Chapter 5

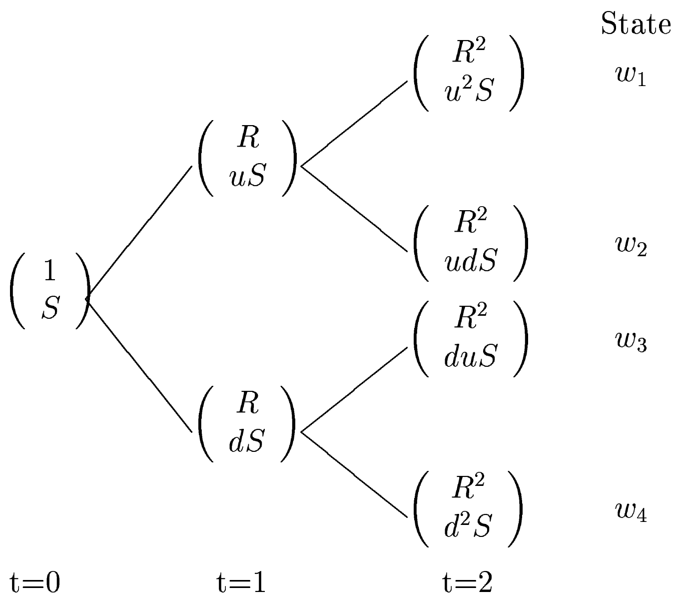
Arbitrage pricing in the multi-period model

5.1 An appetizer

It is fair to argue that to get realism in a model with finite state space we need the number of states to be large. After all, why would the stock take on only two possible values at the expiration date of the option? On the other hand, we know from the previous section that in a model with many states we need many securities to have completeness, which (in arbitrage-free models) is a requirement for pricing every claim. And if we want to price an option using only the underlying stock and a money market account, we only have two securities to work with. Fortunately, there is a clever way out of this.

Assume that over a short time interval the stock can only move to two different values and split up the time interval between 0 and T (the maturity date of an option) into small intervals in which the stock can be traded. Then it turns out that we can have both completeness and therefore arbitrage pricing even if the number of securities is much smaller than the number of states. Again, before we go into the mathematics, we give an example to help with the intuition.

Assume that $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ and that there are three dates: $t \in \{0, 1, 2\}$. We specify the behavior of the stock and the money market account as follows: Assume that $0 < d < R < u$ and that $S > 0$. Consider the following graph:



At time 0 the stock price is S, the money market account is worth 1. At time 1, if the state of the world is ω_1 or ω_2, the prices are uS and R , respectively, whereas if the true state is ω_3 or ω_4, the prices are dS and R. And finally, at time $t = 2$, the prices of the two instruments are as shown in the figure above. Note that $\omega \in \Omega$ describes a whole "sample path" of the stock price process and the money market account, i.e. it tells us not only the final time 2 value, but the entire history of values up to time 2.

Now suppose that we are interested in the price of a European call option on the stock with exercise price K and maturity $T = 2$. At time 2, we know it is worth

$$C_2(\omega) = [S_2(\omega) - K]^+$$

where $S_2(\omega)$ is the value of the stock at time 2 if the true state is ω.

At time 1, if we are in state ω_1 or ω_2, the money market account is worth R and the stock is worth uS, and we know that there are only two possible time 2 values, namely (R^2, u^2S) or (R^2, duS). But then we can use the argument of the one period example to see that at time 1 in state ω_1 or ω_2 we can replicate the calls payoff by choosing a suitable portfolio of stock and money market account: Simply solve the system:

$$au^2S + bR^2 = \left[u^2S - K\right]^+ \equiv C_{uu}$$

$$aduS + bR^2 = [duS - K]^+ \equiv C_{du}$$

for (a, b) and compute the price of forming the portfolio at time 1. We find

$$a = \frac{C_{uu} - C_{du}}{uS(u-d)}, \quad b = \frac{uC_{du} - dC_{uu}}{(u-d)R^2}.$$

The price of this portfolio is

$$\begin{aligned} auS + bR &= \frac{R}{R}\frac{(C_{uu} - C_{du})}{(u-d)} + \frac{uC_{du} - dC_{uu}}{(u-d)R} \\ &= \frac{1}{R}\left[\frac{(R-d)}{(u-d)}C_{uu} + \frac{(u-R)}{(u-d)}C_{ud}\right] =: C_u \end{aligned}$$

This is clearly what the call is worth at time $t = 1$ if we are in ω_1 or ω_2, i.e. if the stock is worth uS at time 1. Similarly, we may define $C_{ud} := [udS - K]^+$ (which is equal to C_{du}) and $C_{dd} = [d^2S - K]^+$. And now we use the exact same argument to see that if we are in state ω_3 or ω_4, i.e. if the stock is worth dS at time 1, then at time 1 the call should be worth C_d where

$$C_d := \frac{1}{R}\left[\frac{(R-d)}{(u-d)}C_{ud} + \frac{(u-R)}{(u-d)}C_{dd}\right].$$

Now we know what the call is worth at time 1 depending on which state we are in: If we are in a state where the stock is worth uS, the call is worth C_u and if the stock is worth dS, the call is worth C_d.

