

Chapter 7

The Black-Scholes formula

7.1 Black-Scholes as a limit of binomial models

So far we have not specified the parameters p , u , d and R which are of course critical for the option pricing model. Also, it seems reasonable that if we want the binomial model to be a realistic model for stock prices over a certain interval of time we should use a binomial model which divides the (calendar) time interval into many sub-periods. In this chapter we will first show that if one divides the interval into finer and finer periods and choose the parameters carefully, the value of the option converges to a limiting formula, the Black-Scholes formula, which was originally derived in a continuous time framework. We then describe that framework and show how to derive the formula in it.

Our starting point is an observed stock price whose logarithmic return satisfies

$$E^P \left[\ln \left(\frac{S_t}{S_{t-1}} \right) \right] = \mu$$

and

$$V^P \left(\ln \left(\frac{S_t}{S_{t-1}} \right) \right) = \sigma^2,$$

where S_t is the price of the stock t years after the starting date 0. Also, assume that the money market account has a continuously compounded return of r , i.e. an amount of 1 placed in the money market account grows to $\exp(r)$ in one year. Note that since $R^T = \exp(T \ln(R))$, a yearly rate of $R = 1.1$ (corresponding to a yearly rate of 10%) translates into the continuous compounding analogue $r = \ln(1.1)$ and this will be a number smaller than 0.1.

Consider pricing an option on this stock with time to maturity T years in a binomial model. Divide each year into n periods. This gives a binomial model with nT periods. In this tree, which we label the n th tree, choose

$$\begin{aligned} u_n &= \exp\left(\sigma\sqrt{\frac{1}{n}}\right), \\ d_n &= \exp\left(-\sigma\sqrt{\frac{1}{n}}\right) = \frac{1}{u_n}, \\ R_n &= \exp\left(\frac{r}{n}\right), \end{aligned}$$

and

$$p_n = \frac{1}{2} + \frac{1}{2} \frac{\mu}{\sigma} \sqrt{\frac{1}{n}}.$$

With the setup in the n th model specified above you may show by simple computation that the one-year logarithmic return satisfies

$$E^P \left[\ln \left(\frac{S_1}{S_0} \right) \right] = n \{ p_n \ln(u_n) + (1 - p_n) \ln(d_n) \} = \mu$$

and

$$V^P \left(\ln \left(\frac{S_1}{S_0} \right) \right) = \sigma^2 - \frac{1}{n} \mu^2,$$

so the log-return of the price process has the same mean and almost the same variance as the process we have observed. And since

$$V^P \left(\ln \left(\frac{S_1}{S_0} \right) \right) \rightarrow \sigma^2 \text{ for } n \rightarrow \infty,$$

it is presumably so that large values of n brings us closer to to “desired” model.

The above story was primarily motivational. Let us now investigate precisely what happens to stock and call prices when n tends to infinity. For each n we may compute the price of a call option with maturity T in the binomial model and we know that it is given as

$$C^n = S_0 \Psi(a_n; nT; q'_n) - \frac{K}{(R_n)^T} \Psi(a_n; nT; q_n) \quad (7.1)$$

where

$$q_n = \frac{R_n - d_n}{u_n - d_n}, \quad q'_n = \frac{u_n}{R_n} q_n$$

and a_n is the smallest integer larger than $\ln(K/(S_0 d_n^{Tn})) / \ln(u_n/d_n)$. Note that alternatively we may write (7.1) as

$$C^n = S_0 Q'(S_n(T) > K) - K e^{-rT} Q(S_n(T) > K) \quad (7.2)$$

where $S_n(T) = S_0 u_n^j d_n^{Tn-j}$ and $j \stackrel{Q}{\sim} \text{bi}(Tn, q_n)$ and $j \stackrel{Q'}{\sim} \text{bi}(Tn, q'_n)$. It is easy to see that

$$\begin{aligned} M_n^Q &:= E^Q(\ln S_n(T)) = \ln S_0 + Tn(q_n \ln u_n + (1 - q_n) \ln d_n) \\ V_n^Q &:= V^Q(\ln S_n(T)) = Tnq_n(1 - q_n)(\ln u_n - \ln d_n)^2, \end{aligned}$$

