# Lecture Notes for the course
# Investerings- og Finansieringsteori.

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# Chapter 1

# Preface

These notes are intended for the introductory course 'Investerings- og Finansieringsteori' given in the third year of the joint mathematics-economics program at the University of Copenhagen. At this stage they are still far from complete. The notes (the dominant part of which are written by DL) aim to fill a gap between elementary textbooks such as Copeland and Weston[1] or Brealey and Myers[2], and more advanced books which require knowledge of finance theory and often cover continuous-time modelling, such as Duffie[3] and Campbell, Lo and MacKinlay[4] and Leroy and Werner.[5]

Except for a brief introduction to the Black-Scholes model, the aim is to present important parts of the theory of finance through discrete-time models emphasizing definitions and setups which prepare the students for the study of continuous-time models.

At this stage the notes have no historical accounts and hardly references any original papers or existing standard textbooks. This will be remedied in later versions but at this stage, in addition to the books already mentioned, we would like to acknowledge having included things we learned from the classic Hull [6], the also recommendable Luenberger[7], as well as Jarrow and

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[1] T. Copeland and F. Weston: Financial Theory and Corporate Policy

[2] Brealey and Myers: Principles of Corporate Finance.McGraw-Hill 4th ed. 1991.

[3] Duffie, D: Dynamic Asset Pricing Theory.
3rd ed. Princeton 2001.

[4] Campbell, J., A. Lo and A.C. MacKinlay: The Econometrics of Financial Markets. Princeton 1997.

[5] LeRoy, S. L. and J. Werner: Principles of Financial Economics, Cambridge 2001.

[6] Hull, J.: Options, Futures and Other Derivative Securities. Prentice-Hall. 4th ed. 1999

[7] Luenberger, D., "Investment Science", Oxford, 1997.

Turnbull[8], and Jensen. [9]

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[8] Jarrow R. and S. Turnbull: Derivative Securities.Cincinnati: South-Western (1996).

[9] Jensen, B.A. Rentesregning. DJØFs forlag. 2001.

# Chapter 2

# Introduction

A student applying for student loans is investing in his or her human capital. Typically, the income of a student is not large enough to cover living expenses, books etc., but the student is hoping that the education will provide future income which is more than enough to repay the loans. The government subsidizes students because it believes that the future income generated by highly educated people will more than compensate for the costs of subsidy, for example through productivity gains and higher tax revenues.

A first time home buyer is typically not able to pay the price of the new home up front but will have to borrow against future income and using the house as collateral.

A company which sees a profitable investment opportunity may not have sufficient funds to launch the project (buy new machines, hire workers) and will seek to raise capital by issuing stocks and/or borrowing money from a bank.

The student, the home buyer and the company are all in need of money to invest now and are confident that they will earn enough in the future to pay back loans that they might receive.

Conversely, a pension fund receives payments from members and promises to pay a certain pension once members retire.

Insurance companies receive premiums on insurance contracts and delivers a promise of future payments in the events of property damage or other unpleasant events which people wish to insure themselves against.

A new lottery millionaire would typically be interested in investing his or her fortune in some sort of assets (government bonds for example) since this will provide a larger income than merely saving the money in a mattress.

The pension fund, the insurance company and the lottery winner are all looking for profitable ways of placing current income in a way which will provide income in the future.

A key role of financial markets is to find efficient ways of connecting the demand for capital with the supply of capital. The examples above illustrated the need for economic agents to substitute income intertemporally. An equally important role of financial markets is to allow risk averse agents (such as insurance buyers) to share risk.

