CHAPTER 17

Diversification and Asset Allocation

Intuitively, we all know that diversification is important for managing investment risk. But how exactly does diversification work, and how can we be sure we have an efficiently diversified portfolio? Insightful answers can be gleaned from the modern theory of diversification and asset allocation.

In this chapter, we examine the role of diversification and asset allocation in investing. Most of us have a strong sense that diversification is important. After all, “Don’t put all your eggs in one basket” is a bit of folk wisdom that seems to have stood the test of time quite well. Even so, the importance of diversification has not always been well understood. For example, noted author and market analyst Mark Twain recommended: “Put all your eggs in the one basket and—WATCH THAT BASKET!” This chapter shows why this was probably not Twain’s best piece of advice.[1]

As we will see, diversification has a profound effect on portfolio risk and return. The role and impact of diversification were first formally explained in the early 1950's by financial pioneer Harry Markowitz, who shared the 1986 Nobel Prize in Economics for his insights. The primary goal of this chapter is to explain and explore the implications of Markowitz’s remarkable discovery.

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[1] This quote has been attributed to both Mark Twain (*The Tragedy of Pudd’nhead Wilson*, 1894) and Andrew Carnegie (*How to Succeed in Life*, 1903).

17.1 Expected Returns and Variances

In Chapter 1, we discussed how to calculate average returns and variances using historical data. We now begin to discuss how to analyze returns and variances when the information we have concerns future possible returns and their probabilities.

Expected Returns

We start with a straightforward case. Consider a period of time such as a year. We have two stocks, say Netcap and Jmart. Netcap is expected to have a return of 25 percent in the coming year; Jmart is expected to have a return of 20 percent during the same period.

In a situation such as this, if all investors agreed on these expected return values, why would anyone want to hold Jmart? After all, why invest in one stock when the expectation is that another will do better? Clearly, the answer must depend on the different risks of the two investments. The return on Netcap, although it is *expected* to be 25 percent, could turn out to be significantly higher or lower. Similarly, Jmart's *realized* return could be significantly higher or lower than expected.

For example, suppose the economy booms. In this case, we think Netcap will have a 70 percent return. But if the economy tanks and enters a recession, we think the return will be -20 percent. In this case, we say that there are *two states of the economy*, which means that there are two possible outcomes. This scenario is oversimplified of course, but it allows us to illustrate some key ideas without a lot of computational complexity.

Suppose we think boom and recession are equally likely to happen, that is, a 50-50 chance of each outcome. Table 17.1 illustrates the basic information we have described and some additional information about Jmart. Notice that Jmart earns 30 percent if there is a recession and 10 percent if there is a boom.

Table 17.1 States of the Economy and Stock Returns

		Security Returns if State Occurs	
State of Economy	**Probability of State of Economy**	**Netcap**	**Jmart**
Recession	.50	-20%	30%
Boom	.50	70	10
	1.00		

Obviously, if you buy one of these stocks, say, Jmart, what you earn in any particular year depends on what the economy does during that year. Suppose these probabilities stay the same through time. If you hold Jmart for a number of years, you'll earn 30 percent about half the time and 10 percent the other half. In this case, we say your **expected return** on Jmart, $E(R_J)$, is 20 percent:

$$E(R_J) = .50 \times 30\% + .50 \times 10\% = 20\%$$

In other words, you should expect to earn 20 percent from this stock, on average.

(*marg. def.* **expected return** Average return on a risky asset expected in the future.)

For Netcap, the probabilities are the same, but the possible returns are different. Here we lose 20 percent half the time, and we gain 70 percent the other half. The expected return on Netcap, $E(R_N)$ is thus 25 percent:

$$E(R_N) = .50 \times -20\% + .50 \times 70\% = 25\%$$

Table 17.2 illustrates these calculations.

Table 17.2 Calculating Expected Returns

		Netcap		Jmart	
(1) State of Economy	**(2) Probability of State of Economy**	**(3) Return if State Occurs**	**(4) Product (2) × (3)**	**(5) Return if State Occurs**	**(6) Product (2) × (5)**
Recession	0.50	-20%	-.10	30%	.15
Boom	0.50	70	.35	10	.05
	1.00	$E(R_N)$ =	25%	$E(R_J)$ =	20%

In Chapter 1, we defined a risk premium as the difference between the returns on a risky investment and a risk-free investment, and we calculated the historical risk premiums on some different investments. Using our projected returns, we can calculate the *projected* or *expected risk premium* as the difference between the expected return on a risky investment and the certain return on a risk-free investment.

For example, suppose risk-free investments are currently offering 8 percent. We will say that the risk-free rate, which we label R_f, is 8 percent. Given this, what is the projected risk premium on Jmart? On Netcap? Since the expected return on Jmart, $E(R_J)$, is 20 percent, the projected risk premium is

$$\begin{aligned}\text{Risk premium} &= \text{Expected return - Risk-free rate} \qquad [17.1]\\ &= E(R_J) - R_f\\ &= 20\% - 8\%\\ &= 12\%\end{aligned}$$

Similarly, the risk premium on Netcap is 25% - 8% = 17%.

In general, the expected return on a security or other asset is simply equal to the sum of the possible returns multiplied by their probabilities. So, if we have 100 possible returns, we would multiply each one by its probability and then add up the results. The sum would be the expected return. The risk premium would then be the difference between this expected return and the risk-free rate.

Example 17.1 Unequal Probabilities. Look again at Tables 17.1 and 17.2. Suppose you thought a boom would occur 20 percent of the time instead of 50 percent. What are the expected returns on Netcap and Jmart in this case? If the risk-free rate is 10 percent, what are the risk premiums?

The first thing to notice is that a recession must occur 80 percent of the time (1 - .20 = .80) since there are only two possibilities. With this in mind, Jmart has a 30 percent return in 80 percent of the years and a 10 percent return in 20 percent of the years. To calculate the expected return, we just multiply the possibilities by the probabilities and add up the results:

$$E(R_J) = .80 \times 30\% + .20 \times 10\% = 26\%$$

Table 17.3 summarizes the calculations for both stocks. Notice that the expected return on Netcap is -2 percent.

Table 17.3 Calculating Expected Returns

		Netcap		Jmart	
(1) State of Economy	**(2) Probability of State of Economy**	**(3) Return if State Occurs**	**(4) Product (2) × (3)**	**(5) Return if State Occurs**	**(6) Product (2) × (5)**
Recession	0.80	-20%	-.16	30%	.24
Boom	0.20	70	.14	10	.02
	1.00	$E(R_N)$ =	-2%	$E(R_J)$ =	26%

The risk premium for Jmart is 26% - 10% = 16% in this case. The risk premium for Netcap is negative: -2% - 10% = -12%. This is a little unusual, but, as we will see, it's not impossible.