and that similar expressions (with q'_n instead of q_n) hold for Q' -moments. Now rewrite the expression for M_n^Q in the following way:

$$\begin{aligned} M_n^Q - \ln S_0 &= Tn \left(\frac{\sigma}{\sqrt{n}} \frac{e^{r/n} - e^{-\sigma/\sqrt{n}}}{e^{\sigma/\sqrt{n}} - e^{-\sigma/\sqrt{n}}} - \frac{\sigma}{\sqrt{n}} \frac{e^{\sigma/\sqrt{n}} - e^{r/n}}{e^{\sigma/\sqrt{n}} - e^{-\sigma/\sqrt{n}}} \right) \\ &= T\sqrt{n}\sigma \left(\frac{2e^{r/n} - e^{\sigma/\sqrt{n}} - e^{-\sigma/\sqrt{n}}}{e^{\sigma/\sqrt{n}} - e^{-\sigma/\sqrt{n}}} \right). \end{aligned}$$

Recall the Taylor-expansion to the second order for the exponential function: $\exp(\pm x) = 1 \pm x + x^2/2 + o(x^2)$. From this we get

$$\begin{aligned} e^{r/n} &= 1 + r/n + o(1/n) \\ e^{\pm\sigma/\sqrt{n}} &= 1 \pm \sigma/\sqrt{n} + \sigma^2/(2n) + o(1/n). \end{aligned}$$

Inserting this in the M_n^Q expression yields

$$\begin{aligned} M_n^Q - \ln S_0 &= T\sqrt{n}\sigma \left(\frac{2r/n - \sigma^2/n + o(1/n)}{2\sigma/\sqrt{n} + o(1/n)} \right) \\ &= T\sigma \left(\frac{2r - \sigma^2 + o(1)}{2\sigma + o(1/\sqrt{n})} \right) \\ &\rightarrow T \left(r - \frac{\sigma^2}{2} \right) \quad \text{for } n \rightarrow \infty. \end{aligned}$$

Similar Taylor expansions for V_n^Q , $M_n^{Q'}$ and $V_n^{Q'}$ show that

$$\begin{aligned} V_n^Q &\rightarrow \sigma^2 T, \\ M_n^{Q'} - \ln S_0 &\rightarrow T \left(r + \frac{\sigma^2}{2} \right) \quad (\text{note the change of sign on } \sigma^2), \\ V_n^{Q'} &\rightarrow \sigma^2 T. \end{aligned}$$

So now we know what the Q/Q' moments converge to. Yet another way to think of $\ln S_n(T)$ is as a sum of Tn independent Bernoulli-variables with possible outcomes $(\ln d_n, \ln u_n)$ and probability parameter q_n (or q'_n). This means that we have a sum of (well-behaved) independent random variables for which the first and second moments converge. Therefore we can use a version of the Central Limit Theorem¹ to conclude that the limit of the sum is normally distributed, i.e.

$$\ln S_n(T) \xrightarrow{Q/Q'} N(\ln S_0 + (r \pm \sigma^2/2)T, \sigma^2 T).$$

This means (almost by definition of the form of convergence implied by CLT) that when determining the limit of the probabilities on the right hand side of (7.2) we can (or: have to) substitute $\ln S_n(T)$ by a random variable X such that

$$X \stackrel{Q/Q'}{\sim} N(\ln S_0 + (r \pm \sigma^2/2)T, \sigma^2 T) \Leftrightarrow \frac{X - \ln S_0 - (r \pm \sigma^2/2)T}{\sigma\sqrt{T}} \stackrel{Q/Q'}{\sim} N(0, 1).$$