In understanding the way financial markets allocate capital we must understand the chief mechanism by which it performs this allocation, namely through prices. Prices govern the flow of capital, and in financial markets investors will compare the price of some financial security with its promised future payments. A very important aspect of this comparison is the riskiness of the promised payments. We have an intuitive feeling that it is reasonable for government bonds to give a smaller expected return than stocks in risky companies, simply because the government is less likely to default. But exactly how should the relationship between risk and reward (return on an investment) be in a well functioning market? Trying to answer that question is a central part of this course. The best answers delivered so far are in a set of mathematical models developed over the last 40 years or so. One set of models, CAPM and APT, consider expected return and variance on return as the natural definitions of reward and risk, respectively and tries to answer how these should be related. Another set of models are based on arbitrage pricing, which is a very powerful application of the simple idea, that two securities which deliver the same payments should have the same price. This is typically illustrated through option pricing models and in the modelling of bond markets, but the methodology actually originated partly in work which tried to answer a somewhat different question, which is an essential part of financial theory as well: How should a firm finance its investments? Should it issue stocks and/or bonds or maybe something completely different? How should it (if at all) distribute dividends among shareholders? The so-called Modigliani-Miller theorems provide a very important starting point for studying these issues which currently are by no means resolved.

A historical survey of how finance theory has evolved will probably be more interesting at the end of the course since we will at that point understand versions of the central models of the theory.

But let us start by considering a classical explanation of the significance of financial markets in a microeconomic setting.

## 2.1 The Role of Financial Markets

Consider the definition of a private ownership economy as in Debreu (1959): Assume for simplicity that there is only one good and one firm with production set $Y$. The $i$th consumer is characterized by a consumption set $X_i$, a preference preordering $\preceq_i$, an endowment $\omega_i$ and shares in the firm $\theta_i$. Given a price system $p$, and given a profit maximizing choice of production $y$, the firm then has a profit of $\pi(p) = p \cdot y$ and this profit is distributed to shareholders such that the wealth of the $i$th consumer becomes

$$w_i = p \cdot \omega_i \ + \theta_i \pi(p) \tag{2.1}$$

The definition of an equilibrium in such an economy then has three seemingly natural requirements: The firm maximizes profits, consumers maximize utility subject to their budget constraint and markets clear, i.e. consumption equals the sum of initial resources and production. But why should the firm maximize its profits? After all, the firm has no utility function, only consumers do. But note that given a price system $p$, the shareholders of the firm all agree that it is desirable to maximize profits, for the higher profits the larger the consumers wealth, and hence the larger is the set of feasible consumption plans, and hence the larger is the attainable level of utility. In this way the firm's production choice is separated from the shareholders' choice of consumption. There are many ways in which we could imagine shareholders disagreeing over the firm's choice of production. Some examples could include cases where the choice of production influences on the consumption sets of the consumers, or if we relax the assumption of price taking behavior, where the choice of production plan affects the price system and thereby the initial wealth of the shareholders. Let us, by two examples, illustrate in what sense the price system changes the behavior of agents.

**Example 1** Consider a single agent who is both a consumer and a producer. The agent has an initial endowment $e_0 > 0$ of the date 0 good and has to divide this endowment between consumption at date 0 and investment in production of a time 1 good. Assume that only non-negative consumption is allowed. Through investment in production, the agent is able to transform an input of $i_0$ into $f(i_0)$ units of date 1 consumption. The agent has a utility function $U(c_0, c_1)$ which we assume is strictly increasing. The agent's problem is then to maximize utility of consumption, i.e. to maximize $U(c_0, c_1)$ subject to the constraints $c_0 + i_0 \leq e_0$ and $c_1 = f(i_0)$ and we may rewrite this problem as

$$\begin{aligned} \max\ v(c_0) \ &\equiv\ U(c_0, f(e_0 - c_0)) \\ s.t.\ c_0 \ &\leq\ e_0 \end{aligned}$$

If we impose regularity conditions on the functions $f$ and $U$ (for example that they are differentiable and strictly concave and that utility of zero consumption in either period is $-\infty$) then we know that at the maximum $c_0^*$ we will have $0 < c_0^* < e_0$ and $v'(c_0^*) = 0$ i.e.

$$D_1U(c_0^*, f(e_0 - c_0^*)) \cdot 1 - D_2U(c_0^*, f(e_0 - c_0^*))f'(e_0 - c_0^*) = 0$$

where $D_1$ means differentiation after the first variable. Defining $i_0^*$ as the optimal investment level and $c_1^* = f(e_0 - c_0^*)$, we see that

$$f'(i_0^*) = \frac{D_1U(c_0^*, c_1^*)}{{}_2U(c_0^*, c_1^*)}$$

and this condition merely says that the marginal rate of substitution in production is equal to the marginal rate of substitution of consumption.