Looking at time 0 now, we know that all we need at time 1 to be able to "create the call", is to have C_u when the stock goes up to uS and C_d when it goes down. But that we can accomplish again by using the one-period example: The cost of getting $\binom{C_u}{C_d}$ is

$$C_0 := \frac{1}{R}\left[\frac{(R-d)}{(u-d)}C_u + \frac{(u-R)}{(u-d)}C_d\right].$$

If we let $q = \frac{R-d}{u-d}$ and if we insert the expressions for C_u and C_d, noting that $C_{ud} = C_{du}$, we find that

$$C_0 = \frac{1}{R^2}\left[q^2 C_{uu} + 2q(1-q)C_{ud} + (1-q)^2 C_{dd}\right]$$

which the reader will recognize as a discounted expected value, just as in the one period example. (Note that the representation as an expected value does not hinge on $C_{ud} = C_{du}$.)

The important thing to understand in this example is the following: Starting out with the amount C_0, an investor is able to form a portfolio in the stock and the money market account which produces the payoffs C_u or C_d

at time 1 depending on where the stock goes. Now without any additional costs, the investor can rearrange his/her portfolio at time 1, such that at time 2, the payoff will match that of the option. Therefore, at time 0 the price of the option must be C_0.

This "dynamic hedging" argument is the key to pricing derivative securities in discrete-time, finite state space models. We now want to understand the mathematics behind this example.

5.2 Price processes, trading and arbitrage

Given a probability space $(\Omega, \mathcal{F}, P)$ with Ω finite, let $\mathcal{F} := 2^{\Omega}$ (i.e. the set of all subsets of Ω) and assume that $P(\omega) > 0$ for all $\omega \in \Omega$. Also assume that there are T+1 dates, starting at date 0, ending at date T. To formalize how information is revealed through time, we introduce the notion of a *filtration*:

Definition 23 *A filtration* $\mathbb{F} = \{\mathcal{F}_t\}_{t=0}^{T}$ *is an increasing sequence of* σ-*algebras contained in* $\mathcal{F}$: $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \ldots \subseteq \mathcal{F}_T$.

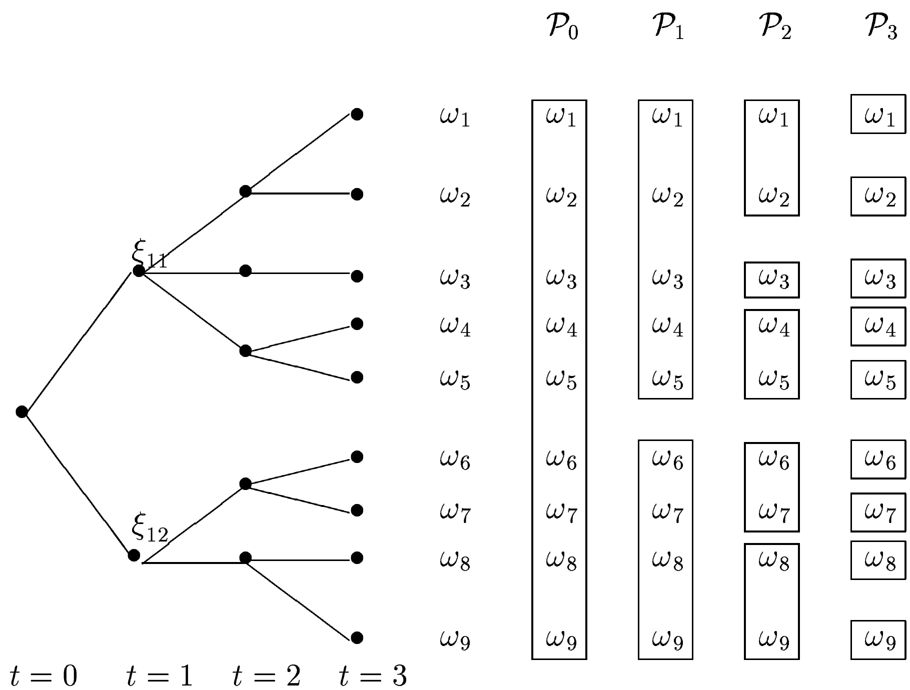
We will always assume that $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_T = \mathcal{F}$. Since Ω is finite, it will be easy to think of the $\sigma-$algebras in terms of *partitions:*

Definition 24 *A partition* $\mathcal{P}_t$ *of* Ω *is a collection of non-empty subsets of* Ω *such that*

- $\bigcup_{P_i \in \mathcal{P}_t} P_i = \Omega$
- $P_i \cap P_j = \emptyset$ whenever $i \neq j, P_i, P_j \in \mathcal{P}_t$.

Because Ω is finite, there is a one-to-one correspondence between partitions and $\sigma-$algebras: The elements of $\mathcal{P}_t$ corresponds to *the atoms* of $\mathcal{F}_t$.

The concepts we have just defined are well illustrated in an *event-tree:*



The event tree illustrates the way in which we imagine information about the true state being revealed over time. At time $t = 1$, for example, we may find ourselves in one of two nodes: ξ_{11} or ξ_{12}. If we are in the node ξ_{11}, we know that the true state is in the set $\{\omega_1, \omega_2, \ldots, \omega_5\}$, but we have no more knowledge than that. In ξ_{12}, we know (only) that $\omega \in \{\omega_6, \omega_7, \ldots, \omega_9\}$. At time $t = 2$ we have more detailed knowledge, as represented by the partition $\mathcal{P}_2$. Elements of the partition $\mathcal{P}_t$ are events which we can decide as having occurred or not occurred at time t, *regardless* of what the true ω is. At time 1, we will always know whether $\{\omega_1, \omega_2, \ldots, \omega_5\}$ has occurred or not, regardless of the true ω. If we are at node ξ_{12}, we would be able to rule out the event $\{\omega_1, \omega_2\}$ also at time 1, but if we are at node ξ_{11}, we will not be able to decide whether this event has occurred or not. Hence $\{\omega_1, \omega_2\}$ is not a member of the partition.