The final analysis:

$$\begin{aligned} \lim_{n \rightarrow \infty} C^n &= \lim_{n \rightarrow \infty} (S_0 Q'(\ln S_n(T) > \ln K) - K e^{-rT} Q(\ln S_n(T) > \ln K)) \\ &= S_0 Q'(X > \ln K) - K e^{-rT} Q(X > \ln K) \\ &= S_0 Q' \left(\frac{X - \ln S_0 - (r + \sigma^2/2)T}{\sigma\sqrt{T}} > \frac{\ln K - \ln S_0 - (r + \sigma^2/2)T}{\sigma\sqrt{T}} \right) \\ &\quad - K e^{-rT} Q \left(\frac{X - \ln S_0 - (r - \sigma^2/2)T}{\sigma\sqrt{T}} > \frac{\ln K - \ln S_0 - (r - \sigma^2/2)T}{\sigma\sqrt{T}} \right) \end{aligned}$$

Now multiply by -1 inside the Q 's (hence reversing the inequalities), use that the $N(0, 1)$ -variables on the left hand sides are symmetric and continuous, and that $\ln(x/y) = \ln x - \ln y$. This shows that

$$\lim_{n \rightarrow \infty} C^n = S_0 \Phi(d_1) - K e^{-rT} \Phi(d_2),$$

where Φ is the standard normal distribution function and

$$\begin{aligned} d_1 &= \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}, \\ d_2 &= \frac{\ln\left(\frac{S_0}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}. \end{aligned}$$

¹Actually you cannot quite make do with the De Moivre-version that you know from Stat 0 because we do not have a scaled sum of *identically* distributed random variables. You need the notion of a triangular array and the Lindeberg-Feller-version of the Central Limit Theorem. Yet another reason to take Stat 2b.

This formula for the call price is called the Black-Scholes formula. So far we can see it just as an artifact of going to the limit in a particular way in a binomial model. But the formula is so strikingly beautiful and simple that there must be more to it than that. In particular, we are interested in the question: Does that exist a “limiting” model in which the above formula is the exact call option price? The answer is: Yes. In the next section we describe what this “limiting” model looks like, and show that the Black-Scholes formula gives the exact call price in the model. That does involve a number of concepts, objects and results that we cannot possibly make rigorous in this course, but the reader should still get a “net benefit” and hopefully an appetite for future courses in financial mathematics.

7.2 The Black-Scholes model

The Black-Scholes formula for the price of a call option on a non-dividend paying stock is one of the most celebrated results in financial economics. In this chapter we will indicate how the formula is derived. A rigorous derivation requires some fairly advanced mathematics which is beyond the scope of this course. Fortunately, the formula is easy to interpret and to apply. Even if there are some technical details left over for a future course, the rigorous understanding we have from our discrete-time models of how arbitrage pricing works will allow us to apply the formula safely.

The formula is formulated in a continuous time framework with random variables that have continuous distribution. The continuous-time and infinite state space setup will not be used elsewhere in the course.² But let us mention that if one wants to develop a theory which allows random variables with continuous distribution and if one wants to obtain results similar to those of the previous chapters, then one has to allow continuous trading as well. By ‘continuous trading’ we mean that agents are allowed to readjust portfolios continuously through time.

If X is normally distributed $X \sim N(\alpha, \sigma^2)$, then we say that $Y := \exp(X)$ is *lognormally* distributed and write $Y \sim LN(\alpha, \sigma^2)$. There is one thing you must always remember about lognormal distributions:

$$\text{If } Y \sim LN(\alpha, \sigma^2) \text{ then } E(Y) = \exp\left(\alpha + \frac{\sigma^2}{2}\right).$$

²A setup which combines *discrete time* and continuous distributions will be encountered later when discussing CAPM and APT, but the primary focus of these models will be to explain stock price behavior and not – as we are now doing – determining option prices for a given behavior of stock prices

