamental method of constructing arbitrage-free systems of bond prices. Once this method is understood the reader will be able to build models for the evolution of the term structure and price interest rate related contingent claims.

We also consider a few topics which are related to term structure modelling and which we can discuss rigorously with our arbitrage pricing technology. These topics are the difference between forwards and futures and the role of 'convexity effects' - or Jensen's inequality - can rule out various properties of term structure evolutions. We also look briefly at so-called swap contracts which are quite important in bond markets.

8.2 Constructing an arbitrage free model

Our goal is to model *prices of zero coupon bonds* of different maturities and through time. Let $P(t, T_i)$, $0 \leq t \leq T_i \leq T$, denote the price at time t of a zero coupon bond with maturity T_i. To follow the notation which is most commonly used in the literature we will deviate slightly from the notation of Chapter 5. To be consistent with Chapter 5 we should write $P(t, T_i)$ for the price of the bond prior to maturity. i.e. when $t < T_i$ and then have a dividend payment $\delta(T_i) = 1$ at maturity and a price process satisfying $P(t, T_i) = 0$ for $t \geq T_i$. We will instead write the dividend into the price and let

$$P(t, t) = 1$$

for all t. (You should have gotten used to this deceptive notation in Chapters 6 and 7.)

We will consider models of bond prices which use the spot rate process $\rho = (\rho_t)_{t=0,\ldots,T-1}$ as the fundamental modelling variable. Recall that the money market account is a process with value 1 and dividend at date $t < T$ given by ρ_{t-1} and a dividend of $1 + \rho_T$ at time T. We will need our simple notation for returns obtained by holding money over several periods in the money market account:

Definition 40 *The return of the money market account from period t to u is*

$$R_{t,u} = (1 + \rho_t)(1 + \rho_{t+1}) \cdots (1 + \rho_{u-1}), \quad \textit{for } t < u$$

Make sure you understand that $R_{t,t+1}$ is known at time t, whereas $R_{t,t+2}$ is not!

From the fundamental theorem of asset pricing (Theorem 15) we know that the system consisting of the money market account and zero coupon bonds will be arbitrage free if and only if

$$\left(\frac{P(t, T_i)}{R_{0,t}} \right)_{0 \leq t \leq T_i}$$

is a martingale for every T_i under some measure Q. Here, we use the fact that the zero coupon bonds only pay one dividend at maturity and we have denoted this dividend $P(T_i, T_i)$ for the bond maturing at date T_i. It is not easy, however, to specify a family of sensible and consistent bond prices. If T is large there are many maturities of zero coupon bonds to keep track of. They all should end up having price 1 at maturity, but that is about all we know. How do we ensure that the large system of prices admits no arbitrage opportunities?

What is often done is the following: We simply construct bond prices as expected discounted values of their terminal price 1 under a measure Q which we specify in advance (as opposed to derive from bond prices). More precisely:

Proposition 27 *Given a spot rate process $\rho = (\rho_t)_{t=0,\ldots,T-1}$. Let*

$$\mathcal{F}_t = \sigma(\rho_0, \rho_1, \ldots, \rho_T).$$

For a given Q define

$$P(t, T_i) = E_t^Q \left[\frac{1}{R_{t,T_i}} \right] \ \ for \ 0 \le t \le T_i \le T,$$

where $E_t^Q[\cdot]$ is short hand for $E^Q[\cdot \mid \mathcal{F}_t]$. Then the system consisting of the money market account and the bond price processes $(P(t, T_i))_{t=0,\ldots,T}$ is arbitrage free.
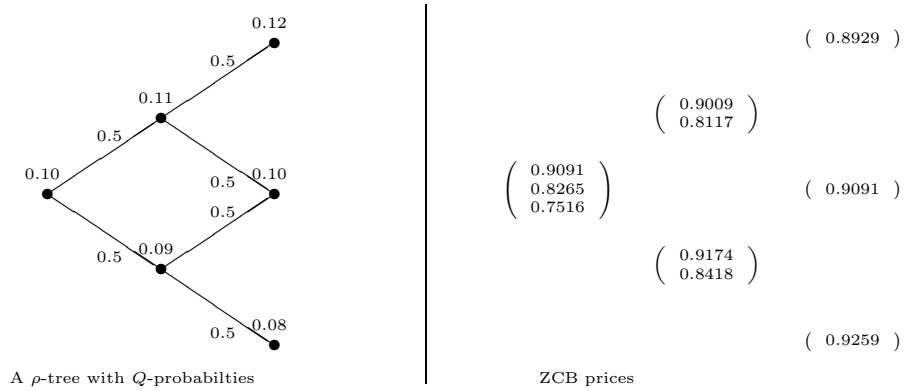
Proof. The proof is an immediate consequence of the definition of prices, since

$$\frac{P(t, T_i)}{R_{0,t}} = \frac{1}{R_{0,t}} E_t^Q \left[\frac{1}{R_{t,T_i}} \right] = E_t^Q \left[\frac{1}{R_{0,T_i}} \right]$$

and this we know defines a martingale for each T_i by Lemma 13. ∎

It is important to note that we take Q as given. Another way of putting this is that a P-specification of the short rate (however well it may fit the data) is not enough to determine Q, bond prices and the Q-dynamics of the short rate. If you only have a short rate process, the only traded asset is the bank account and you cannot replicate bonds with that. Later courses will explain this in more detail.

Example 10 Here is a simple illustration of the procedure in a model where the spot rate follows a binomial process.

