Chapter 6

Option pricing

The classical application of the arbitrage pricing machinery we have developed is to the pricing of options. The pricing models we obtain are used with minor modifications all over the world as the basis for trading billions of dollars worth of contracts every day. For students planning to become traders of financial derivatives this of course gives plenty of motivation for learning these models. But recent collapses of financial institutions have also reminded us that financial managers and executives must understand the way the derivatives markets work. A manager who understands the markets well may use them for effective risk management and will be able to implement effective control mechanisms within a firm to make sure that traders use the markets in accordance with the firm's overall objectives.

From a theoretical perspective, options are very important in several areas of finance. We will see later in the course how they are indispensable for our understanding of a firm's choice of capital structure. Also, a modern theory of capital budgeting relies critically on recognizing options involved in projects, so-called *real options*. And in actuarial science options appear when modelling reinsurance contracts.

6.1 Terminology

A *European (American) call option* on an *underlying security* S, with *strike price* K and *expiration date* T, gives the owner the right, but **not** the obligation, to buy S at a price of K at (up to and including) time T.

A *European (American) put option* on an *underlying security* S, with *strike price* K and *expiration date* T, gives the owner the right, but **not** the obligation, to sell S at a price of K at (up to and including) time T.

The strike price is also referred to as the *exercise price,* and using the

right to buy or sell is referred to as *exercising* the option.

There is no good reason for the American/European terminology - both types are traded in America and Europe.

In the definition above, we think of the person selling a call option (say), often referred to as the person *writing* an option, as actually delivering the underlying security to the option holder if the option holder decides to exercise. This is referred to as *physical delivery.* In reality, options are often *cash settled.* This means that instead of the option holder paying K to the writer of the call and the writer delivering the stock, the holder merely receives an amount $S_T - K$ from the option writer.

Some common examples of options are *stock options* in which the underlying security is a stock, *currency options* in which the underlying security is a foreign currency and where the strike price is to be thought of as an exchange rate, *bond options* which have bonds as underlying security and *index options* whose underlying security is not really a security but a stock market index (and where the contracts are then typically cash settled.) It will always be assumed that the underlying security has non-negative value.

6.2 Diagrams, strategies and put-call parity

Before we venture into constructing exact pricing models we develop some feel for how these instruments work. In this section we focus on what can be said about options if all we assume is that all securities (stocks, bonds, options) can be bought and sold in arbitrary quantities at the given prices with no transactions costs or taxes. This assumption we will refer to as an assumption of *frictionless* markets. We will also assume that at any time t and for any date $T > t$, there exists a zero coupon bond with maturity T in the market whose price at time t is $d(t, T)$.

An immediate consequence of our frictionless markets assumption is the following

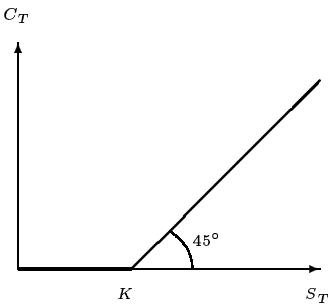
Proposition 20 *The value of an American or European call option at the expiration date T is equal to $C_T = \max(S_T - K, 0)$, where S_T is the price of the underlying security at time T. The value of an American or European put option at the expiration date T is equal to* $\max(K - S_T, 0)$.

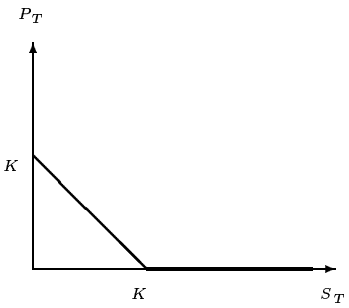
Proof. Consider the call option. If $S_T < K$, we must have $C_T = 0$, for if $C_T > 0$ you would sell the option, receive a positive cash flow, and there would be no exercise.[1] If $S_T \geq K$, we must have $C_T = S_T - K$. For if

[1] Actually, here we need to distinguish between whether the person who bought the

$C_T > S_T - K$ you would sell the option and buy the stock. After the option has been exercised, you are left with a total cash flow of $C_T - S_T + K > 0$, and you would have no future obligations arising from this trade. If $C_T < S_T - K$, buy the option, exercise it immediately, and sell the stock. The total cash flow is $-C_T + S_T - K > 0$, and again there would be no future obligations arising from this trade. The argument for the put option is similar.

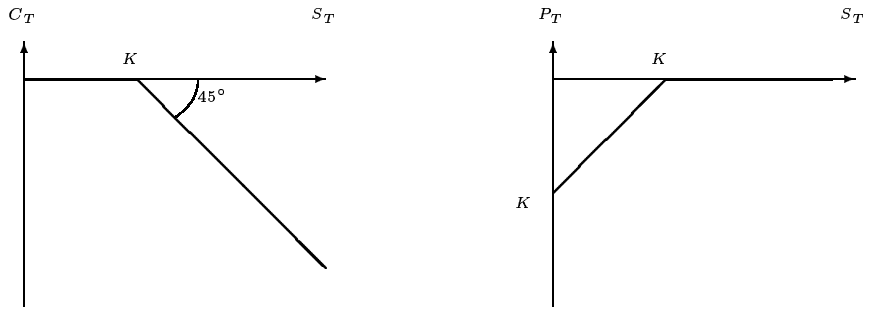
We often represent payoffs of options at an exercise date using payoff diagrams, which show the value of the option as a function of the value of the underlying:



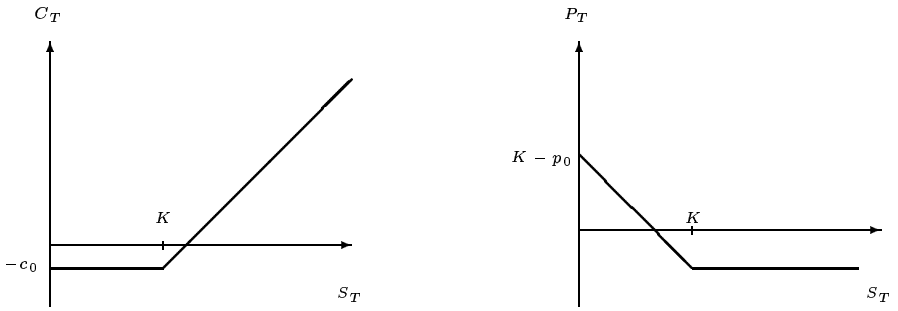


option is an idiot or a complete idiot. Both types are not very smart to pay something for the option at time T. The idiot, however, would realize that there is no reason to pay K to receive the stock which can be bought for less in the market. The complete idiot would exercise the option. Then you as the person having sold the option would have to buy the stock in the market for S_T, but that would be more than financed by the K you received from the complete idiot.