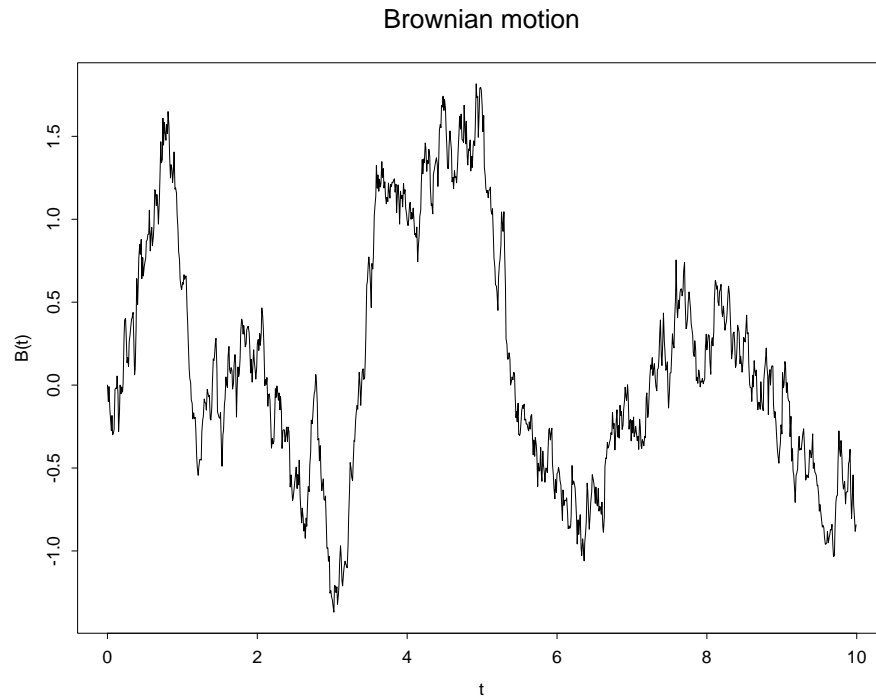
If you have not seen this before, then you are strongly urged to check it. (With that result you should also be able to see why there is no need to use “brain RAM” remembering the variance of a lognormally distributed variable.) Often the lognormal distribution is preferred as a model for stock price distributions since it conforms better with the institutional fact that prices of a stock are non-negative and the empirical observation that the logarithm of stock prices seem to show a better fit to a normal distribution than do prices themselves. However, specifying a distribution of the stock price at time t , say, is not enough. We need to specify the whole process of stock prices, i.e. we need to state what the joint distribution $(S_{t_1}, \dots, S_{t_N})$ is for any $0 \leq t_1 < \dots < t_N$. To do this the following object is central.

Definition 39 *A (standard) Brownian motion ((S)BM) is a stochastic process $B = (B_t)_{t \in [0; \infty[}$ -i.e. a sequence of random variables indexed by t such that:*

1. $B_0 = 0$
2. $B_t - B_s \sim N(0, t - s) \forall s < t$
3. B has independent increments, i.e. for every N and a set of N time points $t_1 < \dots < t_N$, $B_{t_1}, B_{t_2} - B_{t_1}, B_{t_3} - B_{t_2}, \dots, B_{t_N} - B_{t_{N-1}}$ are independent random variables.

That these demands on a process can be satisfied simultaneously is not trivial. But don't worry, Brownian motion does exist. It is, however, a fairly “wild” object. The sample paths (formally the mapping $t \mapsto B_t$ and intuitively simply the graph you get by plotting “temperature/stock price/...” against time) of BM are continuous everywhere but differentiable nowhere. The figure shows a simulated sample path of a BM and should give an indica-

tion of this.



A useful fact following from the independent increment property is that for any measurable $f : \mathbb{R} \rightarrow \mathbb{R}$ for which $E[|f(B_t - B_s)|] < \infty$ we have

$$E[f(B_t - B_s) | \mathcal{F}_s] = E[f(B_t - B_s)] \quad (7.3)$$

where $\mathcal{F}_s = \sigma\{B_u : 0 \leq u \leq s\}$.

The fundamental assumption of the Black-Scholes model is that the stock price can be represented by

$$S_t = S_0 \exp(\alpha t + \sigma B_t) \quad (7.4)$$

where B_t is a SBM. Such a process is called a geometric BM (with drift). Furthermore, it assumes that there exists a riskless asset (a money market account). One dollar invested in the money market account will grow as

$$\beta_t = \exp(rt) \quad (7.5)$$

where r is a constant (typically $r > 0$). Hence β_t is the continuous time analogue of $R_{0,t}$.