A ρ-tree with Q-probabilties   ZCB prices

The spot rate at time 0 is 0.10. At time 1 it becomes 0.11 with probability $\frac{1}{2}$ and 0.09 with probability $\frac{1}{2}$ (both probabilities under Q) Given that it is 0.09 at time 1, it becomes either 0.10 or 0.08 at time 2, both with probability $\frac{1}{2}$. The bond prices have been computed using Proposition 27. Note that a consequence of Proposition 27 is that (check it!)

$$P(t, T_i) = \frac{1}{1 + \rho_t} E_t^Q \left[P(t+1, T_i) \right]$$

and therefore the way to use the proposition is to construct bond prices working backwards through the tree. For a certain maturity T_i we know $P(T_i, T_i) = 1$ regardless of the state. Now the price of this bond at time $T_i - 1$ can be computed as a function of $\rho_{T_i - 1}$, and so forth. The term structure at time 0 is now computed as follows

$$r(0, 1) = \frac{1}{P(0, 1)} - 1 = 0.1$$

$$r(0, 2) = \left(\frac{1}{P(0, 2)} \right)^{\frac{1}{2}} - 1 = 0.09995$$

$$r(0, 3) = \left(\frac{1}{P(0, 3)} \right)^{\frac{1}{3}} - 1 = 0.0998$$

using definitions in Chapter 3. So the term structure in this example is decreasing in t - which is not what is normally seen in the market (but it does happen, for instance in Denmark in 1993 and in the U.S. in 2000). In fact, one calls the term structure "inverted" in this case. Note that when the Q-behavior of r has been specified we can determine not only the current term structure, we can find the term structure in any node of the tree. (Since the model only contains two non-trivial zero-coupon bonds at time 1, the term structure only has two points at time 1.)

So Example 10 shows how the term structure is calculated from a Q-tree of the short rate. But what we (or: practitioners) are really interested in is the reverse question: Given todays (observed) term structure, how do we construct a Q-tree of the short rate that is consistent with the term structure? (By consistent we mean that if we use the tree for ρ in Example 10-fashion we match the observed term structure at the first node.) Such a tree is needed for pricing more complicated contracts (options, for instance).

First, it is easy to see that generally such an "inversion" is in no way unique; a wide variety of ρ-trees give the same term structure. But that is not bad; it means that we impose a convenient structure on the ρ-process and still fit observed term structures. Two such conveniences are that the development of ρ can be represented in a recombining tree (a lattice), or in other words that ρ is Markovian, and that the Q-probability $1/2$ is attached to all branches. (It may not be totally clear that we can do that, but it is easily seen from the next example/subsection.)

8.2.1 Constructing a Q-tree for the short rate that fits the initial term structure

Imagine a situation where two things have been thrust upon us.

1. The almighty ("God "or "The Market") has determined todays term structure,
$$(P(0,1), P(0,2), \ldots, P(0,T)).$$

2. Our not-so-almighty boss has difficulties understanding probability beyond the tossing of a fair coin and wants answers fast, so he(s secretary) has drawn the ρ-lattice in Figure 8.1.

All we have to do is "fill in the blanks'. Optimistically we start, and in the box corresponding to $(t = 0, i = 0)$ we have no choice but to put

$$\rho_0(0) = \frac{1}{P(0,1)} - 1.$$

To fill out boxes corresponding to $(t = 1, i = 0)$ and $(t = 1, i = 1)$ we have the equation

$$P(0,2) = \frac{1}{\rho_0(0)} \left(\frac{1}{2} \times \frac{1}{1 + \rho_1(0)} + \frac{1}{2} \times \frac{1}{1 + \rho_1(1)} \right), \qquad (8.1)$$

which of course has many solutions. (Even many sensible ones.) So we can/have to put more structure on the problem. Two very popular ways of
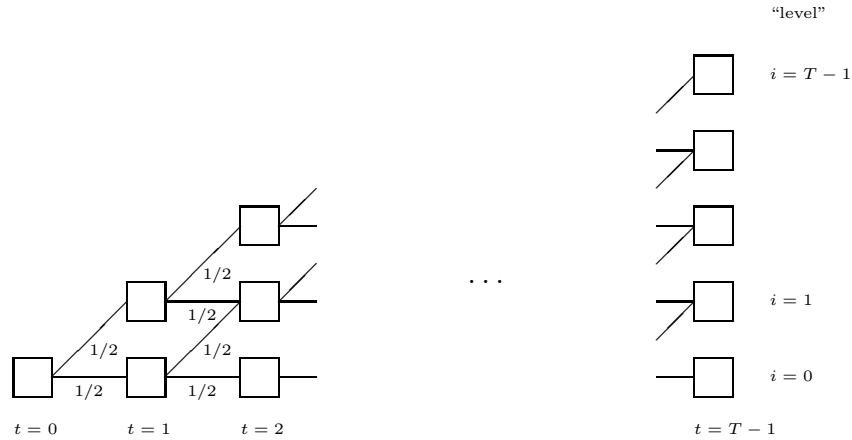
Figure 8.1: The ρ-lattice we must complete.

doing this are these functional forms: [1]

$$\text{Ho/Lee-specification:} \quad \rho_t(i) = a_{imp}(t) + b_{hist}i$$
$$\text{Black/Derman/Toy-specification:} \quad \rho_t(i) = a_{imp}(t)\exp(b_{hist}i)$$

For each t we fit by choosing an appropriate a_{imp}, while b_{hist} is considered a known constant. b_{hist} is called a volatility parameter and is closely related (as you should be able to see) to the conditional variance of the short rate (or its logarithm). This means that it is fairly easy to estimate from historical time series data of the short rate. With b_{hist} fixed, (8.1) can be solved hence determining what goes in the two "$t = 1$"-boxes. We may have to solve the equation determining $a_{imp}(1)$ numerically, but monotonicity makes this an easy task (by bisection or Newton-Raphson, for instance).