Calculating the Variance

To calculate the variances of the returns on our two stocks, we first determine the squared deviations from the expected return. We then multiply each possible squared deviation by its probability. Next we add these up, and the result is the variance.

To illustrate, one of our stocks above, Jmart, has an expected return of 20 percent. In a given year, the return will actually be either 30 percent or 10 percent. The possible deviations are thus 30%-20% = 10% or 10% - 20% = -10%. In this case, the variance is

$$\text{Variance} = \sigma^2 = .50 \times (10\%)^2 + .50 \times (-10\%)^2 = .01$$

The standard deviation is the square root of this:

$$\text{Standard deviation} = \sigma = \sqrt{.01} = .10 = 10\%$$

Table 17.4 summarizes these calculations and the expected return for both stocks. Notice that Netcap has a much larger variance. Netcap has the higher return, but Jmart has less risk. You could get a 70 percent return on your investment in Netcap, but you could also lose 20 percent. Notice that an investment in Jmart will always pay at least 10 percent.

Table 17.4 Expected Returns and Variances

	Netcap	Jmart
Expected return, E(R)	25%	20%
Variance, σ^2	.2025	.0100
Standard deviation, σ	45%	10%

Which of these stocks should you buy? We can't really say; it depends on your personal preferences regarding risk and return. We can be reasonably sure, however, that some investors would prefer one and some would prefer the other.

You've probably noticed that the way we calculated expected returns and variances here is somewhat different from the way we did it in Chapter 1 (and, probably, different from the way you learned it in "sadistics"). The reason is that we were examining historical returns in Chapter 1, so we estimated the average return and the variance based on some actual events. Here, we have projected *future* returns and their associated probabilities, so this is the information with which we must work.

Example 17.2 More Unequal Probabilities. going back to Table 17.3 in Example 17.1, what are the variances on our two stocks once we have unequal probabilities? What are the standard deviations?

We can summarize the needed calculations as follows:

(1) State of Economy	**(2) Probability of State of Economy**	**(3) Return Deviation from Expected Return**	**(4) Squared Return Deviation**	**(5) Product (2) × (4)**
Netcap				
Recession	.80	-.20 - (-.02) = -.18	.0324	.02592
Boom	.20	.70 - (-.02) = .72	.5184	.10368
			σ^2_N =	.12960
Jmart				
Recession	.80	.30 - .26 = .04	.0016	.00128
Boom	.20	.10 - .26 = -.16	.0256	.00512
			σ^2_J =	.00640

Based on these calculations, the standard deviation for Netcap is $\sigma_N = \sqrt{.1296}$ = 36%. The standard deviation for Jmart is much smaller, $\sigma_J = \sqrt{.0064}$, or 8 percent.

CHECK THIS

17.1a How do we calculate the expected return on a security?

17.1b In words, how do we calculate the variance of an expected return?

(*marg. def.* **portfolio** Group of assets such as stocks and bonds held by an investor.)

17.2 Portfolios

Thus far in this chapter, we have concentrated on individual assets considered separately. However, most investors actually hold a **portfolio** of assets. All we mean by this is that investors tend to own more than just a single stock, bond, or other asset. Given that this is so, portfolio return and portfolio risk are of obvious relevance. Accordingly, we now discuss portfolio expected returns and variances.

(*marg. def.* **portfolio weight** Percentage of a portfolio's total value invested in a particular asset.)

Portfolio Weights

There are many equivalent ways of describing a portfolio. The most convenient approach is to list the percentages of the total portfolio's value that are invested in each portfolio asset. We call these percentages the **portfolio weights**.

For example, if we have $50 in one asset and $150 in another, then our total portfolio is worth $200. The percentage of our portfolio in the first asset is $50/$200 = .25. The percentage of our portfolio in the second asset is $150/$200 = .75. Notice that the weights sum up to 1.00 since all of our money is invested somewhere.[2]

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[2]Some of it could be in cash, of course, but we would then just consider cash to be another of the portfolio assets.

Portfolio Expected Returns

Let's go back to Netcap and Jmart. You put half your money in each. The portfolio weights are obviously .50 and .50. What is the pattern of returns on this portfolio? The expected return?

To answer these questions, suppose the economy actually enters a recession. In this case, half your money (the half in Netcap) loses 20 percent. The other half (the half in Jmart) gains 30 percent. Your portfolio return, R_P, in a recession will thus be:

$$R_P = .50 \times -20\% + .50 \times 30\% = 5\%$$

Table 17.5 summarizes the remaining calculations. Notice that when a boom occurs, your portfolio would return 40 percent:

$$R_P = .50 \times 70\% + .50 \times 10\% = 40\%$$

As indicated in Table 17.5, the expected return on your portfolio, $E(R_P)$, is 22.5 percent.

Table 17.5 Expected Portfolio Return

(1) State of Economy	(2) Probability of State of Economy	(3) Portfolio Return if State Occurs	(4) Product (2) × (3)
Recession	.50	.50 × -20% + .50 × 30% = 5%	.025
Boom	.50	.50 × 70% + .50 × 10% = 40%	.200
		$E(R_P)$ =	22.5%

We can save ourselves some work by calculating the expected return more directly. Given these portfolio weights, we could have reasoned that we expect half our money to earn 25 percent (the half in Netcap) and half of our money to earn 20 percent (the half in Jmart). Our portfolio expected return is thus

$$\begin{aligned} E(R_P) &= .50 \times E(R_N) + .50 \times E(R_J) \\ &= .50 \times 25\% + .50 \times 20\% \\ &= 22.5\% \end{aligned}$$

This is the same portfolio return that we calculated in Table 17.5.

This method of calculating the expected return on a portfolio works no matter how many assets there are in the portfolio. Suppose we had n assets in our portfolio, where n is any number at all. If we let x_i stand for the percentage of our money in Asset i, then the expected return is

$$E(R_P) = x_1 \times E(R_1) + x_2 \times E(R_2) + \ldots + x_n \times E(R_n) \qquad [17.2]$$

This says that the expected return on a portfolio is a straightforward combination of the expected returns on the assets in that portfolio. This seems somewhat obvious, but, as we will examine next, the obvious approach is not always the right one.

Example 17.3 More Unequal Probabilities. Suppose we had the following projections on three stocks:

State of Economy	**Probability of State of Economy**	**Returns**		
		Stock A	**Stock B**	**Stock C**
Boom	.50	10%	15%	20%
Bust	.50	8	4	0

We want to calculate portfolio expected returns in two cases. First, what would be the expected return on a portfolio with equal amounts invested in each of the three stocks? Second, what would be the expected return if half of the portfolio were in A, with the remainder equally divided between B and C?