What does (7.4) mean? Note that since $B_t \sim N(0, t)$, S_t has a lognormal distribution and

$$\ln\left(\frac{S_{t_1}}{S_0}\right) = \alpha t_1 + \sigma B_{t_1},$$

$$\ln \left(\frac{S_{t_2}}{S_{t_1}} \right) = \alpha (t_2 - t_1) + \sigma (B_{t_2} - B_{t_1})$$

Since αt , $\alpha (t_2 - t_1)$, and σ are constant, we see that $\ln \left(\frac{S_{t_1}}{S_0} \right)$ and $\ln \left(\frac{S_{t_2}}{S_{t_1}} \right)$ are independent. The *return*, defined in this section as the logarithm of the price relative, that the stock earns between time t_1 and t_2 is independent of the return earned between time 0 and time t_1 , and both are normally distributed. We refer to σ as the *volatility* of the stock - but note that it really describes a property of the logarithmic return of the stock. There are several reasons for modelling the stock price as geometric BM with drift or equivalently all logarithmic returns as independent and normal. First of all, unless it is blatantly unreasonable, modelling “random objects” as “*n iid*” is *the* way to start. Empirically it is often a good approximation to model the logarithmic returns as being normal with fixed mean and fixed variance through time.³ From a probabilistic point of view, it can be shown that if we want a stock price process with continuous sample paths and we want returns to be independent and stationary (but not necessarily normal from the outset), then geometric BM is the only possibility. And last but not least: It gives rise to beautiful financial theory.

If you invest one dollar in the money market account at time 0, it will grow as $\beta_t = \exp(rt)$. Holding one dollar in the stock will give an uncertain amount at time t of $\exp(\alpha t + \sigma B_t)$ and this amount has an expected value of

$$E \exp(\alpha t + \sigma B_t) = \exp\left(\alpha t + \frac{1}{2}\sigma^2 t\right).$$

The quantity $\mu = \alpha + \frac{1}{2}\sigma^2$ is often referred to as the *drift* of the stock. We have not yet discussed (even in our discrete models) how agents determine μ and σ^2 , but for now think of it this way: Risk averse agents will demand μ to be greater than r to compensate for the uncertainty in the stock’s return. The higher σ^2 is, the higher should μ be.

7.3 A derivation of the Black-Scholes formula

In this section we derive the Black-Scholes model taking as given some facts from continuous time finance theory. The main assertion is that the fundamental theorem of asset pricing holds in continuous time and, in particular, in the Black-Scholes setup:

$$S_t = S_0 \exp(\alpha t + \sigma B_t)$$

³But skeptics would say many empirical analyses of financial data is a case of “believing is seeing” rather than the other way around.

$$\beta_t = \exp(rt)$$

What you are asked to believe in this section are the following facts:

- There is no arbitrage in the model and therefore there exists an *equivalent martingale measure* Q such that the discounted stock price $\frac{S_t}{\beta_t}$ is a martingale under Q . (Recall that this means that $E^Q \left[\frac{S_t}{\beta_t} \mid \mathcal{F}_s \right] = \frac{S_s}{\beta_s}$). The probabilistic behavior of S_t under Q is given by

$$S_t = S_0 \exp \left(\left(r - \frac{1}{2} \sigma^2 \right) t + \sigma \widetilde{B}_t \right), \quad (7.6)$$

where \widetilde{B}_t is a SBM under the measure Q .

- To compute the price of a call option on S with expiration date T and exercise price K , we take the discounted expected value of $C_T = [S_T - K]^+$ assuming the behavior of S_t given by (7.6).