Make sure you understand the following

Remark 2 *A random variable defined on $(\Omega, \mathcal{F}, P)$ is measurable with respect to $\mathcal{F}_t$ precisely when it is constant on each member of $\mathcal{P}_t$.*

A stochastic process $X := (X_t)_{t=0,\ldots,T}$ is a sequence of random variables $X_0, X_1, \ldots, X_T$. The process is *adapted* to the filtration $\mathcal{F}$ if X_t is $\mathcal{F}_t-$measurable (which we will often write: $X_t \in \mathcal{F}_t$) for $t = 0, \ldots, T$. Returning to the event tree setup, it must be the case, for example, that $X_1(\omega_1) = X_1(\omega_5)$ if X is adapted, but we may have $X_1(\omega_1) \neq X_1(\omega_6)$.

Given an event tree, it is easy to construct adapted processes: Just assign the values of the process using the nodes of the tree. For example, at time 1, there are two nodes ξ_{11} and ξ_{12}. You can choose one value for X_1 in ξ_{11} and another in ξ_{12}. The value chosen in ξ_{11} will correspond to the value of X_1 on the set $\{\omega_1, \omega_2, \ldots, \omega_5\}$, the value chosen in ξ_{12} will correspond to the common value of X_1 on the set $\{\omega_6, \ldots, \omega_9\}$. When X_t is constant on an event A_t we will sometimes write $X_t(A_t)$ for this value. At time 2 there are five different values possible for X_2. The value chosen in the top node is the value of X_2 on the set $\{\omega_1, \omega_2\}$.

As we have just seen it is convenient to speak in terms of the event tree associated with the filtration. From now on we will refer to the event tree as the graph Ξ and use ξ to refer to the individual nodes. The notation $p(\xi)$ will denote the probability of the event associated with ξ; for example $P(\xi_{11}) = P(\{\omega_1, \omega_2, \ldots, \omega_5\})$. This graph Ξ will also allow us to identify adapted processes with vectors in $\mathbb{R}^{\Xi}$. The following inner products on the space of adapted processes will become useful later: Let X, Y be adapted processes and define

$$\begin{aligned} \sum_{\xi\in\Xi} X(\xi)Y(\xi) &\equiv \sum_{\{(t,A_u):A_u\in\mathcal{P}_t, 0\le t\le T\}} X_t(A_u)Y_t(A_u) \\ E\sum_{\xi\in\Xi} X(\xi)Y(\xi) &\equiv \sum_{\xi\in\Xi} P(\xi)X(\xi)Y(\xi) \\ &\equiv \sum_{\{(t,A_u):A_u\in\mathcal{P}_t, 0\le t\le T\}} P(A_u)X_t(A_u)Y_t(A_u) \end{aligned}$$

Now we are ready to model financial markets in multi-period models. Given is a vector of adapted *dividend processes*

$$\delta = (\delta^1, \ldots, \delta^N)$$

and a vector of adapted *security price processes*

$$S = (S^1, \dots, S^N).$$

The interpretation is as follows: $S_t^i(\omega)$ is the price of security i at time t if the state is ω. Buying the $i'th$ security at time t ensures the buyer (and obligates the seller to deliver) the remaining dividends $\delta_{t+1}^i, \delta_{t+2}^i, \dots, \delta_T^i$.[1] Hence the security price process is to be interpreted as an *ex-dividend* price process and in particular we should think of S_T as 0. In all models considered in these notes we will also assume that there is a money market account which provides locally riskless borrowing and lending. This is modeled as follows: Given an adapted process - *the spot rate process*

$$\rho = (\rho_0, \rho_1, \dots, \rho_{T-1}).$$

To make the math work, all we need to assume about this process is that it is strictly greater than -1 at all times and in all states, but for modelling purposes it is desirable to have it non-negative. Now we may define the money market account as follows:

Definition 25 *The money market account has the security price process*

$$\begin{aligned} S_t^0 &= 1, \qquad t = 0, 1, \dots, T-1 \\ S_T^0 &= 0. \end{aligned}$$

and the dividend process

$$\begin{aligned} \delta_t^0(\omega) &= \rho_{t-1}(\omega) \text{ for all } \omega \text{ and } t = 1, \dots, T-1, \\ \delta_T^0(\omega) &= 1 + \rho_{T-1}(\omega). \end{aligned}$$

This means that if you buy one unit of the money market account at time t you will receive a dividend of ρ_t at time $t+1$. Since ρ_t is known already at time t, the dividend received on the money market account in the next period $t+1$ is known at time t. Since the price is also known to be 1 you know that placing 1 in the money market account at time t and selling the asset at time $t+1$ will give you $1+\rho_t$. This is why we refer to this asset as a locally riskless asset. You may of course also choose to keep the money in the money market account and receive the stream of dividends. Reinvesting the dividends in the money market account will make this account grow according to the process R defined as

$$R_t = (1 + \rho_0) \cdots (1 + \rho_{t-1}).$$

[1] We will follow the tradition of probability theory and often suppress the ω in the notation.

We will need this process to discount cash flows between arbitrary periods and therefore introduce the following notation:

$$R_{s,t} \equiv (1+\rho_s)\cdots(1+\rho_{t-1}).$$

Definition 26 *A trading strategy is an adapted process*

$$\phi = (\phi_t^0, \ldots, \phi_t^N)_{t=0,\ldots,T-1}.$$

and the interpretation is that $\phi_t^i(\omega)$ is the number of the i'th security held at time t if the state is ω. The requirement that the trading strategy is adapted is very important. It represents the idea that the strategy should not be able to see into the future. Returning again to the event tree, when standing in node ξ_{11}, a trading strategy can base the number of securities on the fact that we are in ξ_{11} (and not in ξ_{12}), but not on whether the true state is ω_1 or ω_2.