Of course, you can turn these hockey sticks around in which case you are looking at the value of a written option:



Note that we are only looking at the situation at an exercise date (i.e. date T for a European option). Sometimes we wish to take into account that the option had an initial cost at date 0, c_0 for a call, p_0 for a put, in which case we get the following profit diagrams:

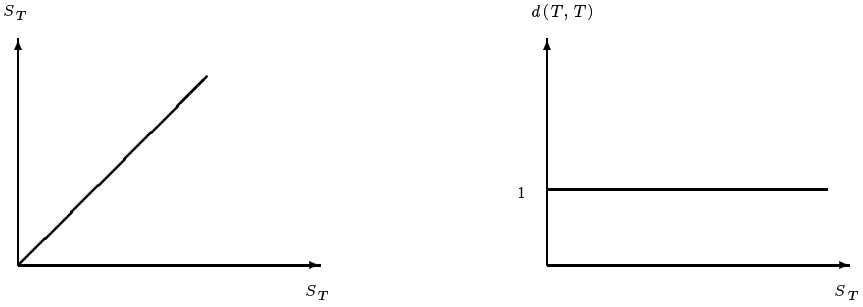


Of course, we are slightly allergic to subtracting payments occurring at different dates without performing some sort of discounting. Therefore, one may also choose to represent the prices of options by their time T forward discounted values $\frac{c_0}{d(0,T)}$ and $\frac{p_0}{d(0,T)}$.

The world of derivative securities is filled with special terminology and here are a few additions to your vocabulary: A call option with strike price K is said to be *(deep) in-the-money* at time t if $S_t > K$ ($S_t \gg K$).The opposite situation $S_t < K$ ($S_t \ll K$) is referred to as the call option being *(deep) out-of-the-money.* If $S_t \approx K$, the option is said to be *at-the-money.* The same terminology applies to put options but with 'opposite signs': A

put option is in-the-money if $S_t < K$.

The diagrams we have seen so far considered positions consisting of just one option. We considered a *long position,* i.e. a position corresponding to holding the option, and we considered a *short position,* i.e. a position corresponding to having written an option. One of the attractive features of options is that they can be combined with positions in other options, the underlying security and bonds to produce more complicated payoffs than those illustrated in the profit diagrams above. We will see examples of this in the exercises. Note that you should think of the payoff diagram for holding the stock and the diagram for holding the bond as being represented by:



Until further notice we will assume that the stock does not pay any dividends in the time interval $[0, T]$. This means that if you own the stock you will not receive any cash unless you decide to sell the stock. With this assumption and the maintained assumption of frictionless markets we will give some restrictions on option prices which follow solely from arbitrage considerations.

The most important relation is the so-called *put-call parity* for European options. Consider the portfolio strategy depicted in the table below and the associated cash flows at time t and time T. Assume that both options are European, expire at date T and have strike price equal to K :

strategy\cashflow	date t	date $T, S_T \leq K$	date $T, S_T > K$
sell 1 call	c_t	0	$K - S_T$
buy 1 put	$-p_t$	$K - S_T$	0
buy stock	$-S_t$	S_T	S_T
sell K bonds	$Kd(t,T)$	$-K$	$-K$
total cash flow	must be 0	0	0

Note that we have constructed a portfolio which gives a payoff of 0 at time T no matter what the value of S_T. Since the options are European we need not consider any time points in (t, T). This portfolio must have price 0, or else there would be an obvious arbitrage strategy. If, for example, the portfolio

had positive value, we would sell the portfolio (corresponding to reversing the strategy in the table) and have no future obligations. In other words we have proved that in a frictionless market we have the following

Proposition 21 *(Put-call parity) The price c_t of a European call and the price p_t of a European put option with expiration date T and exercise price K must satisfy*

$$c_t - p_t = S_t - Kd(t,T).$$

Note one simple but powerful consequence of this result: When deciding which parameters may influence call and put prices the put-call parity gives a very useful way of testing intuitive arguments. If S_t, K and $d(t,T)$ are fixed, then a change in a parameter which produces a higher call price, must produce a higher put-price as well. One would easily for example be tricked into believing that in a model where S_T is stochastic, a higher mean value of S_T given S_t would result in a higher call price since the call option is more likely to finish in-the-money and that it would result in a lower put price since the put is more likely then to finish out-of-the money. But if we assume that S_t and the interest rate are held fixed, put-call parity tells us that this line of reasoning is wrong.

Also note that for $K = \frac{S_t}{d(t,T)}$ we have $c_t = p_t$. This expresses the fact that the exercise price for which $c_t = p_t$ is equal to the *forward* price of S at time t. A *forward* contract is an agreement to buy the underlying security at the expiration date T of the contract at a price of F_t. Note that F_t is specified at time t and that the contract unlike an option forces the holder to buy. In other words you can lose money at expiration on a forward contract. The *forward price* F_t is decided so that the value of the forward contract at date t is 0. Hence the forward price is *not* a price to be paid for the contract at date t. It is more like the exercise price of an option. Which value of F_t then gives the contract a value of 0 at date t? Consider the following portfolio argument:

strategy\cashflow	date t	date T
buy 1 stock	$-S_t$	S_T
sell $\frac{S_t}{d(t,T)}$ bonds	S_t	$-\frac{S_t}{d(t,T)}$
sell 1 forward	0	$F_t - S_T$
total cash flow	0	$F_t - \frac{S_t}{d(t,T)}$

Note that the cash flow at time T is known at time t and since the cash flow by definition of the forward price is equal to 0 at date t, the cash flow at

date T must be 0 as well. Hence

$$F_t = \frac{S_t}{d(t,T)}.$$

Note that buying a call and selling a put, both with exercise price K and expiration date T, is equivalent to buying forward at the price K. Therefore the convention that the forward contract has value 0 at date t is exactly equivalent to specifying K so that $c_t = p_t$.