Recall that in the binomial model we also found that the expected return of the stock under the martingale measure was equal to that of the riskless asset. (7.6) is the equivalent of this fact in the continuous time setup. Before sketching how this expectation is computed note that we have not defined the notion of arbitrage in continuous time. Also we have not justified the form of S_t under Q . But let us check at least that the martingale behavior of $\frac{S_t}{\beta_t}$ seems to be OK (this may explain the " $-\frac{1}{2}\sigma^2 t$ "-term which is in the expression for S_t). Note that

$$\begin{aligned} E^Q \left[\frac{S_t}{\beta_t} \right] &= E^Q \left[S_0 \exp \left(-\frac{1}{2} \sigma^2 t + \sigma \widetilde{B}_t \right) \right] \\ &= S_0 \exp \left(-\frac{1}{2} \sigma^2 t \right) E^Q \left[\exp \left(\sigma \widetilde{B}_t \right) \right]. \end{aligned}$$

But $\sigma \widetilde{B}_t \sim N(0, \sigma^2 t)$ and since we know how to compute the mean of the lognormal distribution we get that

$$E^Q \left[\frac{S_t}{\beta_t} \right] = S_0 = \frac{S_0}{\beta_0}, \text{ since } \beta_0 = 1.$$

By using the property (7.3) of the Brownian motion one can verify that

$$E^Q \left[\frac{S_t}{\beta_t} \mid \mathcal{F}_s \right] = \frac{S_s}{\beta_s}, \text{ } (\mathcal{F}_s = \text{"information at time s"}).$$

but we will not do that here.⁴

Accepting the fact that the call price at time 0 is

$$C_0 = \exp(-rT) E^Q \left[S_0 \exp \left(\left(r - \frac{1}{2} \sigma^2 \right) T + \sigma \tilde{B}_T \right) - K \right]^+$$

we can get the Black-Scholes formula: We know that $\sigma B_T \sim N(0, \sigma^2 T)$ and also “the rule of the unconscious statistician”, which tells us that to compute $E[f(X)]$ for some random variable X which has a density $p(x)$, we compute $\int f(x) p(x) dx$. This gives us

$$C_0 = e^{-rT} \int_{\mathbb{R}} \left[S_0 e^{(r-\sigma^2/2)T+x} - K \right]^+ \frac{1}{\sqrt{2\pi}\sigma\sqrt{T}} e^{-\frac{1}{2}\frac{x^2}{\sigma^2 T}} dx$$

The integrand is different from 0 when

$$S_0 e^{(r-\sigma^2/2)T+x} > K$$

i.e. when⁵

$$x > \ln(K/S_0) - (r - \sigma^2/2) T \equiv d$$

So

$$\begin{aligned} C_0 &= e^{-rT} \int_d^\infty \left(S_0 e^{(r-\frac{1}{2}\sigma^2)T+x} - K \right) \frac{1}{\sqrt{2\pi}\sigma\sqrt{T}} e^{-\frac{1}{2}\frac{x^2}{\sigma^2 T}} dx \\ &= e^{-rT} S_0 \underbrace{\int_d^\infty \frac{1}{\sqrt{2\pi}\sigma\sqrt{T}} e^{(r-\frac{1}{2}\sigma^2)T+x} e^{-\frac{1}{2}\frac{x^2}{\sigma^2 T}} dx}_{:=A} - K e^{-rT} \underbrace{\int_d^\infty \frac{1}{\sqrt{2\pi}\sigma\sqrt{T}} e^{-\frac{1}{2}\frac{x^2}{\sigma^2 T}} dx}_{:=B}. \end{aligned}$$

It is easy to see that $B = K e^{-rT} \text{Prob}(Z > d)$, where $Z \sim N(0, \sigma^2 T)$. So by using symmetry and scaling with $\sigma\sqrt{T}$ we get that

$$B = K e^{-rT} \Phi(d_2),$$

⁴If you want to try it yourself, use

$$\begin{aligned} E \left[\frac{S_t}{\beta_t} \middle| \mathcal{F}_s \right] &= E \left[\frac{S_t \beta_s S_s}{S_s \beta_t \beta_s} \middle| \mathcal{F}_s \right] \\ &= \frac{S_s}{\beta_s} E \left[\frac{S_t \beta_s}{S_s \beta_t} \middle| \mathcal{F}_s \right] \end{aligned}$$

and then see if you can bring (7.3) into play and use

$$E[\exp(\sigma(B_t - B_s))] = \exp\left(\frac{1}{2}\sigma^2(t-s)\right).$$

⁵This should bring up memories of the quantity a which we defined in the binomial model.