The dividend stream generated by the trading strategy ϕ is denoted δ^ϕ and it is defined as

$$\delta_0^\phi = -\phi_0 \cdot S_0$$
$$\delta_t^\phi = \phi_{t-1} \cdot (S_t + \delta_t) - \phi_t \cdot S_t \text{ for } t = 1, \ldots, T.$$

Definition 27 *An arbitrage is a trading strategy for which δ_t^ϕ is a positive process, i.e. always nonnegative and $\delta_t^\phi(\omega) > 0$ for some t and ω. The model is said to be arbitrage-free if it contains no arbitrage opportunities.*

In words, there is arbitrage if we can adopt a trading strategy which at no point in time requires us to pay anything but which at some time in some state gives us a strictly positive payout. Note that since we have included the initial payout as part of the dividend stream generated by a trading strategy, we can capture the definition of arbitrage in this one statement. This one statement captures arbitrage both in the sense of receiving money now with no future obligations and in the sense of paying nothing now but receiving something later.

Definition 28 *A trading strategy ϕ is self-financing if it satisfies*

$$\phi_{t-1} \cdot (S_t + \delta_t) = \phi_t \cdot S_t \quad \textit{for } t = 1, \ldots, T.$$

The interpretation is as follows: Think of forming a portfolio ϕ_{t-1} at time $t-1$. Now as we reach time t, the value of this portfolio is equal to $\phi_{t-1} \cdot (S_t + \delta_t)$, and for a self-financing trading strategy, this is precisely the amount of money which can be used in forming a new portfolio at time t. We will let Φ denote the set of self-financing trading strategies.

5.3 No arbitrage and price functionals

We have seen in the one period model that there is equivalence between the existence of a state price vector and absence of arbitrage. In this section we show the multi-period analogue of this theorem.

The goal of this section is to prove the existence of the multi-period analogue of state-price vectors in the one-period model. Let $\mathbb{L}$ denote the set of adapted processes on the given filtration.

Definition 29 *A pricing functional F is a linear functional*

$$F : \mathbb{L} \to \mathbb{R}$$

which is strictly positive, i.e.

$$\begin{aligned} F(X) &\geq 0 \text{ for } X \geq 0 \\ F(X) &> 0 \text{ for } X > 0. \end{aligned}$$

Definition 30 *A pricing functional F is consistent with security prices if*

$$F(\delta^{\phi}) = 0 \text{ for all trading strategies } \phi.$$

Note that if there exists a consistent pricing functional we may arbitrarily assume that the value of the process $1_{\{t=0\}}$ (i.e. the process which is 1 at time 0 and 0 thereafter) is 1.

By Riesz' representation theorem we can represent the functional F as

$$F(X) = \sum_{\xi \in \Xi} X(\xi) f(\xi)$$

With the convention $F(1_{\{t=0\}}) = 1$, we then note that if there exists a trading strategy ϕ which is initiated at time 0 and which only pays a dividend of 1 in the node ξ, then

$$\phi_0 \cdot S_0 = f(\xi).$$

Hence $f(\xi)$ is the price at time 0 of having a payout of 1 in the node ξ.

Proposition 11 *The model (δ, S) is arbitrage-free if and only if there exists a consistent pricing functional.*

Proof. First, assume that there exists a consistent pricing functional F. Any dividend stream δ^{ϕ} generated by a trading strategy which is positive must have $F(\delta^{\phi}) > 0$ but this contradicts consistency. Hence there is no arbitrage. The other direction requires more work:

Define the sets

$$\begin{aligned}
\mathbb{L}^1 &= \left\{ X \in \mathbb{L} \,\middle|\, X > 0 \text{ and } \sum_{\xi \in \Xi} X(\xi) = 1 \right\} \\
\mathbb{L}^0 &= \left\{ \delta^\phi \in \mathbb{L} \,|\, \phi \text{ trading strategy} \right\}
\end{aligned}$$

and think of both sets as subsets of $\mathbb{R}^\Xi$. Note that $\mathbb{L}^1$is convex and compact and that $\mathbb{L}^0$ is a linear subspace, hence closed and convex. By the no arbitrage assumption the two sets are disjoint. Therefore, there exists a separating hyperplane $H(f;\alpha) := \{x \in \mathbb{R}^\Xi : f \cdot x = \alpha\}$ which separates the two sets strictly and we may choose the direction of f such that $f \cdot x \leq \alpha$ for $x \in \mathbb{L}^0$. Since $\mathbb{L}^0$ is a linear subspace we must have $f \cdot x = 0$ for $x \in \mathbb{L}^0$ (why?). Strict separation then gives us that $f \cdot x > 0$ for $x \in \mathbb{L}^1$, and that in turn implies $f \gg 0$ (why?). Hence the functional

$$F(X) = \sum_{\xi \in \Xi} f(\xi) X(\xi)$$

is consistent. ■

By using the same geometric intuition as in Chapter 2, we note that there is a connection between completeness of the market and uniqueness of the consistent price functional:

Definition 31 *The security model is complete if for every $X \in \mathbb{L}$ there exists a trading strategy ϕ such that $\delta_t^\phi = X_t$ for $t \geq 1$.*

If the model is complete and arbitrage-free, there can only be one consistent price functional (up to multiplication by a scalar). To see this, assume that if we have two consistent price functionals F, G both normed to have $F(1_{\{t=0\}}) = G(1_{\{t=0\}}) = 1$. Then for any trading strategy ϕ we have

$$\begin{aligned}
0 &= -\phi_0 \cdot S_0 + F(1_{\{t>0\}} \delta^\phi) \\
&= -\phi_0 \cdot S_0 + G(1_{\{t>0\}} \delta^\phi)
\end{aligned}$$

hence F and G agree on all processes of the form $1_{\{t>0\}}\delta^\phi$. But they also agree on $1_{\{t=0\}}$ and therefore they are the same since by the assumption of completeness every adapted process can be obtained as a linear combination of these processes.