6.3 Restrictions on option prices

In this section we derive some bounds on call prices which much be satisfied in frictionless markets. The line of reasoning used may of course be used on put options as well.

Consider a European call option with expiration date T and exercise price K. Assume that the underlying security does not pay any dividends during the life of the option. Then the value of the option c_t satisfies

$$S_t \geq c_t \geq \max\left(0, S_t - Kd(t,T)\right). \tag{6.1}$$

Proof. Clearly, $c_t \geq 0$. Also, the corresponding put option satisfies $p_t \geq 0$. Hence

$$c_t \geq c_t - p_t = S_t - Kd(t,T) \tag{6.2}$$

where we have used put-call parity. To see that $S_t \geq c_t$, assume that $S_t < c_t$ and consider the strategy of buying the stock and selling the option. That gives a positive cash flow at time t. If at time T, $S_T > K$ and the option is exercised the stock is delivered to the option holder and K is received. If the option is not exercised, the stock can be sold at non-negative value.

■

It is clear that an American option is more valuable than the corresponding European option, hence we note that the price C_t of an American option also satisfies $C_t \geq S_t - Kd(t,T)$. If interest rates are positive, i.e. $d(t,T) < 1$, this produces the interesting result that the value of the American call is always strictly greater than the immediate exercise value $S_t - K$ when $t < T$. This shows the important result that an American option on a non-dividend paying stock should never be exercised early. Our inequalities above show that it will be better to sell the option. A corresponding result does *not* hold for put options. This is perhaps not so surprising considering that postponing the exercise of a put postpones the receipt of K, whereas delaying the exercise of a call delays the payment of K.

Typically, stocks pay dividends and it is important to take this into account when pricing options. It will often be the case that the option contract does not take into account whether the underlying stock pays dividends. A dividend payment will normally produce a drop in the stock price and an owner of a call option will be hurt by this drop without receiving the benefit of a dividend. A date t is denoted an *ex-dividend* date if purchasing the stock at time $s < t$ gives the new owner part in the next dividend payment whereas a purchase at time t does not. For simplicity, we assume in the following that the dividend payment takes place at the ex-dividend date. Furthermore, we will assume that the size of the dividend is known some time before the dividend date. In a world with no taxes it ought to be the case then that the drop in the stock price around the dividend date is equal to the size of the dividend. Assume, for example, that the drop in the stock price is less than D. Then buying the stock right before the dividend date for a price of S_{t-} and selling it for S_{t+} immediately after the dividend date will produce a cash flow of $S_{t+} + D - S_{t-} > 0$. This resembles an arbitrage opportunity and it is our explanation for assuming in the following that $S_{t-} = S_{t+} + D$.

Now let us consider the price at time 0 of a European call option on a stock which is known to pay one dividend D at time t. Then

$$c_0 \geq \max\left(0, S_0 - Kd(0,T) - Dd(0,t)\right).$$

Again, $c_0 \geq 0$ is trivial. Assume $c_0 < S_0 - Kd(0,T) - Dd(0,t)$. Then buy the left hand side and sell the right hand side. At time t, we must pay dividend D on the stock we have sold, but that dividend is exactly received from the D zero coupon bonds with maturity t. At time T the value of the option we have sold is equal to $\max\left(0, S_T - K\right)$. The value of the right hand side is equal to $S_T - K$. If $S_T \geq K$ the total position is 0.If $S_T < K$ the total position has value $K - S_T$. Hence we have constructed a positive cash flow while also receiving money initially. This is an arbitrage opportunity and hence we rule out $c_0 < S_0 - Kd(0,T) - Dd(0,t)$.

There are many possible variations on the dividend theme. If dividends are not known at time 0 we may assume that they fall within a certain interval and then use the endpoints of this interval to bound calls and puts. The reader may verify that the maximal dividend is important for bounding calls and the minimum dividend for bounding put prices.

However, we maintain the assumption of a known dividend and finish this section by another important observation on the early exercise of American calls on dividend paying stocks. Assume that the stock pays a dividend at time t and that we are at time $0 < t$. It is then not optimal to exercise the option at time 0 whereas it may be optimal right before time t. To see that it

is not optimal at time 0, note that the American option contains as a part of its rights an option with expiration date $s \in (0, t)$, and since this option is an option on a non-dividend paying stock we know that its value is larger than $S_0 - K$, which is the value of immediate exercise. Therefore, the American option is also more worth than $S_0 - K$ and there is no point in exercising before t. To see that it may be optimal to exercise right before t, consider a firm which pays a liquidating dividend to all its shareholders. The stock will be worthless after the liquidation and so will the call option. Certainly, the option holder is better off to exercise right before the dividend date to receive part of the liquidating dividend.

The picture is much more complicated for puts. In the next section we will see how to compute prices for American puts in binomial models and this will give us the optimal exercise strategy as well.

6.4 Binomial models for stock options

In this section we will go through the binomial model for pricing stock options. Our primary focus is the case where the underlying security is a non-dividend paying stock but it should be transparent that the binomial framework is highly flexible and will easily handle the pricing and hedging of derivative securities with more complicated underlying securities.