where (as before)

$$d_2 = -\frac{d}{\sigma\sqrt{T}} = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}.$$

So “we have half the Black-Scholes formula”. The A -term requires a little more work. First we use the change of variable $y = x/(\sigma\sqrt{T})$ to get (with a few rearrangements, a completion of the square, and a further change of variable ($z = y - \sigma\sqrt{T}$))

$$\begin{aligned} A &= S_0 e^{-T\sigma^2/2} \int_{-d_2}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\sigma\sqrt{T}y - y^2/2} dy \\ &= S_0 e^{-T\sigma^2/2} \int_{-d_2}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(y - \sigma\sqrt{T})^2/2 + T\sigma^2/2} dy \\ &= S_0 \int_{-d_1}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz, \end{aligned}$$

where as per usual $d_1 = d_2 + \sigma\sqrt{T}$. But the last integral we can write as $\text{Prob}(Z > d_1)$ for a random variable $Z \sim N(0, 1)$, and by symmetry we get

$$A = S_0 \Phi(d_1),$$

which yields the “promised” result.

Theorem 26 *The unique arbitrage-free price of a European call option on a non-dividend paying stock in the Black-Scholes framework is given by*

$$C_0 = S_0 \Phi(d_1) - K e^{-rT} \Phi(d_2)$$

where

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

and

$$d_2 = d_1 - \sigma\sqrt{T},$$

where Φ is the cumulative distribution function of a standard normal distribution.

As stated, the Black-Scholes formula says only what the call price is at time 0. But it is not hard to guess what happens if we want the price at some time $t \in [0; T]$: The same formula applies with S_0 substituted by S_t and T substituted by $T - t$. You may want to “try your hand” with conditional expectations and properties of BM by proving this.

7.3.1 Hedging the call

There is one last thing about the Black-Scholes model/formula you should know. Just as in the binomial model the call option can be hedged in the Black-Scholes model. This means that there exists a self-financing trading strategy involving the stock and the bond such that the value of the strategy at time T is exactly equal to the payoff of the call, $(S_T - K)^+$. (This is in fact the very reason we can talk about a *unique* arbitrage-free price for the call.) It is a general fact that if we have a contract whose price at time t can be written as

$$\pi(t) = F(t, S_t)$$

for some deterministic function F , then the contract is hedged by a strategy consisting of

$$\phi^1(t) = \left. \frac{\partial F}{\partial x}(t, x) \right|_{x=S_t}$$

units of the stock and $\phi^0(t) = \pi(t) - \phi^1(t)S_t$ \$ in the bank account. Note that this is a strategy that is continuously adjusted.

For the Black-Scholes model this applies to the call with

$$\begin{aligned} F^{BS\text{call}}(t, x) &= x\Phi\left(\frac{\ln\left(\frac{x}{K}\right) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right) \\ &\quad - Ke^{-r(T-t)}\Phi\left(\frac{\ln\left(\frac{x}{K}\right) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right). \end{aligned}$$

The remarkable result (and what you must forever remember) is that the partial derivative (wrt. x) of this lengthy expression is simple:⁶

$$\frac{\partial F^{BS\text{call}}}{\partial x}(t, x) = \Phi\left(\frac{\ln\left(\frac{x}{K}\right) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right) = \Phi(d_1),$$

where the last part is standard and understandable but slightly sloppy notation. So to hedge the call option in a Black-Scholes economy you have to hold (at any time t) $\Phi(d_1)$ units of the stock. This quantity is called *the delta* (or: Δ) *hedge ratio* for the call option. The “lingo” comes about because of the intimate relation to partial derivatives; Δ is approximately the amount that the call price changes, when the stock price changes by 1. In this course we will use computer simulations to illustrate, justify, and hopefully to some degree understand the result.

⁶At one time or another you are bound to be asked to verify this, so you may as well do it right away. Note that if you just look at the B-S formula, forget that S_0 (or x) also appears inside the Φ 's, and differentiate, then you get the right result with a wrong proof.