The key property to note in this example is that what determines the production plan in the absence of prices is the preferences for consumption of the consumer. If two consumers with no access to trade owned shares in the same firm, but had different preferences and identical initial endowments, they would bitterly disagree on the level of the firm's investment.

**Example 2** Now consider the setup of the previous example but assume that a price system $(p_0, p_1)$ (whose components are strictly positive) gives the consumer an additional means of transferring date 0 wealth to date 1 consumption. Note that by selling one unit of date 0 consumption the agent acquires $\frac{p_0}{p_1}$ units of date 1 consumption, and we define $1 + r = \frac{p_0}{p1}$. The initial endowment must now be divided between three parts: consumption at date 0 $c_0$, input into production $i_0$ and $s_0$ which is sold in the market and whose revenue can be used to purchase date 1 consumption in the market.

With this possibility the agent's problem becomes that of maximizing $U(c_0, c_1)$ subject to the constraints

$$\begin{aligned} c_0 + i_0 + s_0 &\leq e_0 \\ c_1 &\leq f(i_0) + (1+r)s_0 \end{aligned}$$

and with monotonicity constraints the inequalities may be replaced by equalities. Note that the problem then may be reduced to having two decision variables $c_0$ and $i_0$ and maximizing

$$v(c_0, i_0) \equiv U(c_0, f(i_0) + (1+r)(e_0 - c_0 - i_0)).$$

Again we may impose enough regularity conditions on $U$ (strict concavity, twice differentiability, strong aversion to zero consumption) to ensure that it attains its maximum in an interior point of the set of feasible pairs $(c_0, i_0)$ and that at this point the gradient of $v$ is zero, i.e.

$$\begin{aligned} D_1U(c_0^*, c_1^*) \cdot 1 - D_2U(c_0^*, f(i_0^*) + (1+r)(e_0 - c_0^* - i_0^*))(1+r) &= 0 \\ D_2U(c_0^*, f(i_0^*) + (1+r)(e_0 - c_0^* - i_0^*))(f'(i_0^*) - (1+r)) &= 0 \end{aligned}$$

With the assumption of strictly increasing $U$, the only way the second equality can hold, is if

$$f^{'}(i_0^*) = (1+r)$$

and the first equality holds if

$$\frac{D_1U(c_0^*, c_1^*)}{D_2U(c_0^*, c_1^*)} = (1+r)$$

We observe two significant features:

First, the production decision is independent of the utility function of the agent. Production is chosen to a point where the marginal benefit of investing in production is equal to the 'interest rate' earned in the market. The consumption decision is separate from the production decision and the marginal condition is provided by the market price. In such an environment we have what is known as Fisher Separation where the firm's decision is independent of the shareholder's utility functions. Such a setup rests critically on the assumptions of the perfect competitive markets where there is price taking behavior and a market for both consumption goods at date 0. Whenever we speak of firms having the objective of maximizing shareholders' wealth we are assuming an economy with a setup similar to that of the private ownership economy of which we may think of the second example as a very special case.

Second, the solution to the maximization problem will typically have a higher level of utility for the agent at the optimal point: Simply note that any feasible solution to the first maximization problem is also a solution to the second. This is an improvement which we take as a 'proof' of the significance of the existence of markets. If we consider a private ownership economy equilibrium, the equilibrium price system will see to that consumers and producers coordinate their activities simply by following the price system and they will obtain higher utility than if each individual would act without a price system as in example 1.