Given a security price system (π, D), the converse is shown in a way very similar to the one-period case. Assume the market is arbitrage-free and incomplete. Then there exists a process π in $\mathbb{L}$, whose restriction to time

$t \geq 1$ is orthogonal to any dividend process generated by a trading strategy. By letting $\pi_0 = 0$ and choosing a sufficiently small $\varepsilon > 0$, the functional defined by

$$(F + \varepsilon\pi)(\delta^{\phi}) = \sum_{\xi \in \Xi} (f(\xi) + \varepsilon\pi(\xi))\, \delta^{\phi}(\xi)$$

is consistent. Hence we have shown:

Proposition 12 *If the market is arbitrage-free, then the model is complete if and only if the consistent price functional is unique.*

5.4 Conditional expectations and martingales

Consistent price systems turn out to be less interesting for computation when we look at more general models, and they do not really explain the strange probability measure q which we saw earlier. We are about to remedy both problems, but first we need to make sure that we can handle conditional expectations in our models and that we have a few useful computational rules at our disposal.

Definition 32 *The conditional expectation of an $\mathcal{F}_u$−measurable random variable X_u given $\mathcal{F}_t$, where $\mathcal{F}_t \subseteq \mathcal{F}_u$, is given by*

$$E(X_u \,|\mathcal{F}_t)(\omega) = \frac{1}{P(A_t)} \sum_{A_v \in \mathcal{P}_u : A_v \subseteq A_t} P(A_v) X_u(A_v) \text{ for } \omega \in A_t$$

where we have written $X_u(A_v)$ for the value of $X_u(\omega)$ on the set A_v and where $A_t \in \mathcal{P}_t$.

We will illustrate this definition in the exercises. Note that we obtain an $\mathcal{F}_t$−measurable random variable since it is constant over elements of the partition $\mathcal{P}_t$. The definition above does not work when the probability space becomes uncountable. Then one has to adopt a different definition which we give here and which the reader may check is satisfied by the random variable given above in the case of finite sample space:

Definition 33 *The conditional expectation of an $\mathcal{F}_u$−measurable random variable X_u given $\mathcal{F}_t$ is a random variable $E(X_u \,|\mathcal{F}_t)$ which is $\mathcal{F}_t$−measurable and satisfies*

$$\int_{A_t} E(X_u \,|\mathcal{F}_t) dP = \int_{A_t} X_u dP$$

for all $A_t \in \mathcal{F}_t$.

It is easy to see that the conditional expectation is linear, i.e. if $X_u, Y_u \in \mathcal{F}_u$ and $a, b \in \mathbb{R}$, then

$$E(aX_u + bY_u \,|\mathcal{F}_t) = aE(X_u \,|\mathcal{F}_t) + bE(Y_u \,|\mathcal{F}_t).$$

We will also need the following computational rules for conditional expectations:

$$E(E(X_u \,|\mathcal{F}_t)) = EX_u \tag{5.1}$$
$$E(Z_t X_u \,|\mathcal{F}_t) = Z_t E(X_u \,|\mathcal{F}_t) \text{ whenever } Z_t \in \mathcal{F}_t \tag{5.2}$$
$$E(E(X_u \,|\mathcal{F}_t) \,|\mathcal{F}_s) = E(X_u \,|\mathcal{F}_s) \text{ whenever } s < t < u \tag{5.3}$$

Note that a consequence of (5.2) obtained by letting $X_u = 1$, is that

$$E(Z_t \,|\mathcal{F}_t) = Z_t \quad \text{whenever} \quad Z_t \in \mathcal{F}_t. \tag{5.4}$$

Now we can state the important definition:

Definition 34 *A stochastic process X is a martingale with respect to the filtration $\mathbb{F}$ if it satisfies*

$$E(X_t \,|\mathcal{F}_{t-1}) = X_{t-1} \quad \textit{all } t = 1, \ldots, T.$$

You can try out the definition immediately by showing:

Lemma 13 *A stochastic process defined as*

$$X_t = E(X \,|\mathcal{F}_t) \quad t = 0, 1, \ldots, T$$

where $X \in \mathcal{F}_T$, is a martingale.

Proof. Use (5.3)! ■

Let $E^P(Y; A) \equiv \int_A Y dP$ for any random variable Y and $A \in \mathcal{F}$. Using this notation and the definition (33) of a martingale, this lemma says that

$$E(X; A) = E(X_t; A) \qquad \text{for all } t \text{ and } A \in \mathcal{F}_t$$

When there can be no confusion about the underlying filtration we will often write $E_t(X)$ instead of $E(X \,|\mathcal{F}_t)$.

Two probability measures are equivalent when they assign zero probability to the same sets and since we have assumed that $P(\omega) > 0$ for all ω, the measures equivalent to P will be the ones which assign strictly positive probability to all events.