We consider a model with T periods and assume throughout that the following two securities trade:

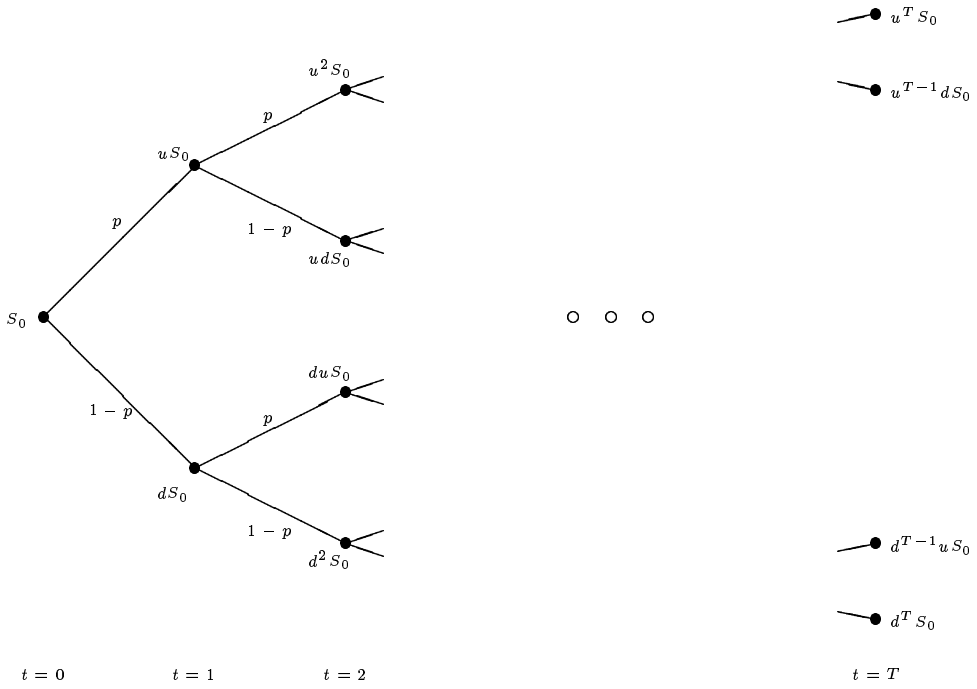
1. A money market account with a constant spot rate process ρ. Let $1 + \rho_t = R$, where $R \geq 1$. Hence we have for $s < t$

$$R_{s,t} = R^{t-s}.$$

2. A stock[2] S, which pays no dividends[3], whose price at time 0 is S_0 and whose evolution under the measure P is described in the tree (where we have assumed that $u > R > d > 0$) shown below.

[2]Since there is only one stock we will write S instead of S^1.

[3]To comply with the mathematical model of the previous chapter we should actually say that the stock pays a liquidating dividend of S_T at time T. We will however speak of S_T as the price at time T of the stock.



The mathematical description of the process is as follows: Let $U_1, \ldots, U_T$ be a sequence of i.i.d. Bernoulli variables, let $p = P(U_1 = 1)$ and define

$$N_t = \sum_{i=1}^{t} U_i.$$

Think of N_t as the number of up-jumps that the stock has had between time 0 and t. Clearly, this is a binomially distributed random variable. Let $u > R > d > 0$ be constants. Later, we will see how these parameters are chosen in practice. Then

$$S_t = S_0 u^{N_t} d^{t-N_t}. \tag{6.3}$$

Using the results on one-period submodels it is clear that the model is arbitrage free and complete and that the equivalent martingale measure is given in terms of conditional probabilities as

$$\begin{aligned} Q(S_t = uS_{t-1} \,|\, S_{t-1}) &\equiv q = \frac{R-d}{u-d} \\ Q(S_t = dS_{t-1} \,|\, S_{t-1}) &= 1-q = \frac{u-R}{u-d}. \end{aligned}$$

6.5 Pricing the European call

We now have the martingale measure Q in place and hence the value at time t of a European call with maturity T is given in an arbitrage-free model by

$$C_t = \frac{1}{R^{T-t}} E^Q \left(\max(0, S_T - K) \, | \mathcal{F}_t \right).$$

Using this fact we get the following

Proposition 22 *Let the stock and money market account be as described in section 6.4. Then the price of a European call option with exercise price K and maturity date T is given as*

$$C_t = \frac{1}{R^{T-t}} \sum_{i=0}^{T-t} \binom{T-t}{i} q^i (1-q)^{T-t-i} \max(0, S_t u^i d^{T-t-i} - K).$$

Proof. Since the money market account and S_0 are deterministic, we have that we get all information by observing just stock-prices, or equivalently the U's, i.e. $\mathcal{F}_t = \sigma(S_1, \ldots, S_t) = \sigma(U_1, \ldots, U_t)$. By using (6.3) twice we can write

$$S_T = S_t u^{(N_T - N_t)} d^{(T-t)-(N_T - N_t)} = S_t u^Z d^{(T-t)-Z},$$

where $Z = N_T - N_t = \sum_{j=t+1}^{T} U_j \overset{Q}{\sim} \text{bi}(q; (T-t))$, and Z is independent of $\mathcal{F}_t$ (because the U's are independent). Therefore

$$R^{T-t} C_t = E^Q((S_T - K)^+ | \mathcal{F}_t) = E^Q((S_t u^Z d^{(T-t)-Z} - K)^+ | \mathcal{F}_t).$$

At this point in the narrative we need something called "the useful rule". It states the following: Suppose we are given a function $f : \mathbb{R}^2 \mapsto \mathbb{R}$, a σ-algebra $\mathcal{F}$, an $\mathcal{F}$-measurable random variable X and a random variable Y that is independent of $\mathcal{F}$. Define the function $g : \mathbb{R} \mapsto \mathbb{R}$ by $g(x) = E(f(x, Y))$. Then $E(f(X, Y)|\mathcal{F}) = g(X)$. We then use this in the above expression with S_t playing the role of X, Z as Y, and $f(x, y) = (x u^y d^{(T-t)-y} - K)^+$. By using the general transformation rule for discrete random variables $E(h(Y)) = \sum_{y_i} h(y_i) P(Y = y_i)$, and the fact that Z is Q-binomially distributed we get in the notation of "the useful rule" that

$$g(x) = \sum_{i=0}^{T-t} \left(\begin{array}{c} T-t \\ i \end{array} \right) q^i (1-q)^{(T-t)-i} (x u^i d^{(T-t)-i} - K)^+,$$

and the desired result follows. ■

We rewrite the expression for C_0 using some handy notation. Let a be the smallest number of upward jumps needed for the option to finish in the money, i.e.