From our earlier discussion, the expected returns on the individual stocks are

$$E(R_A) = 9.0\%$$
$$E(R_B) = 9.5\%$$
$$E(R_C) = 10.0\%$$

Check these for practice. If a portfolio has equal investments in each asset, the portfolio weights are all the same. Such a portfolio is said to be *equally weighted*. Since there are three stocks in this case, the weights are all equal to **a**. The portfolio expected return is thus

E(R_P) = **a** × 9.0% + **a** × 9.5% + **a** × 10.0% = 9.5%

In the second case, check that the portfolio expected return is 8.4%.

Portfolio Variance

From the preceding discussion, the expected return on a portfolio that contains equal investments in Netcap and Jmart is 22.5 percent. What is the standard deviation of return on this portfolio? Simple intuition might suggest that half of our money has a standard deviation of 45 percent, and the other half has a standard deviation of 10 percent. So the portfolio's standard deviation might be calculated as follows:

$$\sigma_P = .50 \times 45\% + .50 \times 10\% = 27.5\%$$

Unfortunately, this approach is *completely* incorrect!

Let's see what the standard deviation really is. Table 17.6 summarizes the relevant calculations. As we see, the portfolio's variance is about .031, and its standard deviation is less than we thought—it's only 17.5 percent. What is illustrated here is that the variance on a portfolio is *not* generally a simple combination of the variances of the assets in the portfolio.

Table 17.6 Calculating Portfolio Variance

(1) State of Economy	(2) Probability of State of Economy	(3) Portfolio Returns if State Occurs	(4) Squared Deviation from Expected Return	(5) Product (2) × (4)
Recession	.50	5%	$(.05 - .225)^2 = .030625$	.0153125
Boom	.50	40	$(.40 - .225)^2 = .030625$	.0153125
			$\sigma^2_P =$	.030625
			$\sigma_P = \sqrt{.030625} =$	17.5%

We can illustrate this point a little more dramatically by considering a slightly different set of portfolio weights. Suppose we put 2/11 (about 18 percent) in Netcap and the other 9/11 (about 82 percent) in Jmart. If a recession occurs, this portfolio with have a return of

$$R_P = 2/11 \times -20\% + 9/11 \times 30\% = 20.91\%$$

If a boom occurs, this portfolio will have a return of

$$R_P = 2/11 \times 70\% + 9/11 \times 10\% = 20.91\%$$

Notice that the return is the same no matter what happens. No further calculation is needed: This portfolio has a *zero* variance and no risk!

This is a nice bit of financial alchemy. We take two quite risky assets and by mixing them just right, we create a riskless portfolio. It seems very clear that combining assets into portfolios can substantially alter the risks faced by an investor. This is a crucial observation, and we will begin to explore its implications in the next section.[3]

[3]Earlier, we had a risk-free rate of 8 percent. Now we have, in effect, a 20.91 percent risk-free rate. If this situation actually existed, there would be a very profitable arbitrage opportunity! In reality, we expect that all riskless investments would have the same return.

Example 17.4 Portfolio Variance and Standard Deviations. In Example 17.3, what are the standard deviations of the two portfolios?

To answer, we first have to calculate the portfolio returns in the two states. We will work with the second portfolio, which has 50 percent in Stock A and 25 percent in each of Stocks B and C. The relevant calculations are summarized as follows:

State of the Economy	Probability of State of the Economy	Returns			
		Stock A	**Stock B**	**Stock C**	**Portfolio**
Boom	.50	10%	15%	20%	13.75%
Bust	.50	8	4	0	5.00

The portfolio return when the economy booms is calculated as

$$R_P = .50 \times 10\% + .25 \times 15\% + .25 \times 20\% = 13.75\%$$

The return when the economy goes bust is calculated the same way. Check that it's 5 percent and also check that the expected return on the portfolio is 8.5 percent. The variance is thus

$$\sigma^2_P = .40 \times (.1375 - .085)^2 + .60 \times (.05 - .085)^2 = .0018375$$

The standard deviation is thus about 4.3 percent. For our equally weighted portfolio, redo these calculations and check that the standard deviation is about 5.4 percent.

CHECK THIS

17.2a What is a portfolio weight?

17.2b How do we calculate the variance of an expected return?

17.3 Diversification and Portfolio Risk

Our discussion to this point has focused on some hypothetical securities. We've seen that portfolio risks can, in principle, be quite different from the risks of the assets that make up the portfolio. We now look more closely at the risk of an individual asset versus the risk of a portfolio of many different assets. As we did in Chapter 1, we will examine some stock market history to get an idea of what happens with actual investments in U.S. capital markets.

The Effect of Diversification: Another Lesson from Market History

In Chapter 1, we saw that the standard deviation of the annual return on a portfolio of large common stocks was about 20 percent per year. Does this mean that the standard deviation of the annual return on a typical stock in that group is about 20 percent? As you might suspect by now, the answer is no. This is an extremely important observation.

To examine the relationship between portfolio size and portfolio risk, Table 17.7 illustrates typical average annual standard deviations for equally weighted portfolios that contain different numbers of randomly selected NYSE securities.

Table 17.7 about here

In column 2 of Table 17.7, we see that the standard deviation for a "portfolio" of one security is just under 50 percent per year at 49.24 percent. What this means is that if you randomly select a single NYSE stock and put all your money into it, your standard deviation of return would typically have been about 50 percent per year. Obviously, such a strategy has significant risk! If you were to randomly select two NYSE securities and put half your money in each, your average annual standard deviation would have been about 37 percent.

The important thing to notice in Table 17.7 is that the standard deviation declines as the number of securities is increased. By the time we have 100 randomly chosen stocks (and 1 percent invested in each), the portfolio's volatility has declined by 60 percent, from 50 percent per year to 20 percent per year. With 500 securities, the standard deviation is 19.27 percent per year, similar to the 20 percent per year we saw in Chapter 1 for large common stocks. The small difference exists because the portfolio securities, portfolio weights, and the time periods covered are not identical.

The Principle of Diversification

Figure 17.1 illustrates the point we've been discussing. What we have plotted is the standard deviation of the return versus the number of stocks in the portfolio. Notice in Figure 17.1 that the benefit in terms of risk reduction from adding securities drops off as we add more and more. By the time we have 10 securities, most of the diversification effect is already realized, and by the time we get to 30 or so, there is very little remaining benefit. In other words, the benefit of further diversification increases at a decreasing rate, so the "law of diminishing returns" applies here as it does in so many other places.

Figure 17.1 about here

(*marg. def.* **principle of diversification** Spreading an investment across a number of assets will eliminate some, but not all, of the risk.)

Figure 17.1 illustrates two key points. First, some of the riskiness associated with individual assets can be eliminated by forming portfolios. The process of spreading an investment across assets (and thereby forming a portfolio) is called *diversification*. The **principle of diversification** tells us that spreading an investment across many assets will eliminate some of the risk. Not surprisingly, risks that can be eliminated by diversification are called "diversifiable" risks.