We will need a way to translate conditional expectations under one measure to conditional expectations under an equivalent measure. To do this we need the *density process:*

Definition 35 *Let the density process Z be defined as*

$$Z_T(\omega) = \frac{Q(\omega)}{P(\omega)}$$

and

$$Z_t = E^P(Z_T | \mathcal{F}_t) \qquad t = 0, 1, \ldots, T.$$

We will need (but will not prove) the following result of called the *Abstract Bayes Formula.*

Proposition 14 *Let X be a random variable on $(\Omega, \mathcal{F})$. Then*

$$E^Q(X | \mathcal{F}_t) = \frac{1}{Z_t} E^P(X Z_T | \mathcal{F}_t).$$

5.5 Equivalent martingale measures

In this section we state and prove what is sometimes known as the fundamental theorem of asset pricing. This theorem will explain the mysterious $q-$probabilities which arose earlier and it will provide an indispensable tool for constructing arbitrage-free models and pricing contingent claims in these models.

We maintain the setup with a money market account generated by the spot rate process ρ and N securities with price- and dividend processes $S = (S^1, \ldots, S^N), \delta = (\delta^1, \ldots, \delta^N)$. Define the corresponding discounted processes $\widetilde{S}, \widetilde{\delta}$ by defining for each $i = 1, \ldots, N$

$$\widetilde{S}_t^i = \frac{S_t^i}{R_{0,t}} \qquad t = 0, \ldots, T,$$

$$\widetilde{\delta}_t^i = \frac{\delta_t^i}{R_{0,t}} \qquad t = 1, \ldots, T.$$

Definition 36 *A probability measure Q on $\mathcal{F}$ is an equivalent martingale measure (EMM) if $Q(\omega) > 0$ all ω and for all $i = 1, \ldots, N$*

$$\widetilde{S}_t^i = E_t^Q \left(\sum_{j=t+1}^{T} \widetilde{\delta}_j^i \right) \qquad t = 0, \ldots, T-1. \tag{5.5}$$

The term martingale measure has the following explanation: Given a (one-dimensional) security price process S whose underlying dividend process only pays dividend δ_T at time T. Then the existence of an EMM will give us

$$\widetilde{S}_t = E_t^Q \left(\widetilde{\delta}_T \right) \qquad t = 0, \ldots, T-1.$$

and therefore the process $(\widetilde{S}_0, \widetilde{S}_1, \ldots, \widetilde{S}_{T-1}, \widetilde{\delta}_T)$ is a martingale, cf. Lemma (13).

We are now ready to formulate and prove what is sometimes known as 'the fundamental theorem of asset pricing' in a version with discrete time and finite state space:

Theorem 15 *In our security market model the following statements are equivalent:*

1. *There are no arbitrage opportunities.*
2. *There exists an equivalent martingale measure.*

Proof. We have already seen that no arbitrage is equivalent to the existence of a consistent price functional F. Therefore, what we show in the following is that there is a one-to-one correspondence between consistent price functionals (up to multiplication by a positive scalar) and equivalent martingale measures. We will need the following notation for the restriction of F to an $\mathcal{F}_t-$measurable random variable: Let δ^X be a dividend process whose only payout is X at time t. Define

$$F_t(X) = F(\delta^X).$$

If we assume (as we do from now on) that $F_0(1) = 1$, we may think of $F_t(1_A)$ as the price a time 0 of a claim (if it trades) paying off 1 at time t if $\omega \in A$. Note that since we have a assumed the existence of a money market account, we have

$$F_T(R_{0,T}) = 1 \tag{5.6}$$

First, assume there is no arbitrage and let F be a consistent price functional. Our candidate as equivalent martingale measure is defined as follows:

$$Q(A) = F_T(1_A R_{0,T}) \qquad A \in \mathcal{F} \equiv \mathcal{F}_T. \tag{5.7}$$

By the strict positivity, linearity and (5.6) we see that Q is a probability measure which is strictly positive on all ω.We may write (5.7) as

$$E^Q 1_A = F_T(1_A R_{0,T}) \qquad A \in \mathcal{F} \equiv \mathcal{F}_T$$

and by writing a random variable X as a sum of constants times indicator functions, we note that

$$E^Q(X) = F_T(X R_{0,T}) \tag{5.8}$$

Now we want to check the condition (5.5). By definition (33) this is equivalent to showing that for every security we have

$$E^Q(1_A \widetilde{S}_t^i) = E^Q\left(1_A \sum_{j=t+1}^{T} \widetilde{\delta}_j^i\right) \qquad t = 1, \dots, T. \tag{5.9}$$

Consider for given $A \in \mathcal{F}_t$ the following trading strategy ϕ:

- Buy one unit of stock i at time 0 (this costs S_t^i). Invest all dividends before time t in the money market account and keep them there at least until time t.
- At time t, if $\omega \in A$ (and this we know at time t since $A \in \mathcal{F}_t$) sell the security and invest the proceeds in the money market account, i.e. buy S_t^i units of the $0'$th security and roll over the money until time T.
- If $\omega \notin A$, then hold the i'th security to time T.