$$\begin{aligned} a &= \min_{j\in\mathbf{N}}\{j|S_0u^jd^{T-j} > K\} \\ &= \min_{j\in\mathbf{N}}\{j|j\ln u + (T-j)\ln d > \ln(K/S_0)\} \\ &= \min_{j\in\mathbf{N}}\{j|j > \ln(K/(S_0d^T))/\ln(u/d)\} \\ &= \left\lfloor \frac{\ln\left(\frac{K}{S_0d^T}\right)}{\ln\left(\frac{u}{d}\right)} \right\rfloor + 1. \end{aligned}$$

Letting

$$\Psi(a;T,q) = \sum_{i=a}^{T} \left(\begin{array}{c} T \\ i \end{array} \right) q^i(1-q)^{T-i},$$

we may write (you may want to check the first term on the RHS)

$$C_0 = S_0\Psi\left(a;T,q'\right) - \frac{K}{R^T}\Psi\left(a;T,q\right) \tag{6.4}$$

where

$$q' = \frac{u}{R}q.$$

Using put-call parity gives us the price of the European put:

Corollary 23 *The price of a European put option with T periods to maturity, exercise price T and the stocks as underlying security has a price at time 0 given by*

$$P_0 = \frac{K}{R^T}\left(1-\Psi\left(a;T,q\right)\right) - S_0\left(1-\Psi\left(a;T,q'\right)\right)$$

Note that our option pricing formulae use T to denote the number of periods until maturity. Later, we will be more explicit in relating this to actual calendar time.

6.6 Hedging the European call

We have already seen in a two period model how the trading strategy replicating a European call option may be constructed. In this section we simply state the result for the case with T periods and we then note an interesting

way of expressing the result. We consider the case with a money market account and one risky asset S and assume that the market is complete and arbitrage-free. The European call option has a payout at maturity of

$$\delta_T^c = \max(S_T - K, 0).$$

Proposition 24 *A self-financing trading strategy replicating the dividend process of the option from time* 1 *to* T *is constructed recursively as follows: Find* $\phi_{T-1} = (\phi_{T-1}^0, \phi_{T-1}^1)$ *such that*

$$\phi_{T-1}^0 R + \phi_{T-1}^1 S_T = \delta_T^c.$$

For $t = T-2, T-3, \ldots, 1$ *find* $\phi_t = (\phi_t^0, \phi_t^1)$ *such that*

$$\phi_t^0 R + \phi_t^1 S_{t+1} = \phi_{t+1}^0 + \phi_{t+1}^1 S_{t+1}.$$

The trading strategy is self-financing by definition, replicates the call and its initial price of $\phi_0^0 + \phi_0^1 S_0$ is equal to the arbitrage-free price of the option. We may easily extend to the case where both the underlying and the contingent claim have dividends other than the one dividend of the option considered above. In that case the procedure is the following:Find $\phi_{T-1} = (\phi_{T-1}^0, \phi_{T-1}^1)$ such that

$$\phi_{T-1}^0 R + \phi_{T-1}^1 (S_T + \delta_T) = \delta_T^c.$$

For $t = T-2, T-3, \ldots, 1$ find $\phi_t = (\phi_t^0, \phi_t^1)$ such that

$$\phi_t^0 R + \phi_t^1 (S_{t+1} + \delta_{t+1}) = \phi_{t+1}^0 + \phi_{t+1}^1 S_{t+1} + \delta_{t+1}^c.$$

In this case the trading strategy is not self-financing in general but it matches the dividend process of the contingent claim, and the initial price of the contingent claim is still $\phi_0^0 + \phi_0^1 S_0$.

An additional insight into the hedging strategy is given by the proposition below.

Recall the notation

$$\widetilde{S}_t = \frac{S_t}{R_{0,t}}$$

for the discounted price process of the stock. Let C_t denote the price process of a contingent claim whose dividend process is δ^c and let

$$\begin{aligned} \widetilde{C}_t &= \frac{C_t}{R_{0,t}} \\ \widetilde{\delta}_t^c &= \frac{\delta_t^c}{R_{0,t}} \end{aligned}$$

denote the discounted price and dividend processes of the contingent claim. Define the conditional covariance under the martingale measure Q as follows:

$$\mathrm{Cov}^Q\left(X_{t+1}, Y_{t+1}\,|\mathcal{F}_t\right) = E^Q\left(\left(X_{t+1}-X_t\right)\left(Y_{t+1}-Y_t\right)|\,\mathcal{F}_t\right)$$

One may then show the following (but we will omit the proof):

Proposition 25 *Assume that the stock pays no dividends during the life of the option. The hedging strategy which replicates δ^c is computed as follows:*

$$\phi_t^1 = \frac{\mathrm{Cov}^Q\left(\widetilde{S}_{t+1}, \widetilde{C}_{t+1}+\widetilde{\delta}^c_{t+1}\,\middle|\,\mathcal{F}_t\right)}{\mathrm{VAR}^Q\left(\widetilde{S}_{t+1}\,|\mathcal{F}_t\right)} \qquad t=0,1,\ldots,T-1$$

$$\phi_t^0 = \widetilde{C}_t - \phi_t^1\widetilde{S}_t \qquad t=0,1,\ldots,T-1$$

Note the similarity with regression analysis! We will not go further into this at this stage. But this way of looking at hedging is important when defining so-called risk minimal trading strategies in incomplete markets.

The number of stocks held at time t in the replicating strategy is called the *hedge ratio*. The hedge ratio for a call option is a number between 0 and 1, and it is larger the more in-the-money the call is.