And now can can do the same for $t = 2, \ldots, T - 1$ and we can put our computer to work and go to lunch. Well, yes and no. Even though we take a long lunch there is a good chance that the computer is not finished when we get back. Why? Note that as it stands, every time we make a guess at $a_{imp}(t)$ (and since a numerical solution is involved we are likely to be making a number of these) we have to work our way backward trough the lattice all the way down to 0. And this we have to do for each t. While not a computational catastrophe (a small calculation shows that the computation time grows as T^3), it does not seem totally efficient. We would like to go through the lattice only once (as it was the case when the initial term structure was determined

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[1]Of course there is a reason for the names attached. As so often before, this is for later courses to explain.

from a known ρ-lattice). Fortunately there is a way of doing this. We need the following lemma.

Lemma 28 *Consider the binomial ρ-lattice in Figure 8.1. Let $\psi(t,i)$ be the price at time 0 of a security that pays 1 at time t if state/level i occurs at that time. Then $\psi(0,0) = 1$, $\psi(0,i) = 0$ for $i > 0$ and the following* forward equation *holds:*

$$
\psi(t+1,i) = \begin{cases} \frac{\psi(t,i)}{2(1+\rho_t(i))} + \frac{\psi(t,i-1)}{2(1+\rho_t(i-1))} & 0 < i < t+1, \\ \frac{\psi(t,i-1)}{2(1+\rho_t(i-1))} & i = t+1, \\ \frac{\psi(t,i)}{2(1+\rho_t(i))} & i = 0. \end{cases}
$$

Proof. We do the proof only for the "$0 < i < t+1$"-case, the others are similar. Recall that we can think of $\mathcal{F}_t$-measurable random variables (of the type considered here) as vectors in in $\mathbb{R}^{t+1}$. Since conditional expectation is linear, we can (for $s \leq t$) think of the $\mathcal{F}_s$-conditional expectation of an $\mathcal{F}_t$-measurable random variable as a linear mapping from $\mathbb{R}^{t+1}$ to $\mathbb{R}^{s+1}$. In other words it can be represented by a $(s+1) \times (t+1)$-matrix. In particular the time $t-1$ price of a contract with time t price X can be represented as

$$
E_t^Q \left(\frac{X}{1+\rho_{t-1}} \right) = \Pi_{t-1} X
$$

Now note that in the binomial model there are only two places to go from a given point, so the Π_{t-1}-matrices have the form

$$
\Pi_{t-1} = \left. \begin{bmatrix} \frac{1-q}{1+\rho_{t-1}(0)} & \frac{q}{1+\rho_{t-1}(0)} & & & 0 \\ & \frac{1-q}{1+\rho_{t-1}(1)} & \frac{q}{1+\rho_{t-1}(1)} & & \\ & & \ddots & \ddots & \\ 0 & & & \frac{1-q}{1+\rho_{t-1}(t-1)} & \frac{q}{1+\rho_{t-1}(t-1)} \end{bmatrix} \right\} t \text{ rows}
$$

$$
\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad}_{t+1 \text{ columns}}
$$

Let $e_i(t)$ be the i'th vector of the standard base in $\mathbb{R}^t$. The claim that pays 1 in state i at time $t+1$ can be represented in the lattice by $e_{i+1}(t+2)$ and by iterated expectations we have

$$
\psi(t+1,i) = \Pi_0 \Pi_1 \cdots \Pi_{t-1} \Pi_t e_{i+1}(t+2).
$$

But we know that multiplying a matrix by $e_i(t)$ from the right picks out the i'th column. For $0 < i < t+1$ we may write the $i+1$'st column of Π_t as (look at $i = 1$)

$$
\frac{1-q}{1+\rho_t(i-1)} e_i(t+1) + \frac{q}{1+\rho_t(i)} e_{i+1}(t+1).
$$

Hence we get

$$
\begin{aligned}
\psi(t+1, i) &= \Pi_0\Pi_1\cdots\Pi_{t-1}\left(\frac{1-q}{1+\rho_t(i-1)}e_i(t+1) + \frac{q}{1+\rho_t(i)}e_{i+1}(t+1)\right) \\
&= \frac{1-q}{1+\rho_t(i-1)}\underbrace{\Pi_0\Pi_1\cdots\Pi_{t-1}e_i(t+1)}_{\psi(t,i-1)} \\
&\quad + \frac{q}{1+\rho_t(i)}\underbrace{\Pi_0\Pi_1\cdots\Pi_{t-1}e_{i+1}(t+1)}_{\psi(t,i)},
\end{aligned}
$$

and since $q = 1/2$, this ends the proof. ■

Since $P(0,t) = \sum_{i=0}^{t}\psi(t,i)$, we can use the following algorithm to fit the initial term structure.

1. Let $\psi(0,0) = 1$ and put $t = 1$.

2. Let $\lambda_t(a_{imp}(t-1)) = \sum_{i=0}^{t}\psi(t,i)$ where $\psi(t,i)$ is calculated from the $\psi(t-1,\cdot)$'s using the specified $a_{imp}(t-1)$-value in the forward equation from Lemma 28.
   Solve $\lambda_t(a_{imp}(t-1)) = P(0,t)$ numerically for $a_{imp}(t-1)$.

3. Increase t by one. If $t \leq T$ then go to 2., otherwise stop.

An inspection reveals that the computation time of this procedure only grows as T^2, so we have "gained an order", which can be quite significant when T is large. And don't worry: There will be exercises to help you understand and implement this algorithm.

8.3 On the impossibility of flat shifts of flat term structures

Now let us demonstrate that in our term structure modelling framework it is *impossible to have only parallel shifts of a flat term structure*. In other words, in a model with no arbitrage we cannot have bond prices at time 0 given as

$$
P(0,t) = \frac{1}{(1+r)^t}
$$

for some $r \geq 0$, $t = 1,\ldots,T$ and

$$
P(1,t) = \frac{1}{(1+\widetilde{r})^{t-1}}, \; t = 2,\ldots,T,
$$

where $\widetilde{r}$ is a random variable (which takes on at least two different values with positive probability). To assign meaning to a "flat term structure" at time 1 we should have $T \geq 3$.