This strategy clearly only requires an initial payment of S_0^i. The dividend process generated by this strategy is non-zero only at time 0 and at time T. At time T the dividend is

$$\begin{aligned}\delta_T^\phi &= 1_A R_{t,T}\left(S_t^i + \sum_{j=1}^{t} \delta_j^i R_{j,t}\right) + 1_{A^c} \sum_{j=1}^{T} \delta_j^i R_{j,T} \\ &= 1_A R_{0,T}\left(\widetilde{S}_t^i + \sum_{j=1}^{t} \widetilde{\delta}_j^i\right) + 1_{A^c} \sum_{j=1}^{T} \delta_j^i R_{j,T}\end{aligned}$$

One could also choose to just buy the i'th security and then roll over the dividends to time T. Call this strategy ψ.This would generate a terminal dividend which we may write in a complicated but useful way as

$$\begin{aligned}\delta_T^\psi &= 1_A \sum_{j=1}^{T} \delta_j^i R_{j,T} + 1_{A^c} \sum_{j=1}^{T} \delta_j^i R_{j,T} \\ &= 1_A R_{0,T} \sum_{j=1}^{T} \widetilde{\delta}_j^i + 1_{A^c} \sum_{j=1}^{T} \delta_j^i R_{j,T}\end{aligned}$$

The dividend stream of both strategies at time 0 is $-S_0^i$. We therefore have

$$F_T(\delta_T^\phi) = F_T(\delta_T^\psi)$$

which in turn implies

$$F_T\left(1_A R_{0,T}\left(\widetilde{S}_t^i + \sum_{j=1}^{t} \widetilde{\delta}_j^i\right)\right) = F_T\left(1_A R_{0,T} \sum_{j=1}^{T} \widetilde{\delta}_j^i\right)$$

i.e.

$$F_T\left(1_A R_{0,T} \widetilde{S}_t^i\right) = F_T\left(1_A R_{0,T} \sum_{j=t+1}^{T} \widetilde{\delta}_j^i\right).$$

Now use (5.8) to conclude that

$$E^Q(1_A \widetilde{S}_t^i) = E^Q(1_A \sum_{j=t+1}^{T} \widetilde{\delta}_j^i)$$

and that is what we needed to show. Q is an equivalent martingale measure.

Now assume that Q is an equivalent martingale measure. Define for an arbitrary dividend process δ

$$F(\delta) = E^Q \sum_{j=0}^{T} \widetilde{\delta}_j$$

Clearly, F is linear and strictly positive. Now consider the dividend process δ^ϕ generated by some trading strategy ϕ. To show consistency we need to show that

$$\phi_0 \cdot \widetilde{S}_0 = E^Q \sum_{j=1}^{T} \widetilde{\delta}_j^\phi.$$

Notice that we know that for individual securities we have

$$\widetilde{S}_0^i = E^Q \sum_{j=1}^{T} \widetilde{\delta}_j^i.$$

We only need to extend that to portfolios. We do some calculations (where we make good use of the rule $E^Q E_j = E^Q E_{j-1}$)

$$\begin{aligned} E^Q \sum_{j=1}^{T} \widetilde{\delta}_j^\phi &= E^Q\left(\sum_{j=1}^{T} \phi_{j-1} \cdot (\widetilde{S}_j + \widetilde{\delta}_j) - \phi_j \cdot \widetilde{S}_j\right) \\ &= E^Q\left(\sum_{j=1}^{T} \phi_{j-1} \cdot \left(E_j^Q\left(\sum_{k=j}^{T} \widetilde{\delta}_k\right)\right) - \phi_j \cdot E_j^Q\left(\sum_{k=j+1}^{T} \widetilde{\delta}_k\right)\right) \end{aligned}$$

$$
\begin{aligned}
&= E^Q\left(\sum_{j=1}^{T}\phi_{j-1}\cdot\left(E_{j-1}^Q\left(\sum_{k=j}^{T}\widetilde{\delta}_k^i\right)\right)-\sum_{j=2}^{T}\phi_{j-1}\cdot E_{j-1}^Q\left(\sum_{k=j}^{T}\widetilde{\delta}_k^i\right)\right)\\
&= E^Q\left(\phi_0\cdot\left(E_0^Q\sum_{k=1}^{T}\widetilde{\delta}_k^i\right)\right)\\
&= E^Q\left(\phi_0\cdot\widetilde{S}_0\right)\\
&= \phi_0\cdot\widetilde{S}_0
\end{aligned}
$$

This completes the proof. ■

Earlier, we established a one-to-one correspondence between consistent price functionals (normed to 1 at date 0) and equivalent martingale measures. Therefore we have also proved the following

Corollary 16 *Assume the security model is arbitrage-free. Then the market is complete if and only if the equivalent martingale measure is unique.*

Another immediate consequence from the definition of consistent price functionals and equivalent martingale measures is the following

Corollary 17 *Let the security model defined by (S,δ) (including the money market account) on $(\Omega,P,\mathcal{F},\mathbb{F})$ be arbitrage-free and complete. Then the augmented model obtained by adding a new pair (S^{N+1},δ^{N+1}) of security price and dividend processes is arbitrage-free if and only if*

$$
\widetilde{S}_t^{N+1}=E_t^Q\left(\sum_{j=t+1}^{T}\widetilde{\delta}_j^{N+1}\right) \tag{5.10}
$$

i.e.

$$
\frac{S_t^{N+1}}{R_{0,t}}=E_t^Q\left(\sum_{j=t+1}^{T}\frac{\delta_j^{N+1}}{R_{0,j}}\right)
$$

where Q is the unique equivalent martingale measure for (S,δ).