6.7 Recombining tree representation

If the number of time periods T is large it the tree representing the stock price evolution grows very rapidly. The number of nodes at time t is equal to 2^t, and since for example $2^{20} = 1048576$ we see that when you implement this model in a spreadsheet and you wish to follow C_t and the associated hedging strategy over time, you may soon run out of space. Fortunately, in many cases there is a way around this problem: If your security price process is *Markov* and the contingent claim you wish to price is *path-independent*, you can use a *recombining* tree to do all of your calculations. Let us look at each property in turn[4]: The process S is a *Markov chain* under Q if it satisfies

$$Q(S_{t+1}=s_{t+1}\,|S_t=s_t,\ldots S_1=s_1,S_0=s_0) = Q(S_{t+1}=s_{t+1}\,|S_t=s_t)$$

for all t and all $(s_{t+1}, s_t, \ldots, s_1, s_0)$. Intuitively, standing at time t, the current value of the process s_t is sufficient for describing the distribution of the

[4]These properties are interesting to consider for the stock only since the money market account trivially has all nice properties discussed in the following.

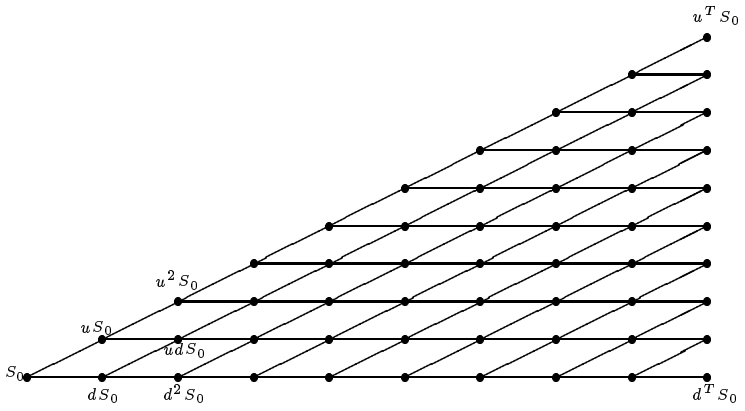


Figure 6.1: A lattice, i.e. a recombining tree.

process at time $t+1$. The binomial model of this chapter is clearly a Markov chain. An important consequence of this is that when $\mathcal{F}_t = \sigma(S_0, \ldots, S_t)$ then for any (measurable) function f and time points $t < u$ there exists a function g such that

$$E^Q\left(f(S_u)\,|\mathcal{F}_t\right) = g(S_t). \tag{6.5}$$

In other words, conditional expectations of functions of future values given everything we know at time t can be expressed as a function of the value of S_t at time t. The way S arrived at S_t is not important. We used this fact in the formula for the price of the European call: There, the conditional expectation given time t information became a function of S_t. The past did not enter into the formula. We can therefore represent the behavior of the process S in a *recombining tree*, also known as a *lattice*, as shown in Figure 6.1 in which one node at time t represents exactly one value of S_t. Another way of stating this is to say that the tree keeps track of the number of up-jumps that have occurred, not the order in which they occurred. A full event tree would keep track of the exact timing of the up-jumps.

To see what can go wrong, Figure 6.2 shows a process that is not Markov.

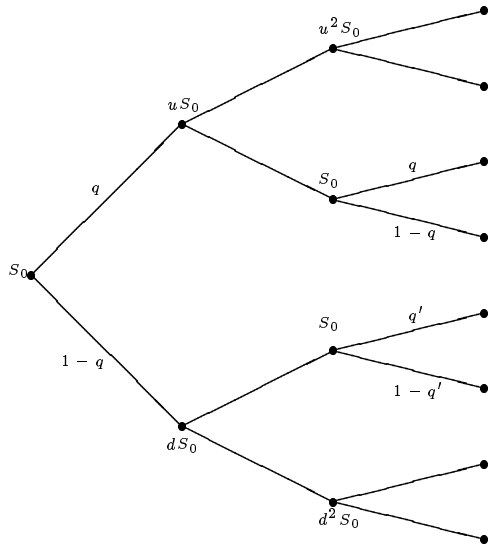


Figure 6.2: A tree that's not a lattice.

The problem is at time 2 when the value of the process is S_0, we need to know the pre-history of S to decide whether the probability of going up to uS is equal to q or q'. In standard binomial models such behavior is normally precluded.

Note that now the number of nodes required at time t is only $t+1$, and then using several hundred time periods is no problem for a spreadsheet.

A technical issue which we will not address here is the following: Normally we specify the process under the measure P, and it need not be the case that the Markov property is preserved under a change of measure. However, one may show that if the price process is Markov under P and the model is complete and arbitrage-free, then the price process is Markov under the equivalent martingale measure Q as well.

A second condition for using a recombining tree to price a contingent claim is a condition on the contingent claim itself:

Definition 37 *A contingent claim with dividend process δ^c is path independent if $\delta_t = f_t(S_t)$ for some (measurable) function f.*

Indeed if the claim is path independent and the underlying process is Markov, we have

$$C_t = R_{0,t}E\left(\sum_{i=t+1}^{T}\widetilde{\delta}_i^c \,\middle|\, \mathcal{F}_t\right)$$

$$= R_{0,t}E\left(\sum_{i=t+1}^{T} f_i(S_i)\middle|\mathcal{F}_t\right)$$
$$= R_{0,t}E\left(\sum_{i=t+1}^{T} f_i(S_i)\middle|S_t\right)$$

and the last expression is a function of S_t by the Markov property. A European option with expiration date T is path-independent since its only dividend payment is at time T and is given as $\max(S_T - K, 0)$.

The *Asian option* is an example of a contingent claim which is *not* path-independent. An Asian option on the stock, initiated at time 0, expiration date T and exercise price K has a payoff at date T given by

$$C_T^{asian} = \max\left(0, \left(\frac{1}{T+1}\sum_{t=0}^{T} S_t\right) - K\right)$$

Hence the average of the stock price over the period determines the option price. Clearly, S_T is not sufficient to describe the value of the Asian option at maturity. To compute the average value one needs the whole path of S. As noted above, even in a binomial model keeping track of the whole path for, say, 50 periods becomes intractable.