Now consider the zero-coupon bonds with maturity dates 2 and 3. If the term structure is flat at time 0 we have for some $r \geq 0$

$$P(0,2) = \frac{1}{(1+r)^2} \text{ and } P(0,3) = \frac{1}{(1+r)^3}$$

and if it remains flat at time 1, there exist a random variable $\widetilde{r}$ such that

$$P(1,2) = \frac{1}{1+\widetilde{r}} \text{ and } P(1,3) = \frac{1}{(1+\widetilde{r})^2}.$$

Furthermore, in an arbitrage-free model it will be the case that

$$\begin{aligned} P(0,2) &= \frac{1}{1+r} E^Q\left[P(1,2)\right] \\ &= \frac{1}{1+r} E^Q\left[\frac{1}{1+\widetilde{r}}\right] \end{aligned}$$

and

$$\begin{aligned} P(0,3) &= \frac{1}{1+r} E^Q\left[P(1,3)\right] \\ &= \frac{1}{1+r} E^Q\left[\frac{1}{(1+\widetilde{r})^2}\right] \end{aligned}$$

Combining these results, we have

$$\frac{1}{1+r} = E^Q\left[\frac{1}{(1+\widetilde{r})}\right]$$

and

$$\frac{1}{(1+r)^2} = E^Q\left[\frac{1}{(1+\widetilde{r})^2}\right]$$

which contradicts Jensen's inequality, for if

$$\frac{1}{1+r} = E^Q\left[\frac{1}{(1+\widetilde{r})}\right]$$

then since $u \longmapsto u^2$ is strictly convex and $\widetilde{r}$ not constant we must have

$$\frac{1}{(1+r)^2} < E^Q\left[\frac{1}{(1+\widetilde{r})^2}\right].$$

Note that the result does not say that it is impossible for the term structure to be flat. But it is inconsistent with no arbitrage to have a flat term structure and *only* have the possibility of moves to other flat term structures.

This explains what goes "wrong" in the example in Section 3.5.3. There the term structure was flat. We then created a position that had a value of 0 at that level of interest rates, but a strictly positive value with at flat term structure at any other level. But if interest rates are really stochastic then an arbitrage-free model cannot have only flat shifts of flat structure.

8.4   On forwards and futures

A forward and a futures contract are very similar contracts: The buyer (seller) of either type of contract is obligated to buy (sell) a certain asset at some specified date in the future for a price - the delivery price - agreed upon today. The forward/futures price of a certain asset is the delivery price which makes the forward/futures contract have zero value initially. It is very important to see that a forward/futures price is closer in spirit to the exercise price of an option than to the price of an option contract. Whereas an option always has positive value (and usually strictly positive) initially, both futures and forwards have zero value initially because the delivery price is used as a balancing tool.

The following example might clarify this: If a stock trades at \$100 today and we were to consider buying a futures contract on the stock with delivery in three months and if we had an idea that this stock would not move a lot over the next three months, then we would be happy to pay something for a contract which obligated us to buy the stock in three months for, say, \$50. Even though things could go wrong and the stock fall below \$50 in three months we consider that a much smaller risk of loss than the chance of gaining a lot from the contract. Similarly, we would not obligate ourselves to buying the stock in three months for, say, \$150 without receiving some money now. Somewhere in between \$50 and \$150 is a delivery price at which we would neither pay nor insist on receiving money to enter into the contract.

In a market with many potential buyers and sellers there is an equilibrium price at which supply meets demand: The number of contracts with that delivery price offered at zero initial cost equals the number of contracts demanded. This equilibrium price is the forward/futures price (depending on which contract we consider). In the following we will look at this definition in a more mathematical way and we will explain in what sense futures and forwards are different. Although they produce different cash flows (see below) that only results in a price difference when interest rates are stochas-

tic. Therefore, we will illustrate this difference with an example involving futures/forwards on bonds. We will ignore margin payments (i.e. payments that one or both sides of the contract have to make initially to guarantee future payments) in this presentation.

First, let us look at the key difference between forwards and futures by illustrating the cash flows involved in both types of contracts: Let F_t denote the forward price at time t for delivery of an underlying asset at time T and let Φ_t denote the futures price of the same asset for delivery at T, where $t \leq T$. Strictly speaking, we should write $F_{t,T}$ and $\Phi_{t,T}$ instead of F_t and Φ_t respectively, since it is important to keep track of both the date at which the contract is entered into and the delivery date. But we have chosen to consider the particular delivery date T and then keep track of how the futures and forward prices change as a function of t. The cash flows produced by the two types of contracts, if bought at time t, are as follows:

	t	t+1	t+2	$\cdots$	T-1	T
Forward	0	0	0	$\cdots$	0	$S_T - F_t$
Futures	0	$\Phi_{t+1} - \Phi_t$	$\Phi_{t+2} - \Phi_{t+1}$	$\cdots$	$\Phi_{T-1} - \Phi_{T-2}$	$S_T - \Phi_{T-1}$

where S_T is the price of the underlying asset at time T. The forward cash flow is self-explanatory. The futures cash flow can be explained as follows: If you buy a futures contract at date t you agree to buy the underlying asset at time T for Φ_t. At time $t + 1$ markets may have changed and the price at which futures trade changed to Φ_{t+1}. What happens is now a *resettlement of the futures contract*. If Φ_{t+1} is bigger than Φ_t you (the buyer of the futures at time t) receive the amount $\Phi_{t+1} - \Phi_t$ from the seller at time t+1 whereas you pay the difference between Φ_{t+1} and Φ_t to the seller if $\Phi_{t+1} < \Phi_t$. The story continues as shown in the figure.