The second point is equally important. There is a minimum level of risk that cannot be eliminated by simply diversifying. This minimum level is labeled "nondiversifiable risk" in Figure 17.1. Taken together, these two points are another important lesson from financial market history: Diversification reduces risk, but only up to a point. Put another way, some risk is diversifiable and some is not.

CHECK THIS

17.3a What happens to the standard deviation of return for a portfolio if we increase the number of securities in the portfolio?

17.3b What is the principle of diversification?

17.4 Correlation and Diversification

We've seen that diversification is important. What we haven't discussed is how to get the most out of diversification. For example, in our previous section, we investigated what happens if we simply spread our money evenly across randomly chosen stocks. We saw that significant risk reduction resulted from this strategy, but you might wonder whether even larger gains could be achieved by a more sophisticated approach. As we begin to examine that question here, the answer is yes.

(*marg. def.* **correlation** The tendency of the returns on two assets to move together.)

Why Diversification Works

Why diversification reduces portfolio risk as measured by the portfolio standard deviation is important and worth exploring in some detail. The key concept is **correlation**, which is the extent to which the returns on two assets move together. If the returns on two assets tend to move up and down together, we say they are *positively* correlated. If they tend to move in opposite directions, we say they are *negatively* correlated. If there is no particular relationship between the two assets, we say they are *uncorrelated*.

The *correlation coefficient*, which we use to measure correlation, ranges from -1 to +1, and we will denote the correlation between the returns on two assets, say A and B, as Corr(R_A, R_B). The Greek letter ρ (rho) is often used to designate correlation as well. A correlation of +1 indicates that the two assets have a *perfect* positive correlation. For example, suppose that whatever return Asset A realizes, either up or down, Asset B does the same thing by exactly twice as much. In this case, they are perfectly correlated because the movement on one is completely predictable from the movement on the other. Notice, however, that perfect correlation does not necessarily mean they move by the same amount.

A zero correlation means that the two assets are uncorrelated. If we know that one asset is up, then we have no idea what the other one is likely to do; there simply is no relation between them. Perfect negative correlation (Corr(R_A, R_B) = -1) indicates that they always move in opposite directions. Figure 17.2 illustrates the three benchmark cases of perfect positive, perfect negative, and zero correlation.

Figure 17.2 about here

Diversification works because security returns are generally not perfectly correlated. We will be more precise about the impact of correlation on portfolio risk in just a moment. For now, it is useful to simply think about combining two assets into a portfolio. If the two assets are highly correlated (the correlation is near +1), then they have a strong tendency to move up and down together. As a result, they offer limited diversification benefit. For example, two stocks from the same industry, say, General Motors and Ford, will tend to be relatively highly correlated since the companies are in essentially the same business, and a portfolio of two such stocks is not likely to be very diversified.

In contrast, if the two assets are negatively correlated, then they tend to move in opposite directions; whenever one zigs, the other tends to zag. In such a case, there will be substantial diversification benefit because variation in the return on one asset tends to be offset by variation in the opposite direction from the other. In fact, if two assets have a perfect negative correlation (Corr(R_A, R_B) = -1) then it is possible to combine them such that all risk is eliminated. Looking back at our example involving Jmart and Netcap in which we were able to eliminate all of the risk, what we now see is that they must be perfectly negatively correlated.

Table 17.8 Annual Returns on Stocks A and B

Year	Stock A	Stock B	Portfolio AB
1995	10%	15%	12.5%
1996	30	-10	10
1997	-10	25	7.5
1998	5	20	12.5
1999	10	15	12.5
Average returns	9%	13%	11%
Standard deviations	14.3%	13.5%	2.2%

To further illustrate the impact of diversification on portfolio risk, suppose we observed the actual annual returns on two stocks, A and B, for the years 1995 - 1999. We summarize these returns in Table 17.8: In addition to actual returns on stocks A and B, we also calculated the returns on an equally weighted portfolio of A and B. We label this portfolio as AB. In 1996, for example, Stock A returned 10 percent and Stock B returned 15 percent. Since Portfolio AB is half invested in each, its return for the year was

$$\tfrac{1}{2} \times 10\% + \tfrac{1}{2} \times 15\% = 12.5\%$$

The returns for the other years are calculated similarly.

At the bottom of Table 17.8, we calculated the average returns and standard deviations on the two stocks and the equally-weighted portfolio. These averages and standard deviations are calculated just as they were in Chapter 1 (check a couple just to refresh your memory). The impact of diversification is apparent. The two stocks have standard deviations in the 13 percent to 14 percent per year range, but the portfolio's volatility is only 2.2 percent. In fact, if we compare the portfolio to Stock B, it has a higher return (11 percent versus 9 percent) and much less risk.

Figure 17.3 illustrates in more detail what is occurring with our example. Here we have three bar graphs showing the year-by-year returns on Stocks A and B and Portfolio AB. Examining the graphs, we see that in 1996, for example, Stock A earned 30 percent while Stock B lost 10 percent. The following year, Stock B earned 25 percent while A lost 10 percent. These ups and downs tend to cancel out in our portfolio, however, with the result that there is much less variation in return from year to year. In other words, the correlation between the returns on stocks A and B is relatively low.

Figure 17.3 about here

Calculating the correlation between stocks A and B is not difficult, but it would require us to digress a bit. Instead, we will explain the needed calculation in the next chapter where we build on the principles developed here.

Calculating Portfolio Risk

We've seen that correlation is an important determinant of portfolio risk. To further pursue this issue, we need to know how to calculate portfolio variances directly. For a portfolio of two assets, *A* and *B*, the variance of the return on the portfolio, σ_P^2, is given by Equation 17.3:

$$\sigma_P^2 = x_A^2\sigma_A^2 + x_B^2\sigma_B^2 + 2x_A x_B \sigma_A \sigma_B Corr(R_A,R_B) \qquad [17.3]$$

In this equation, x_A and x_B are the percentages invested in assets A and B. Notice that $x_A + x_B = 1$ (Why?).

Equation 17.3 looks a little involved, but its use is straightforward. For example, suppose Stock A has a standard deviation of 40 percent per year and Stock B has a standard deviation of 60 percent per year. The correlation between them is .15. If you put half your money in each, what is your portfolio standard deviation?

To answer, we just plug the numbers in to Equation 17.3. Note that x_A and x_B are each equal to .50, while σ_A and σ_B are .40 and .60, respectively. Taking Corr(R_A, R_B) = .15, we have

$$\begin{aligned}\sigma_P^2 &= .50^2\times.40^2 + .50^2\times.60^2 + 2\times.50\times.50\times.40\times.60\times.15 \\ &= .25\times.16 + .25\times.36 + .018 \\ &= .148\end{aligned}$$

Thus, the portfolio variance is .148. As always, variances are not easy to interpret since they are based on squared returns, so we calculate the standard deviation by taking the square root:

$$\sigma_P = \sqrt{.148} = .3847 = 38.47\%$$

Once again, we see the impact of diversification. This portfolio has a standard deviation of 38.47 percent, which is less than either of the standard deviations on the two assets that are in the portfolio.