In the special case where the discount rate is deterministic the expression simplifies somewhat. For ease of notation assume that the spot interest rate is not only deterministic but also constant and let $R=1+\rho$. Then (5.10) becomes

$$
\begin{aligned}
S_t^{N+1} &= R^tE_t^Q\left(\sum_{j=t+1}^{T}\frac{\delta_j^{N+1}}{R_{0,j}}\right) \qquad (5.11)\\
&= E_t^Q\left(\sum_{j=t+1}^{T}\frac{\delta_j^{N+1}}{R^{j-t}}\right)
\end{aligned}
$$

5.6 One-period submodels

Before we turn to applications we note a few results for which we do not give proofs. The results show that the one-period model which we analyzed earlier actually is very useful for analyzing multi-period models as well.

Given the market model with the $N-$dimensional security price process S and dividend process δ and assume that a money market account exists as well. Let $A_t \in \mathcal{P}_t$ and let

$$N(A_t) \equiv \left|\{B \in \mathcal{P}_{t+1} : B \subseteq A_t\}\right|.$$

This number is often referred to as the splitting index at A_t. In our graphical representation where the set A_t is represented as a node in a graph, the splitting index at A_t is simply the number of vertices leaving that node. At each such node we can define a one-period submodel as follows: Let

$$\pi(t, A_t) \equiv \left(1, S_t^1(A_t), \dots, S_t^N(A_t)\right).$$

Denote by $B_1, \dots, B_{N(A_t)}$ the members of $\mathcal{P}_{t+1}$ which are subsets of A_t and define

$$D(t, A_t) \equiv \begin{pmatrix} 1+\rho_t(A_t) & \cdots & 1+\rho_t(A_t) \\ S_{t+1}^0(B_1)+\delta_{t+1}^0(B_1) & \vdots & S_{t+1}^0(B_{N(A_t)})+\delta_{t+1}^0(B_{N(A_t)}) \vdots \\ \vdots & & \vdots \\ S_{t+1}^N(B_1)+\delta_{t+1}^N(B_1) & \cdots & S_{t+1}^N(B_{N(A_t)})+\delta_{t+1}^N(B_{N(A_t)}) \end{pmatrix}.$$

Then the following results hold:

Proposition 18 *The security market model is arbitrage-free if and only if the one-period model $(\pi(t, A_t), D(t, A_t))$ is arbitrage-free for all (t, A_t) where $A_t \in \mathcal{P}_t$.*

Proposition 19 *The security market model is complete if and only if the one-period model $(\pi(t, A_t), D(t, A_t))$ is complete for all (t, A_t) where $A_t \in \mathcal{P}_t$.*

In the complete, arbitrage-free case we obtain from each one-period submodel a unique state price vector $\psi(t, A_t)$ and by following the same procedure as outlined in chapter (4) we may decompose this into a discount factor, which will be $1+\rho_t(A_t)$, and a probability measure $q_1, \dots, q_{N(A_t)}$. The probabilities thus obtained are then the conditional probabilities $q_i = Q(B_i \,|A_t)$ for $i = 1, \dots, N(A_t)$. From these conditional probabilities the martingale measure can be obtained.

The usefulness of these local results is that we often build multi-period models by repeating the same one-period structure.We may then check absence of arbitrage and completeness by looking at a one-period submodel instead of the whole tree.

5.7 The multi-period model on matrix form

As a final curiosity we note that it is in fact possible to embed a multi-period model into one giant one-period model by stacking the one-period submodels defined above into a giant matrix. Instead of giving the abstract notation for how this is done, we indicate for the two-period model of chapter (5) how this is done. Consider the following table in which we have defined a set of 'elementary securities':

	0	1,1	1,2	2,1	2,2	2,3	2,4
S_1^0	-1	R	R	0	0	0	0
S_1^1	$-S$	uS	dS	0	0	0	0
$S^0(\{\omega_1,\omega_2\})$	0	$-R$	0	R^2	R^2	0	0
$S^1(\{\omega_1,\omega_2\})$	0	$-uS$	0	u^2S	duS	0	0
$S^0(\{\omega_3,\omega_4\})$	0	0	$-R$	0	0	R^2	R^2
$S^1(\{\omega_3,\omega_4\})$	0	0	$-dS$	0	0	udS	d^2S

Each elementary security is to be thought of as arising from buying the security at one node and selling at the successor nodes. The pairs 1,1 ; 1,2 etc. in the top row are to be read as date 1, partition element 1; date 1 partition element 2, etc. Note that the setup is very much as in the application of Stiemke's lemma to one-period models in that we include negative prices for one date and positive prices for the subsequent date. Define

$$D_{big} = \begin{pmatrix} -1 & R & R & 0 & 0 & 0 & 0 \\ -S & uS & dS & 0 & 0 & 0 & 0 \\ 0 & -R & 0 & R^2 & R^2 & 0 & 0 \\ 0 & -uS & 0 & u^2S & duS & 0 & 0 \\ 0 & 0 & -R & 0 & 0 & R^2 & R^2 \\ 0 & 0 & -dS & 0 & 0 & udS & d^2S \end{pmatrix}$$

What you can check for yourself now is that we can define a trading strategy as a vector $\theta \in \mathbb{R}^6$ and then interpret $D_{big}^{top}\theta$ as the dividend process generated by the trading strategy. A self-financing strategy would be one for which the dividend process was non-zero at all dates $1, \ldots, T-1$ (although this could easily be relaxed to a definition of self-financing up to a liquidation date $t < T$). Arbitrage may then be defined as a trading strategy generating a positive, non-zero dividend stream. If the market is arbitrage-free, the corresponding vector of state prices is an element of $\mathbb{R}^7$ and if we normalize the first component to be 1, the state prices correspond to time-zero prices of securities delivering one unit of account at nodes of the tree.

We will not go further into this but note that it may be a useful way of representing a multi-period model when one wants to introduce short-selling constraints into the model.