We have already seen that if the underlying asset trades at time t and a zero coupon bond with maturity T also trades then the forward price is given as

$$F_t = \frac{S_t}{P(t,T)}$$

i.e.

$$F_t = S_t \left(1 + r\left(t,T\right)\right)^{T-t} \tag{8.2}$$

where $r(t,T)$ is the internal rate of return on the zero coupon bond.

To see what Φ_t is requires a little more work: First of all to avoid arbitrage we must have $\Phi_T = S_T$. Now consider Φ_{T-1}. In an arbitrage free system there exists an equivalent martingale measure Q. The futures price Φ_{T-1} is such

that the cash flow promised by the contract (bought at $T - 1$) has value 0. We must therefore have

$$0 = E^Q_{T-1} \left[\frac{S_T - \Phi_{T-1}}{R_{T-1,T}} \right]$$

but since $R_{T-1,T}$ is $\mathcal{F}_{T-1}$-measurable this implies

$$0 = \frac{1}{R_{T-1,T}} E^Q_{T-1} [S_T - \Phi_{T-1}]$$

i.e.

$$\Phi_{T-1} = E^Q_{T-1} [S_T] \tag{8.3}$$

Since Q is a martingale measure recall that

$$\frac{S_{T-1}}{R_{0,T-1}} = E^Q_{T-1} \left[\frac{S_T}{R_{0,T}} \right]$$

i.e.

$$S_{T-1} = \frac{1}{1 + \rho_{T-1}} E^Q_{T-1} [S_T]$$

hence we can write (8.3) as

$$\Phi_{T-1} = \left(1 + \rho_{T-1} \right) S_{T-1}$$

and that is the same as (8.2) since the yield on a one period zero coupon bond is precisely the spot rate. So we note that with one time period remaining we have $\Phi_{T-1} = F_{T-1}$. But that also follows trivially since with one period remaining the difference in cash flows between forwards and futures does not have time to materialize.

Now consider Φ_{T-2}. By definition Φ_{T-2} should be set such that the cash flow of the futures contract signed at $T - 2$ has zero value:

$$0 = E^Q_{T-2} \left[\frac{\Phi_{T-1} - \Phi_{T-2}}{R_{T-2,T-1}} + \frac{S_T - \Phi_{T-1}}{R_{T-2,T}} \right] \tag{8.4}$$

Now note that using the rule of iterated expectations and the expression for Φ_{T-1} we find

$$E^Q_{T-2} \left[\frac{S_T - \Phi_{T-1}}{R_{T-2,T}} \right]$$
$$= \frac{1}{R_{T-2,T-1}} E^Q_{T-2} \left[E^Q_{T-1} \left[\frac{S_T - \Phi_{T-1}}{R_{T-1,T}} \right] \right]$$
$$= 0$$

so (8.4) holds precisely when

$$
\begin{aligned}
0 &= E^Q_{T-2}\left[\frac{\Phi_{T-1}-\Phi_{T-2}}{R_{T-2,T-1}}\right]\\
&= \frac{1}{R_{T-2,T-1}}E^Q_{T-2}\left[\Phi_{T-1}-\Phi_{T-2}\right]
\end{aligned}
$$

i.e. we have

$$
\Phi_{T-2}=E^Q_{T-2}\left[\Phi_{T-1}\right]=E^Q_{T-2}\left[S_T\right].
$$

This argument can be continued backwards and we arrive at the expression

$$
\Phi_t = E^Q_t\left[S_T\right] \tag{8.5}
$$

Note that (8.5) is not in general equal to (8.2):

Under Q, we have $S_t = E^Q_t\left[\frac{S_T}{R_{t,T}}\right]$ so if $\frac{1}{R_{t,T}}$ and S_T are uncorrelated under Q we may write

$$
S_t = E^Q_t\left[\frac{1}{R_{t,T}}\right]E^Q_t\left[S_T\right]=P(t,T)\Phi_t
$$

which would imply that

$$
\Phi_t = \frac{S_t}{P(t,T)} = F_t
$$

Hence, *if $\frac{1}{R_{t,T}}$ and S_T are uncorrelated under Q, the forward price F_t and the futures price Φ_t are the same.* A special case of this is *when interest rates are deterministic*, i.e. all future spot rates and hence $R_{t,T}$ are known at time t.

Note that in general,

$$
\begin{aligned}
\Phi_t - F_t &= \frac{1}{P(t,T)}\left(P(t,T)E^Q_t\left[S_T\right]-S_t\right)\\
&= \frac{1}{P(t,T)}\left(E^Q_t\left[\frac{1}{R_{t,T}}\right]E^Q_t\left[S_T\right]-S_t\right)\\
&= \frac{1}{P(t,T)}\left(E^Q_t\left(\frac{S_T}{R_{t,T}}\right)-Cov^Q_t\left(\frac{1}{R_{t,T}},S_T\right)-S_t\right)\\
&= \frac{-1}{P(t,T)}\left(Cov^Q_t\left(\frac{1}{R_{t,T}},S_T\right)\right).
\end{aligned}
$$

Note that margin payments go to the holder of a futures contract when spot prices rise, i.e. in states where S_T is high. If $\frac{1}{R(t,T)}$ is negatively correlated with S_T, then interest rates tend to be high when the spot price is high and hence the holder of a futures contract will receive cash when interest rates are high. Hence a futures contract is more valuable in that case and the futures price should therefore be set higher to keep the contract value at 0.

8.5 On swap contracts

A swap contract is an agreement to exchange one stream of payments for another. A wide variety of swaps exists in financial markets; they are often tailor-made to the specific need of a company/an investor and can be highly complex. However, we consider only the valuation of the simplest[2] interest rate swap where fixed interest payments are exchanged for floating rate interest payments.