Example 17.5 Portfolio Variance and Standard Deviation. In the example we just examined, Stock A has a standard deviation of 40 percent per year and Stock B has a standard deviation of 60 percent per year. Suppose now that the correlation between them is .35. Also suppose you put one-fourth of your money in Stock A. What is your portfolio standard deviation?

If you put ¼ (or .25) in Stock A, you must have ¾ (or .75) in Stock B, so $x_A = .25$ and $x_B = .75$. Making use of our portfolio variance equation (17.3), we have

$$\begin{aligned}\sigma_P^2 &= .25^2\times.40^2 + .75^2\times.60^2 + 2\times.25\times.75\times.40\times.60\times.35\\ &= .0625\times.16 + .5625\times.36 + .0315\\ &= .244\end{aligned}$$

Thus the portfolio variance is .244. Taking the square root, we get

$$\sigma_P = \sqrt{.244} = .49396 \approx 49\%$$

This portfolio has a standard deviation of 49 percent, which is between the individual standard deviations. This shows that a portfolio's standard deviation isn't necessarily less than the individual standard deviations.

To illustrate why correlation is an important, practical, real-world consideration, suppose that as a very conservative, risk-averse investor, you decide to invest all of your money in a bond mutual fund. Based on your analysis, you think this fund has an expected return of 6 percent with a standard deviation of 10 percent per year. A stock fund is available, however, with an expected return of 12 percent, but the standard deviation of 15 percent is too high for your taste. Also, the correlation between the returns on the two funds is about .10.

Is the decision to invest 100 percent in the bond fund a wise one, even for a very risk-averse investor? The answer is no; in fact, it is a bad decision for any investor. To see why, Table 17.9 shows expected returns and standard deviations available from different combinations of the two mutual funds. In constructing the table, we begin with 100 percent in the stock fund and work our way down to 100 percent in bond fund by reducing the percentage in the stock fund in increments of .05. These calculations are all done just like our examples just above; you should check some (or all) of them for practice.

Table 17.9 Risk and Return with Stocks and Bonds

Portfolio Weights		Expected	Standard
Stocks	**Bonds**	**Return**	**Deviation**
1.00	0.00	12.00%	15.00%
0.95	0.05	11.70	14.31
0.90	0.10	11.40	13.64
0.85	0.15	11.10	12.99
0.80	0.20	10.80	12.36
0.75	0.25	10.50	11.77
0.70	0.30	10.20	11.20
0.65	0.35	9.90	10.68
0.60	0.40	9.60	10.28
0.55	0.45	9.30	9.78
0.50	0.50	9.00	9.42
0.45	0.55	8.70	9.12
0.40	0.60	8.40	8.90
0.35	0.65	8.10	8.75
0.30	0.70	7.80	8.69
0.25	0.75	7.50	8.71
0.20	0.80	7.20	8.82
0.15	0.85	6.90	9.01
0.10	0.90	6.60	9.27
0.05	0.95	6.30	9.60
0.00	1.00	6.00	10.00

Beginning on the first row in Table 17.9, we have 100 percent in the stock fund, so our expected return is 12 percent, and our standard deviation is 15 percent. As we begin to move out of the stock fund and into the bond fund, we are not surprised to see both the expected return and the standard deviation decline. However, what might be surprising to you is the fact that the standard deviation falls only so far and then begins to rise again. In other words, beyond a point, adding more of the lower risk bond fund actually *increases* your risk!

The best way to see what is going on is to plot the various combinations of expected returns and standard deviations calculated in Table 17.9 as do in Figure 17.4. We simply placed the standard deviations from Table 17.9 on the horizontal axis and the corresponding expected returns on the vertical axis.

Figure 17.4 about here

(*marg. def.* **investment opportunity set** Collection of possible risk-return combinations available from portfolios of individual assets)

Examining the plot in Figure 17.4, we see that the various combinations of risk and return available all fall on a smooth curve (in fact, for the geometrically inclined, it's a hyperbola). This curve is called an **investment opportunity set** because it shows the possible combinations of risk and return available from portfolios of these two assets. One important thing to notice is that, as we have shown, there is a portfolio that has the smallest standard deviation (or variance - same thing) of all. It is labeled "minimum variance portfolio" in Figure 17.4. What are (approximately) its expected return and standard deviation?

Now we see clearly why a 100 percent bonds strategy is a poor one. With a 10 percent standard deviation, the bond fund offers an expected return of 6 percent. However, Table 17.9 shows us that a combination of about 60 percent stocks and 40 percent bonds has almost the same standard deviation, but a return of about 9.6 percent. Comparing 9.6 percent to 6 percent, we see that this portfolio has a return that is fully 60 percent greater (6% × 1.6 = 9.6%) with the same risk. Our conclusion? Asset allocation matters.

Given the apparent importance of asset allocation, it is not too surprising to learn that this analysis is becoming widespread in investment practice. The nearby Investment Updates box presents an article from *Forbes* discussing its use for mutual fund investors. Notice how closely the discussion tracks our development.

Investment Update: *Forbes* Egg basket analysis

Going back to Figure 17.4, notice that any portfolio that plots below the minimum variance portfolio is a poor choice because, no matter which one you pick, there is another portfolio with the same risk and a much better return. In the jargon of finance, we say that these undesirable portfolios are *dominated* and/or *inefficient*. Either way, we mean that given their level of risk, the expected return is inadequate compared to some other portfolio of equivalent risk. A portfolio that offers the highest return for its level of risk is said to be an **efficient portfolio**. In Figure 17.4, the minimum variance portfolio and all portfolios that plot above it are therefore efficient.

(*marg. def.* **efficient portfolio** A portfolio that offers the highest return for its level of risk.)

Example 17.6 More Portfolio Variance and Standard Deviation. Looking at Table 17.9, suppose you put 57.627 percent in the stock fund. What is your expected return? Your standard deviation? How does this compare with the bond fund?

If you put 57.627 percent in stocks, you must have 42.373 percent in bonds, so $x_A = .57627$ and $x_B = .42373$. Making use of our portfolio variance equation (17.3), we have

$$\begin{aligned}\sigma_P^2 &= .57627^2 \times .15^2 + .42373^2 \times .10^2 + 2 \times .57627 \times .42373 \times .15 \times .10 \times .10 \\ &= .332 \times .0225 + .180 \times .01 + .0007325 \\ &= .01\end{aligned}$$

Thus the portfolio variance is .01, so the standard deviation is .1 or 10 percent. Check that the expected return is 9.46 percent. Compared to the bond fund, the standard deviation is now identical, but the expected return is almost 350 basis points higher.