6.8 The binomial model for American puts

We describe in this section a simple way of pricing the American put option in a binomial model. Strictly speaking, an American put is not a contingent claim in the sense we have thought of contingent claims earlier. Generally, we have thought of contingent claims as random variables or sometimes as processes but a put is actually not specified until an exercise policy is associated with the put. What we will do in the following is to simultaneously solve for the optimal exercise policy, i.e. the one that maximizes the expected, discounted value of the cash flows under the martingale measure, and the price of the option. The argument given is not a proof but should be enough to convince the reader that the right solution is obtained (it is fairly easy to show that another exercise policy will create arbitrage opportunities for the option writer).

The value of an American put at its maturity is easy enough:[5]

$$P_T = \max(0, K - S_T). \tag{6.6}$$

[5]Or is it? As it stands P_t is really the value at time t given that the put has not been exercised at times $0, 1, t-1$. But that will most often be exactly what we are interested in; if we exercised the put to years ago, we really don't care about it anymore.

Now consider the situation one period before maturity. If the put has not been exercised at that date, the put option holder has two possibilities: Exercise the put at time $T-1$ or hold the put to maturity. The value of holding the put to maturity is given as the discounted (back to time $T-1$) value of (6.6), whereas the value at time $T-1$ of exercising immediately is $K - S_{T-1}$ something only to be considered of course if $K > S_{T-1}$. Clearly, the put option holder has a contract whose value is given by the maximal value of these two strategies, i.e.[6]

$$P_{T-1} = \max\left(K - S_{T-1}, E^Q\left(\frac{P_T}{R}\middle|\mathcal{F}_{T-1}\right)\right).$$

Now continue in this fashion by working backwards through the tree to obtain the price process of the American put option given by the recursion

$$P_{t-1} = \max\left(K - S_{t-1}, E^Q\left(\frac{P_t}{R}\middle|\mathcal{F}_{t-1}\right)\right) \qquad t = 1, \ldots, T.$$

Once this price process is given we see that the optimal exercise strategy is to exercise the put the first time t for which

$$K - S_t > E^Q\left(\frac{P_{t+1}}{R}\middle|\mathcal{F}_t\right).$$

This way of thinking is easily translated to American call options on dividend paying stocks for which early exercise is something to consider.

6.9 Implied volatility

We assume in this section that the Black-Scholes formula is known to the reader: The price at time t of a European call option maturing at time T, when the exercise price is K and the underlying security is a non-dividend paying stock with a price of S_t, is given in the Black-Scholes framework by

$$C_t = S_t\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2)$$

where

$$d_1 = \frac{\log\left(\frac{S_t}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

and

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

[6] We do not need 0 in the list of arguments of max since positivity is assured by $P_T \geq 0$.

where Φ is the cumulative distribution function of a standard normal distribution.

Consider the Black-Scholes formula for the price of a European call on an underlying security whose value at time 0 is S_0: Recall that Φ is a distribution function, hence $\Phi(x) \to 1$ as $x \to \infty$ and $\Phi(x) \to 0$ as $x \to -\infty$. Assume throughout that $T > 0$. From this it is easy to see that $c_0 \to S_0$ as $\sigma \to \infty$. By considering the cases $S_0 < K\exp(-rT)$, $S_0 = K\exp(-rT)$ and $S_0 > K\exp(-rT)$ separately, it is easy to see that as $\sigma \to 0$, we have $c_0 \to \max(0, S_0 - K\exp(-rT))$. By differentiating c_0 with respect to σ, one may verify that c_0 is strictly increasing in σ. Therefore, the following definition makes sense:

Definition 38 *Given a security with price S_0. Assume that the risk free rate (i.e. the rate of the money market account) is equal to r. Assume that the price of a call option on the security with exercise price K and time to maturity T is observed to have a price of c^{obs} with*

$$\max(0, S_0 - K\exp(-rT)) < c^{obs} < S_0.$$

Then the implied volatility of the option is the unique value of σ for which

$$c_0(S_0, K, T, \sigma, r) = c^{obs}.$$

In other words, the implied volatility is the unique value of the volatility which makes the Black-Scholes model 'fit' c^{obs}. Clearly, we may also associate an implied volatility to a put option whose observed price respects the appropriate arbitrage bounds.

A very important reason for the popularity of implied volatility is the way in which it allows a transformation of option prices which are hard to compare into a common scale. Assume that the price of a stock is 100 and the riskfree rate is 0.1. If one observed a price of 9.58 on a call option on the stock with exercise price 100 and 6 months to maturity and a price of 2.81 on a put option on the stock with exercise price 95 and 3 months to maturity then it would require a very good knowledge of the Black-Scholes model to see if one price was in some way higher than the other. However, if we are told that the implied volatility of the call is 0.25 and the implied volatility of the put is 0.30, then at least we know that compared to the Black-Scholes model, the put is more expensive than the call. This way of comparing is in fact so popular that traders in option markets typically do not quote prices in (say) dollars, but use 'vols' instead.

If the Black-Scholes model were true the implied volatility of all options written on the same underlying security should be the same, namely equal

to the volatility of the stock and this volatility would be a quantity we could estimate from historical data. In short, in a world where the Black-Scholes model holds, historical volatility (of the stock) is equal to implied volatility (of options written on the stock). In practice this is not the case - after all the Black-Scholes model is only a model. The expenses of hedging an option depend on the volatility of the stock during the life of the option. If, for example, it is known that, after a long and quiet period, important news about the underlying stock will arrive during the life of the option, the option price should reflect the fact that future fluctuations in the stock price might be bigger than the historical ones. In this case the implied volatility would be higher than the historical.

However, taking this knowledge of future volatility into account one could still imagine that all implied volatilities of options on the same underlying were the same (and equal to the 'anticipated' volatility). In practice this is not observed either. To get an idea of why, we consider the notion of portfolio insurance.