This swap you may see referred to as anything from "basis" to "forward starting ???monthly payer swap settled in arrears". Fortunately the payments are easier to describe. For a set of equidistant dates $(T_i)_{i=0}^n$, say δ apart, it is a contract with cash flow (per unit of notational principal)

$$\left(\underbrace{\frac{1}{P(T_{i-1},T_i)} - 1}_{\text{floating leg}} - \underbrace{\delta\kappa}_{\text{fixed leg}} \right) \quad \text{at date } T_i \quad \text{for } i = 1, \ldots, n,$$

where κ is a constant (an interest rate with δ-compounding quoted on yearly basis.) You should convince yourself why the so-called floating leg does in fact correspond to receiving floating interest rate payments. The term $(1/P(T_{i-1},T_i) - 1)/\delta$ is often called the $(12^*\delta)$-month LIBOR (which an acronym for London Interbank Offer Rate, and does not really mean anything nowadays, it is just easy to pronounce). Note that the payment made at T_i is known at T_{i-1}.

It is clear that since the payments in the fixed leg are deterministic, they have a value of

$$\delta\kappa \sum_{i=1}^n P(t,T_i).$$

The payments in the floating leg are not deterministic. But despite this, we can find their value without a stochastic model for bond prices/interest rates. Consider the following simple portfolio strategy:

Time	Action	Net cash flow
t	Sell 1 T_i-ZCB	
	Buy 1 T_{i-1}-ZCB	$P(t,T_i) - P(t,T_{i-1})$
T_{i-1}	Use principal received from T_{i-1}-ZCB	
	to buy $1/P(T_{i-1},T_i)$ T_i-ZCBs	0
T_i	Close position	$1/P(T_{i-1},T_i) - 1$

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[2]Simple *objects* are often referred to as plain vanilla *objects*. But what is seen as simple depends very much on who is looking.

This means that the T_i-payment in the floating leg has a value of $P(t, T_{i-1}) - P(t, T_i)$, so when summing over i see that the value of the floating leg is

$$P(t, T_0) - P(t, T_n).$$

In the case where $t = T_0$ this is easy to remember/interpret. A bullet-like bond that has a principal of 1 pays a coupon that is the short rate must have a price of 1 (lingo: "it is trading at par"). The only difference between this contract and the floating leg is the payment of the principal at time T_n; the time t value of this is $P(t, T_n)$ hence the value of the floating leg is $1 - P(t, T_n)$.

All in all the swap has a value of

$$V = P(t, T_0) - P(t, T_n) - \delta\kappa \sum_{i=1}^{n} P(t, T_i)).$$

But there is a further twist; these basis swaps are only traded with one κ (for each length; each n), namely the one that makes the value 0. This rate is called the swap rate (at a given date for a given maturity)

$$\kappa_n(t) = \frac{P(t, T_0) - P(t, T_n)}{\delta \sum_{i=1}^{n} P(t, T_i)}. \tag{8.6}$$

In practice (8.6) is often used "backwards", meaning that swap rates for swaps of different lengths (called the "swap curve") are used to infer discount factors/the term structure. Note that this is easy to do recursively if we can "get started", which is clearly the case if $t = T_0$.[3]

The main point is that the basis swap can be priced without using a full dynamic model, we only need today's term structure. But it takes only minor changes in the contract specification for this conclusion to break down. For instance different dynamic models with same current term structure give different swap values if the ith payment in the basis swap is transferred to date T_{i-1} (where it is first known; this is called settlement in advance) or if we swap every 3 months against the 6-month LIBOR.

The need for a swap-market can also be motivated by the following example showing swaps can offer comparative advantages. In its swap-formulation it is very inspired by Hull's book, but you you should recognize the idea from introductory economics courses (or David Ricardo's work of 1817, whichever came first). Consider two firms, A and B, each of which wants to borrow

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[3]There should be a "don't try this at work" disclaimer here. In the market different day count conventions are often used on the two swap legs, so things may not be quite what they seem.

$10M for 5 years. Firm A prefers to pay a floating rate, say one that is adjusted every year. It could be that the cash-flows generated by the investment (that it presumably needs the $10M for) depend (positively) on the interest rate market conditions. So from their point of view a floating rate loan removes risk. Firm B prefers to borrow at a fixed rate. In this way is knows in advance exactly how much it has to pay over the 5 years, which it is quite concievable that someone would want. The firms contact their banks and recieve the following loan offers: (Lingo: "bp" means basispoints (pronounced "beeps" if you're really cool) and is one hundredth of a percentage point, i.e. "100bp = 1%")

Firm	Fixed	Floating
A	5Y-ZCB-rate + 50bp	1Y-ZCB-rate + 30bp
B	5Y-ZCB-rate + 170bp	1Y-ZCB-rate + 100bp

So B gets a systematically "worse deal" than A, which could be because is of lower credit quality than A. But "less worse" for a floating rate loan, where they only have to pay 70bp more than A compared to 120bp for a fixed rate loan. So A could take the floating rate offer and B the fixed rate offer, and everybody is mildly happy. But consider the following arrangement: A takes the fixed rate offer from the bank and B the floating rate. A then offers to lend B the 10M as a fixed rate loan "at the 5Y-ZCB-rate + 45bp", whereas B offers to lend A its 10M floating rate loan "at the 1Y-ZCB-rate" (and would maybe add "flat" to indicate that there *is* no spread). In other words A and B are exchanging, or swapping, their bank loans. The result:

A: Pays (5Y-ZCB-rate + 50bp) (to bank), Pays 1Y-ZCB-rate (to B) and receives (5Y-ZCB-rate + 45bp) (from B). In net-terms: Pays 1Y-ZCB-rate+5bp

B: Pays (1Y-ZCB-rate + 100bp) (to bank), Pays (5Y-ZCB-rate + 45bp) (to A) and receives (1Y-ZCB-rate) (from A). In net-terms: Pays 5Y-ZCB-rate+145bp

So this swap-arrangement has put both A and B in a better position (by 25bp) than they would have been had they only used the bank.