More on Correlation and the Risk-Return Trade-Off

Given the expected returns and standard deviations on the two assets, the shape of the investment opportunity set in Figure 17.4 depends on the correlation. The lower the correlation, the more bowed to the left the investment opportunity set will be. To illustrate, Figure 17.5 shows the investment opportunity for correlations of -1, 0 , and +1 for two stocks, A and B. Notice that Stock A has an expected return of 12 percent and a standard deviation of 15 percent, while Stock B has an expected return of 6 percent and a standard deviation of 10 percent. These are the same expected returns and standard deviations we used to build Figure 17.4, and the calculations are all done the same way, just the correlations are different. Notice also that we use the symbol ρ to stand for the correlation coefficient.

Figure 17.5 about here

In Figure 17.5, when the correlation is +1, the investment opportunity set is a straight line connecting the two stocks, so, as expected, there is little or no diversification benefit. As the correlation declines to zero, the bend to the left becomes pronounced. For correlations between +1 and zero, there would simply be a less pronounced bend.

Finally, as the correlation becomes negative, the bend becomes quite pronounced, and the investment opportunity set actually becomes two straight-line segments when the correlation hits -1. Notice that the minimum variance portfolio has a *zero* variance in this case.

It is sometimes desirable to be able to calculate the percentage investments needed to create the minimum variance portfolio. We will just state the result here, but a problem at the end of the chapter asks you to show that the weight on Asset A in the minimum variance portfolio, x_A^*, is

$$x_A^* = \frac{\sigma_B^2 - \sigma_A \sigma_B Corr(R_A, R_B)}{\sigma_A^2 + \sigma_B^2 - 2\sigma_A \sigma_B Corr(R_A, R_B)} \qquad [17.4]$$

In Equation 17.4, we will take Asset A to be the one with the larger standard deviation. If the standard deviations happened to be the same, then Asset A could be either

Example 17.7 Finding the Minimum Variance Portfolio Looking back at Table 17.9, what combination of the stock fund and the bond fund has the lowest possible standard deviation? What is the minimum possible standard deviation?

Recalling that the standard deviations for the stock fund and bond fund were .15 and .10 and noting that the correlation was .1, we have

$$\begin{aligned} x_A^* &= \frac{.10^2 - .15\times.10\times.10}{.15^2 + .10^2 - 2\times.15\times.10\times.10} \qquad [17.4] \\ &= .288136 \\ &\approx 28.8\% \end{aligned}$$

Thus the minimum variance portfolio has 28.8 percent in stocks and the balance, 71.2 percent, in bonds. Plugging these into our formula for portfolio variance, we have

$$\begin{aligned} \sigma_P^2 &= .288^2\times.15^2 + .712^2\times.10^2 + 2\times.288\times.712\times.15\times.10\times.10 \\ &= .007551 \end{aligned}$$

The standard deviation is the square root of .007551, about 8.7 percent. Notice that, in Figure 17.5, this is where the minimum occurs.

CHECK THIS

17.4a Fundamentally, why does diversification work?

17.4b If two stocks have positive correlation, what does this mean?

17.4c What is an efficient portfolio?

17.5 The Markowitz Efficient Frontier

In the previous section, we looked closely at the risk-return possibilities available when we consider combining two risky assets. Now we are left with an obvious question: What happens when we consider combining three or more risky assets? As we will see, at least on a conceptual level, the answer turns out to be a straightforward extension of our previous analysis.

Risk and Return with Multiple Assets

When we consider multiple assets, the formula for computing portfolio standard deviation becomes cumbersome; indeed a great deal of calculation is required once we have much beyond two assets. As a result, although the required calculations are not difficult, they can be very tedious and are best relegated to a computer. We therefore will not delve into how to calculate portfolio standard deviations when there are many assets.

Figure 17.6 shows the result of calculating the expected returns and portfolio standard deviations when there are three assets. To illustrate the importance of asset allocation, we calculated expected returns and standard deviations from portfolios composed of three key investment types: U.S. stocks, foreign (non-U.S.) stocks, and U.S. bonds. These asset classes are not highly correlated in general; we assume a zero correlation in all cases. The expected returns and standard deviations are as follows:

	Expected Returns	Standard Deviations
Foreign stocks	18%	35%
U.S. stocks	12	22
U.S. bonds	8	14

Figure 17.6 about here

In Figure 17.6, each point plotted is a possible risk-return combination. Comparing the result with our two-asset case in Figure 17.4, we see that now not only do some assets plot below the minimum variance portfolio on a smooth curve, but we also have portfolios plotting inside as well. Only combinations that plot on the upper left-hand boundary are efficient; all the rest are inefficient. This upper left-hand boundary is called the **Markowitz efficient frontier**, and it represents the set of portfolios with the maximum return for a given standard deviation.

(*marg. def.* **Markowitz efficient frontier** The set of portfolios with the maximum return for a given standard deviation.)

Once again, Figure 17.6 makes it clear that asset allocation matters. For example, a portfolio of 100 percent U.S. stocks is highly inefficient. For the same standard deviation, there is a portfolio with an expected return almost 400 basis points, or 4 percent, higher. Or, for the same expected return, there is a portfolio with about half as much risk!

The analysis in this section can be extended to any number of assets or asset classes. In principle, it is possible to compute efficient frontiers using thousands of assets. As a practical matter, however, this analysis is most widely used with a relatively small number of asset classes. For example, most investment banks maintain so-called model portfolios. These are simply recommended asset allocation strategies typically involving three to six asset categories.

A primary reason that the Markowitz analysis is not usually extended to large collections of individual assets has to do with data requirements. The inputs into the analysis are (1) expected returns on all assets; (2) standard deviations on all assets; and (3) correlations between every pair of assets. Moreover, these inputs have to be measured with some precision, or we just end up with a garbage-in, garbage-out (GIGO) system.

Suppose we just look at 2,000 NYSE stocks. We need 2,000 expected returns and standard deviations. This is already a problem since returns on individual stocks cannot be predicted with precision at all. To make matters worse, however, we need to know the correlation between every *pair* of stocks. With 2,000 stocks, there are 2,000 × 1,999 / 2 = 1,999,000, or almost 2 million unique pairs![4] Also, as with expected returns, correlations between individual stocks are very difficult to predict accurately. We will return to this issue in our next chapter, where we show that there may be an extremely elegant way around the problem.

CHECK THIS

17.5a What is the Markowitz efficient frontier?

17.5b Why is Markowitz portfolio analysis most commonly used to make asset allocation decisions?

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[4]With 2,000 stocks, there are $2{,}000^2$ = 4,000,000 possible pairs. Of these, 2,000 involve pairing a stock with itself. Further, we recognize that the correlation between A and B is the same as the correlation between B and A, so we only need to actually calculate half of the remaining 3,998,000 correlations.

17.6 Summary and Conclusions

In this chapter, we covered the basics of diversification and portfolio risk and return. From this material we saw that:

1. A portfolio's expected return is a simple weighted combination of the expected returns on the assets in the portfolio, but the standard deviation on a portfolio is not.