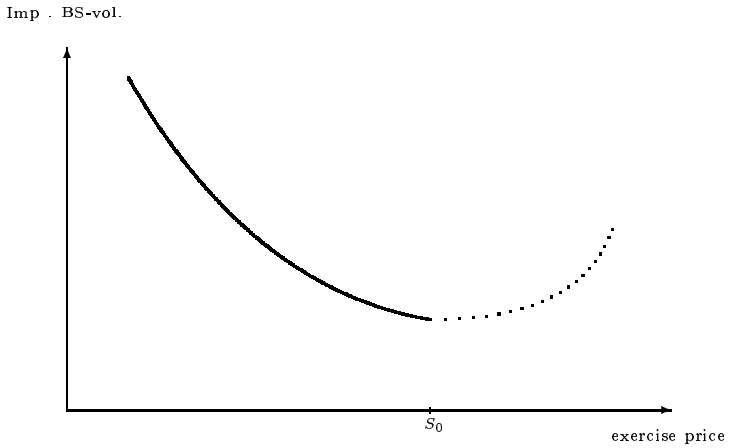
6.10 Portfolio insurance, implied volatility and crash fears

Consider a portfolio manager who manages a portfolio which is diversified so that the value of her portfolio follows that of the market stock index. Assume that the value of her portfolio is 1000 times the value of the index which is assumed to be at 110. The portfolio manager is very worried about losing a large portion of the value of the portfolio over the next year - she thinks that there is a distinct possibility that the market will crash. On the other hand she is far from certain. If she were certain, she could just move the money to a bank at a lower but safer expected return than in the stock market. But she does not want to exclude herself from the gains that a surge in the index would bring. She therefore decides to buy *portfolio insurance* in such a way that the value of her portfolio will never fall below a level of (say) 90.000. More specifically, she decides to buy 1000 put options with one year to maturity and an exercise price of 90 on the underlying index. Now consider the value of the portfolio after a year as a function of the level of the index S_T :

value of index	$S_T \geq 90$	$S_T < 90$
value of stocks	$S_T \times 1000$	$S_T \times 1000$
value of puts	0	$1000 \times (90 - S_T)$
total value	$S_T \times 1000 > 90.000$	90.000

Although it has of course not been costless to buy put options, the portfolio manager has succeeded in preventing the value of her portfolio from falling below 90.000. Since the put options are far out-of-the-money (such contracts are often called "lottery tickets") at the time of purchase they are probably not that expensive. And if the market booms she will still be a successful portfolio manager.

But what if she is not alone with her fear of crashes. We may then imagine a lot of portfolio managers interested in buying out-of-the-money put options hence pushing up the price of these contracts. This is equivalent to saying that the implied volatility goes up and we may experience the scenario shown in the graph below, in which the implied volatility of put options is higher for low exercise price puts:



This phenomenon is called a "smirk". If (as it is often seen from data) the implied volatility is increasing (the dotted part of the curve) for puts that are in the money, then we have what is known as a "smile". Actually options that are deeply in-the-money are rarely traded, so the implied volatility figures used to draw "the other half" of the smile typically comes from out-of-the-money calls. (Why/how? Recall the put-call parity.)

A smirk has been observed before crashes and it is indicative of a situation where the Black-Scholes model is not a good model to use. The typical modification allows for stock prices to jump discontinuously but you will have to wait for future courses to learn about this.

6.11 Debt and equity as options on firm value

In this section we consider a very important application of option pricing. Our goal is to learn a somewhat simplified but extremely useful way of think-

ing about a firm which is financed by debt and equity (see below). A fundamental assumption in this section is that a firm has a market value given by a stochastic process V. In Arrow-Debreu economies in which we know prices and production plans adopted by the firms, it is easy to define the value of a firm as the (net) value of its production. In reality things are of course a lot more complicated. It is hard to know, for example, what the value of Novo-Nordisk is - i.e. what is the market value of the firm's assets (including know-how, goodwill etc.). Part of the problem is of course that it is extremely difficult to model future prices and production levels. But in a sense the actual value does not matter for this section in that the 'sign' of the results that we derive does not depend on what the value of the firm is - only the "magnitude' does.

The fundamental simplification concerns the *capital structure* of the firm. Assume that the firm has raised capital to finance its activities in two ways: It has issued stocks (also referred to as *equity)* and *debt.* The debt consists of zero coupon bonds with face value D maturing at time T. Legally what distinguishes the debt holders from the stock holders is the following: The stock holders control the firm and they decide at time T whether the firm should repay its debt to the bondholders. If the bondholders are not repaid in full they can force the firm into bankruptcy and take over the remaining assets of the firm (which means both controlling and owning it). The stocks will then be worthless. If the stockholders pay back D at maturity to the bondholders, they own the firm entirely. They may then of course decide to issue new debt to finance new projects but we will not worry about that now.

It is clear that the stockholders will have an interest in repaying the bondholders precisely when $V_T > D$. Only then will the expense in paying back the debt be more than outweighed by the value of the firm. If $V_T < D$ (and there are no bankruptcy costs) the stockholders will default on their debt, the firm will go into bankruptcy and the bondholders will take over. In short, we may write the value of debt and equity at time T as

$$\begin{aligned} B_T &= \min(D, V_T) = D - \max(D - V_T, 0) \\ S_T &= \max(V_T - D, 0). \end{aligned}$$

In other words, we may think of equity as a call option on the value of the firm and debt as a zero coupon bond minus a put option on the value of the firm. Assuming then that V behaves like the underlying security in the Black-Scholes model and that there exists a money market account with interest rate r, we can use the Black-Scholes model to price debt and equity at time 0 :

$$B_0 = D\exp(-rT) - p_0(V_0, D, T, \sigma, r)$$

$$S_0 = c_0(V_0, D, T, \sigma, r)$$

where p_0, c_0 are Black-Scholes put and call functions.

Let us illustrate a potential conflict between stockholders and bondholders in this model. Assume that at time 0 the firm has the possibility of adopting a project which will not alter the value of the firm at time 0, but which will have the effect of increasing the volatility of the process V. Since both the value of the call and the put increases when σ increases we see that the stockholders will like this project since it increases the value of the equity whereas the bondholders will not like the project since the put option which they have in a sense written will be a greater liability to them. This is a very clear and very important illustration of so-called *asset substitution,* a source of conflict which exists between stock-and bondholders of a firm. This setup of analyzing the value of debt and equity is useful in a number of contexts and you should make sure that you understand it completely. We will return to this towards the end of the course when discussing corporate finance.