But when used in the finance/interest rate context, there is somewhat of a snag in this story. We argued that the loans offered reflected differences in credit quality. If that is so, then it must mean that default ("going broke") is a possibility that cannot be ignored. It is this risk that the bank is "charging extra" for. With this point of view the reason why the firms get better deals after swapping is that each chooses to take on the credit risk from the other party. If firm B defaults, firm A can forget about (at least part of) what's in the "receives from B"-column, but will (certainly with this construction)

only be able to get out of its obligations to B to a much lesser extent. So the firms are getting lower rates by taking on default risk, which a risk of the type "a large loss with a small probability". One can quite sensibly ask if that is the kind of risks that individual firms want to take.

One could try to remedy the problem by saying that we set up a financial institution through which the swapping takes place. This institution should ensure payments to the non-defaulting party (hence taking "credit risk" $\times$ 2), in return for a share of the possible "lower rate"-gain from the swap, and hope for some "law of large numbers"-diversification effect. But that story is questionable; isn't that what the bank is doing in the first place?

So the morale is two-fold: *i*) If something seems to be too good to be true it usually is. Also in credit risk models. *ii*) The only way to see if the spreads offered to firms A and B are set such that there is no gain without extra risk, i.e. consistent with no arbitrage, is to set up a real dynamic stochastic model of the defaults (something that subsequent courses will do), just as stochastic term structure models help us realize that non-flat yield curves do not imply arbitrage.

8.6   On expectation hypotheses

Recall that the spot rate in our term structure models is a stochastic process. At time 0 we do not know what the spot rate will be at time 1. We may however from current bond prices compute the one-period forward rate $f(1,2)$ and it is natural to think that this rate at least carries some information about the level of the spot rate at time 1. For example, one type of *expectation hypothesis* would argue, that the expected value of spot rates is equal to the corresponding forward rates. As we shall see shortly, there is little reason to think that this is satisfied in arbitrage-free models. There are a number of other *expectation hypotheses* that one can formulate concerning future levels of interest rates, bond prices, yields and forward rates. Although we will no go through all of these in great detail, one point should be clear after this: There is essentially only one expectation hypothesis which follows as a simple consequence of no arbitrage (and an assumption of risk neutral agents). Many other form of expectation hypothesis have little mathematical justification, often there are inconsistent with each other, and very often the same form of the expectation hypotheses cannot hold for different maturities.

But let us begin with the good news. We know that in an arbitrage-free model, we have for any zero coupon bond with maturity T_i that

$$P(t, T_i) = \frac{1}{1 + \rho_t} E_t^Q \left[P(t+1, T_i) \right].$$

Hence standing at time t, the expected return under Q of holding a bond in one period is

$$\frac{E_t^Q [P(t+1,T_i)]}{P(t,T_i)} - 1 = \rho_t$$

and this does not depend on the maturity of the bond. Hence under Q, the one-period return on all bonds is the same. This is a mathematical consequence of no arbitrage. It becomes a hypothesis, which we may call the *local expectations hypothesis,* once we claim that this also holds under the measure P which governs the evolution of interest rates in the real world. This would of course be true if $P = Q$, something which only holds in an economy in which all agents are risk-neutral.

Let us assume that $P = Q$ and consider an extension (called the "return to maturity hypothesis") of this local hypothesis to n periods which equates the expected return from rolling over the money market account in n periods with that of holding an $n-$period bond. This would be equivalent to stating that

$$E_t^Q (R_{t,t+n}) = (1 + y(t,t+n))^n$$

where $y(t,t+n)$ is the yield at time t of a bond maturing at time $t+n$. What if we claim that this holds for all n? Then Jensen's inequality brings us into trouble since from our fundamental pricing relationship we have

$$
\begin{aligned}
P(t,t+n) &= \\
\frac{1}{(1 + y(t,t+n))^n} &= E_t^Q \left[\frac{1}{R_{t,t+n}} \right]
\end{aligned}
$$

and unless interest rates are deterministic we have

$$E_t^Q \left[\frac{1}{R_{t,t+n}} \right] > \frac{1}{E_t^Q R_{t,t+n}}.$$

Finally let us consider another popular hypothesis about the term structure of interest rates, which states that forward rates are unbiased predictors of spot rates. Our discussion of this hypothesis will be much clearer if we have at our disposal the concept of forward measures.

Proposition 29 *Given a term structure model with Q as the martingale measure. Define the random variable Z_T^T as*

$$Z_T^T = \frac{1}{R_{0,T+1} P(0,T+1)}.$$

Then a new probability measure Q^T is defined by letting

$$Q^T(A) = E^Q\left(Z_T^T 1_A\right), \ A \in \mathcal{F}. \tag{8.7}$$

Under this measure, the forward rate process $(f(t,T))_{t=0,\ldots,T}$ is a martingale.
Proof. First note that since $\rho > -1$, $Z_T^T > 0$. Also,

$$E^Q\left(Z_T^T\right) = \frac{1}{P(0, T+1)} E^Q \frac{1}{R_{0,T+1}} = 1$$

and therefore (8.7) defines a new probability measure on Ω. Let

$$
\begin{aligned}
Z_t^T &= E_t^Q\left(Z_T^T\right) \\
&= \frac{1}{R(0,t)P(0,T+1)} E_t^Q\left(\frac{1}{R_{t,T+1}}\right) \\
&= \frac{P(t,T+1)}{R(0,t)P(0,T+1)}.
\end{aligned}
$$

Now note that

$$
\begin{aligned}
E_t^Q\left(\frac{\rho_T}{R_{t,T+1}}\right) &= P(0,T+1)R_{0,t}E_t^Q\left(\frac{\rho_T}{R_{0,T+1}P(0,T+1)}\right) \\
&= \frac{P(t,T+1)}{Z_t^T} E_t^Q\left(Z_T^T \rho_T\right) \\
&= P(t,T+1)E_t^{Q^T}\left(\rho_T\right).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
E_t^{Q^T}\left(\rho_T\right) &= \frac{1}{P(t,T+1)} E_t^Q\left(\frac{\rho_T}{R_{t,T+1}}\right) \\
&= \frac{1}{P(t,T+1)} E_t^Q\left(\frac{1+\rho_T}{R_{t,T+1}} - \frac{1}{R_{t,T+1}}\right) \\
&= \frac{1}{P(t,T+1)} E_t^Q\left(\frac{1}{R_{t,T}} - \frac{1}{R_{t,T+1}}\right) \\
&= \frac{P(t,T)}{P(t,T+1)} - 1 \\
&= f(t,T).
\end{aligned}
$$

This proves the martingale property. ■
    This proposition shows that there exists a measure (and this measure is called the $T-$forward measure) under which the expected spot rate at

time T is equal to the forward rate. Typically, the forward measure is not equal to P, and it is not equal to Q unless interest rates are deterministic. Furthermore, one may check that for $f(t, T)$ and $f(t, T + 1)$ to be unbiased estimators of r_T and r_{T+1}, respectively, the spot interest rate at time T must be deterministic. The moral of all this, is that viewing the forward rate as unbiased estimators of future spot rates is problematic.

8.7 Why $P = Q$ means risk neutrality

In this section we will keep referring to the measure P which is the measure determining the actual evolution of prices. To make sure that the meaning of P is clear, we can say that a statistician estimating parameters of prices is trying to find P. We have seen that the version of the expectations hypothesis known as the local expectations hypothesis holds under the martingale measure Q used for pricing. Recall that the measure Q is a measure which allows us to give convenient expressions for prices of claims and derivative securities but not a measure governing the actual movement of prices.

We have stated earlier somewhat loosely that P and Q are actually the same when agents are risk neutral. Since we have not seen many agents this statement needs some elaboration. A quick sketch of this line of reasoning is the following: Recall that under Q all securities have the same one period returns: They are equal to the short rate. If $Q = P$ it would be the case that *actual* expected returns were the same for all assets, regardless of their variances. This would only be possible in a world where agents are risk neutral and therefore do not care about risk (variance, say) but look only at expected returns and prefer more expected return to less. In fact, if there is as much as one risk neutral agent in the economy and two assets have different expected returns, then this one agent would ruin the equilibrium by demanding infinitely much of the asset with the high expected return and financing the purchase by selling the asset with low expected return in infinite quantities. Therefore, we may say that $Q = P$ follows from risk neutrality of at least one agent. The argument can be made more precise by explicitly modelling the inter-temporal optimization problem of a representative agent who maximizes an additively separable expected utility of consumption over a certain time period. When this is done we can interpret the pricing relation

$$P(t, T_i) = \frac{1}{1 + \rho_t} E_t^Q \left[P(t + 1, T_i) \right]$$

in terms of marginal utilities. The key result is that in equilibrium the prices of bonds adjust in such a way that the increase in marginal utility for the

agent obtained by selling the bond at date t and using the proceeds for consumption is exactly equal to the marginal loss of expected utility at date $t+1$ resulting from the smaller amount of money for consumption available by selling the position in that bond at time $t+1$. Let us consider a one-period case. If we denote by C_0 (known at time 0) and C_1 (stochastic viewed from time 0) the optimal consumption of the agent at dates 0 and 1, it will be the case in equilibrium that the price of the $i'th$ asset satisfies

$$P^i(0)u'(C_0) = E_0^P \left[P^i(1)u'(C_1) \right]$$

i.e.

$$
\begin{aligned}
P^i(0) &= E_0^P \left[\frac{P^i(1)u'(C_1)}{u'(C_0)} \right] \\
&= E_t^P \left[\frac{P^i(1)Z_1}{1+\rho_0} \right]
\end{aligned}
$$

where

$$
\begin{aligned}
Z_1 &= \frac{u'(C_1)}{E_0^P u'(C_1)} \\
1 + \rho_0 &= \frac{u'(C_0)}{E_t^P u'(C_1)}
\end{aligned}
$$

and this we may then write as

$$P^i(0) = E_t^Q \left[\frac{P^i(1)}{1+\rho_0} \right]$$

where Q is defined by

$$Q(A) = E^P(1_A Z_1).$$

This establishes the connection between utility maximization and the equivalent martingale measure. An agent who is risk neutral will have an affine utility function, and hence for such an agent $u'(C_1)$ is constant (i.e. does not vary with ω as C_1 does). In that case $Z_1 = 1$ and $P = Q$.

It is clear that $P = Q$ is sufficient for the local expectation hypothesis to hold but it may seem to be too strong a requirement. After all, it is only an expectation of one random variable that we are referring to and one could imagine that a measure change would not alter this particular expectation. To analyze this question a little further, consider the fundamental definition of a new measure through the random variable Z_1 :

$$Q(A) = E^P(1_A Z_1).$$

For some random variable $X,$ which could be the spot rate at some future date, we have

$$E^Q(X) = E^P(XZ)$$

and therefore $E^Q(X) = E^P(X)$ if and only if

$$E^P(X(Z - 1)) = 0.$$

Since $E(Z - 1) = 0$ this is the same as requiring

$$Cov(X, Z) = 0.$$

Therefore, for the change of measure to preserve a mean value we must have that the variable in question is uncorrelated with the change of measure variable $Z,$ and this will typically not hold in the term structure models we consider.