2. Diversification is a very important consideration. The principle of diversification tells us that spreading an investment across many assets can reduce some, but not all, of the risk. Based on U.S. stock market history, for example, about 60 percent of the risk associated with owning individual stocks can be eliminated by naive diversification.

3. Diversification works because asset returns are not perfectly correlated. All else the same, the lower the correlation, the greater is the gain from diversification.

4. When we consider the possible combinations of risk and return available from portfolios of assets, we find that some are inefficient (or dominated), meaning that they offer too little return for their risk.

5. Finally, for any group of assets, there is a set that is efficient. That set is known as the Markowitz efficient frontier.

The most important thing to carry away from this chapter is an understanding of diversification and why it works. Once you understand this, then the importance of asset allocation follows immediately. Our story is not complete, however, because we have not considered one important asset class: riskless assets. This will be the first task in our next chapter.

Key Terms

expected return	**correlation**
portfolio	**investment opportunity set**
portfolio weight	**efficient portfolio**
principle of diversification	**Markowitz efficient frontier**

Get Real!

This chapter explained diversification, a very important consideration for real-world investors and money managers. The chapter also explore the famous Markowitz efficient portfolio concept, which shows how (and why) asset allocation affects portfolio risk and return.

Building a diversified portfolio is not a trivial task. Of course, as we discussed many chapters ago, mutual funds provide one way for investors to build diversified portfolios, but there are some significant caveats concerning mutual funds as a diversification tool. First of all, investors sometimes assume a fund is diversified simply because it holds a relatively large number of stocks. However, with the exception of some index funds, most mutual funds will reflect a particular style of investing, either explicity as stated in the fund's objective or implicitly as favored by the fund manager. For example, in the mid- to late 1990's, stocks as a whole did very well, but mutual funds that concentrated on smaller stocks generally did not do well at all.

It is tempting to buy a number of mutual funds to ensure broad diversification, but even this may not work. Within a given fund family, the same manager may actually be responsible for multiple funds. In addition, managers within a large fund family frequently have similar views about the market and individual companies.

Thinking just about stocks for the moment, what does an investor need to consider to build a well-diversified portfolio? At a minimum, such a portfolio probably needs to be diversified across industries, with no undue concentrations in particular sectors of the economy, it needs to be diversified by company size (small, midcap, and large), and it needs to be diversified across "growth" (i.e., high P/E) and "value" (low P/E) stocks. Perhaps the most controversial diversification issue concerns international diversification. The correlation between international stock exchanges is surprisingly low, suggesting large benefits from diversifying globally.

Perhaps the most disconcerting fact about diversification is that it leads to the following paradox: A well-diversified portfolio will always be invested in something that does not do well! Put differently, such a portfolio will almost always have both winners and losers. In many ways, that's the whole idea. Even so, it requires a lot of financial discipline to stay diversified when some portion of your portfolio seems to be doing poorly. The payoff is that, over the long run, a well-diversified portfolio should provide much steadier returns and be much less prone to abrupt changes in value.

Chapter 17
Diversification and Asset Allocation

End of Chapter Questions and problems

Review Problems and Self-Test

Use the following table of states of the economy and stock returns to answer the review problems:

		Security Returns if State Occurs	
State of Economy	**Probability of State of Economy**	Roten	Bradley
Bust	.40	-10%	30%
Boom	.60	40	10
	1.00		

1. **Expected Returns** Calculate the expected returns for Roten and Bradley.

2. **Standard Deviations** Calculate the standard deviations for Roten and Bradley.

3. **Portfolio Expected Returns** Calculate the expected return on a portfolio of 50 percent Roten and 50 percent Bradley.

4. **Portfolio Volatility** Calculate the volatility of a portfolio of 50 percent Roten and 50 percent Bradley.

Answers to Self-Test Problems

1. We calculate the expected return as follows:

		Roten		Bradley	
(1) State of Economy	**(2) Probability of State of Economy**	**(3) Return if State Occurs**	**(4) Product (2) × (3)**	**(5) Return if State Occurs**	**(6) Product (2) × (5)**
Bust	.40	-10%	-.04	30%	.12
Boom	.60	40%	.24	10%	.06
		$E(R) =$	20%	$E(R) =$	18%

2. We calculate the standard deviation as follows:

(1) State of Economy	**(2) Probability of State of Economy**	**(3) Return Deviation from Expected Return**	**(4) Squared Return Deviation**	**(5) Product (2) × (4)**
Roten				
Bust	.40	-.30	.09	.036
Boom	.60	.20	.04	.024
			$\sigma^2 =$	.06
Bradley				
Bust	.40	.12	.0144	.00576
Boom	.60	-.08	.0064	.00384
			$\sigma^2 =$	.0096

Taking square roots, the standard deviations are 24.495 percent and 9.798 percent.

3. We calculate the expected return on a portfolio of 50 percent Roten and 50 percent Bradley as follows:

(1) State of Economy	**(2) Probability of State of the Economy**	**(3) Portfolio Return if State Occurs**	**(4) Product (2) × (3)**
Bust	.40	10%	.04
Boom	.60	25%	.15
		$E(R_p) =$	19%

4. We calculate the volatility of a portfolio of 50 percent Roten and 50 percent Bradley as follows:

(1) State of Economy	**(2) Probability of State of Economy**	**(3) Portfolio Returns if State Occurs**	**(4) Squared Deviation from Expected Return**	**(5) Product (2) × (4)**
Bust	.40	.10	.0081	.00324
Boom	.60	.25	.0036	.00216
			σ^2_P =	.0054
			σ_P =	7.3485%

Test Your IQ (Investment Quotient)

1. **Diversification** Netcap has an expected return of 25 percent and Jmart has an expected return of 20 percent. What is the likely investment decision for a risk-averse investor?

a. invest all funds in Netcap
b. invest all funds in Jmart
c. do not invest any funds in Netcap and Jmart
d. invest funds partly in Netcap and partly in Jmart

2. **Return Standard Deviation** Netcap experiences returns of 5 percent or 45 percent, each with an equal probability. What is the return standard deviation for Netcap?

a. 30 percent
b. 25 percent
c. 20 percent
d. 10 percent

3. **Return Standard Deviation** Jmart experiences returns of 0 percent, 25 percent, or 50 percent, each with a one-third probability. What is the approximate return standard deviation for Jmart?

a. 30 percent
b. 25 percent
c. 20 percent
d. 10 percent

4. **Expected Return** An analyst estimates that a stock has the following return probabilities and returns, depending on the state of the economy:

State of Economy	Probability	Return
Good	.1	15%
Normal	.6	13
Poor	.3	7

What is the expected return of the stock? (*1994 CFA Exam*)

a. 7.8 percent
b. 11.4 percent
c. 11.7 percent
d. 13.0 percent

5. **Risk Premium** Netcap has an expected return of 25 percent, Jmart has an expected return of 20 percent, and the risk-free rate is 5 percent. You invest half your funds in Netcap and the other half in Jmart. What is the risk premium for your portfolio?

a. 20 percent
b. 17.5 percent
c. 15 percent
d. 12.5 percent

6. **Return Standard Deviation** Both Netcap and Jmart have the same return standard deviation of 20 percent, and Netcap and Jmart returns have zero correlation. You invest half your funds in Netcap and the other half in Jmart. What is the return standard deviation for your portfolio?

a. 20 percent
b. 14.14 percent
c. 10 percent
d. 0 percent

7. **Return Standard Deviation** Both Netcap and Jmart have the same return standard deviation of 20 percent, and Netcap and Jmart returns have a correlation of +1. You invest half your funds in Netcap and the other half in Jmart. What is the return standard deviation for your portfolio?

a. 20 percent
b. 14.14 percent
c. 10 percent
d. 0 percent

8. **Return Standard Deviation** Both Netcap and Jmart have the same return standard deviation of 20 percent, and Netcap and Jmart returns have a correlation of -1. You invest half your funds in Netcap and the other half in Jmart. What is the return standard deviation for your portfolio?

a. 20 percent
b. 14.14 percent
c. 10 percent
d. 0 percent

9. **Minimum Variance Portfolio** Both Netcap and Jmart have the same return standard deviation of 20 percent, and Netcap and Jmart returns have zero correlation. What is the minimum attainable return variance for a portfolio of Netcap and Jmart?

a. 20 percent
b. 14.14 percent
c. 10 percent
d. 0 percent

10. **Minimum Variance Portfolio** Both Netcap and Jmart have the same return standard deviation of 20 percent, and Netcap and Jmart returns have a correlation of -1. What is the minimum attainable return variance for a portfolio of Netcap and Jmart?

a. 20 percent
b. 14.14 percent
c. 10 percent
d. 0 percent

11. **Minimum Variance Portfolio** Stocks A, B, and C each have the same expected return and standard deviation. The following shows the correlations between returns on these stocks.

	Stock A	Stock B	Stock C
Stock A	+1.0		
Stock B	+0.9	+1.0	
Stock C	+0.1	-0.4	+1.0

Given these correlations, which of the following portfolios constructed from these stocks would have the lowest risk? (*1994 CFA Exam*)

a. equally invested in stocks A and B
b. equally invested in stocks A and C
c. equally invested in stocks B and C
d. totally invested in stock C

12. **Markowitz Efficient Frontier** Which of the following portfolios cannot lie on the efficient frontier as described by Markowitz? (*1994 CFA Exam*)

	Portfolio	Expected Return	Standard Deviation
a.	W	9%	21%
b.	X	5	7
c.	Y	15	36
d.	Z	12	15

Questions and Problems

Core Questions

1. **Expected Returns** Use the following information on states of the economy and stock returns to calculate the expected return for Dingaling Telephone:

State of Economy	Probability of State of Economy	Security Return if State Occurs
Recession	.20	-10%
Normal	.50	20%
Boom	.30	30%
	1.00	

2. **Standard Deviations** Using the information in the previous question, calculate the standard deviation of return.

3. **Expected Returns and Deviations** Repeat Questions 1 and 2 assuming that all three states are equally likely.

Use the following information on states of the economy and stock returns to answer Questions 4 - 7.

		Security Returns if State Occurs	
State of Economy	Probability of State of Economy	Roll	Ross
Bust	0.30	-10%	40%
Boom	0.70	50	10
	1.00		

4. **Expected Returns** Calculate the expected returns for Roll and Ross by filling in the following:

		Roll		Ross	
(1) State of Economy	**(2) Probability of State of Economy**	**(3) Return if State Occurs**	**(4) Product (2) × (3)**	**(5) Return if State Occurs**	**(6) Product (2) × (5)**
		E(R) =		E(R) =	

5. **Standard Deviations** Calculate the standard deviations for Roll and Ross by filling in the following:

(1) State of Economy	**(2) Probability of State of Economy**	**(3) Return Deviation from Expected Return**	**(4) Squared Return Deviation**	**(5) Product (2) × (4)**
Roll				
			σ^2 =	
Ross				
			σ^2 =	

6. **Portfolio Expected Returns** Calculate the expected return on a portfolio of 40 percent Roll and 60 percent Ross by filling in the following:

(1) State of the Economy	**(2) Probability of State of the Economy**	**(3) Portfolio Return if State Occurs**	**(4) Product (2) × (3)**
		E(R_P) =	

7. Portfolio Volatility Calculate the volatility of a portfolio of 70 percent Roll and 30 percent Ross by filling in the following:

(1) State of Economy	(2) Probability of State of Economy	(3) Portfolio Returns if State Occurs	(4) Squared Deviation from Expected Return	(5) Product (2) × (4)
			σ^2_P =	
			σ_P =	

8. Diversification and Market History Based on market history, what is the average annual standard deviation of return for a single, randomly chosen stock? What is the average annual standard deviation for an equally weighted portfolio of many stocks?

9. Interpreting Correlations If the returns on two stocks are highly correlated, what does this mean? If they have no correlation? If they are negatively correlated?

10. Efficient Portfolios What is an efficient portfolio?

11. Portfolio Returns and Volatilities Fill in the missing information in the following table. Assume that Portfolio AB is 40 percent invested in Stock A.

	Annual Returns		
Year	**Stock A**	**Stock B**	**Portfolio AB**
1995	15%	55%	
1996	35	-40	
1997	-15	45	
1998	20	0	
1999	0	10	
Average returns			
Standard deviations			

Intermediate Questions

12. **Portfolio Returns and Volatilities** Given the following information, calculate the expected return and standard deviation for a portfolio that has 40 percent invested in Stock A, 30 percent in Stock B, and the balance in Stock C.

State of Economy	Probability of State of Economy	Returns		
		Stock A	**Stock B**	**Stock C**
Boom	.40	15%	18%	20%
Bust	.60	5	0	-5

13. **Portfolio Variance** Use the following information to calculate the expected return and standard deviation of a portfolio that is 40 percent invested in Kuipers and 60 percent invested in SuCo:

	Kuipers	**SuCo**
Expected return, E(*R*)	30%	26%
Standard deviation, σ	65	45
Correlation	.30	

14. **More Portfolio Variance** In the previous question, what is the standard deviation if the correlation is +1? 0? -1? As the correlation declines from +1 to -1 here, what do you see happening to portfolio volatility? Why?

15. **Minimum Variance Portfolio** In Problem 13, what are the expected return and standard deviation on the minimum variance portfolio?

16. **Asset Allocation** Fill in the missing information assuming a correlation of -.10.

Portfolio Weights		Expected	Standard
Stocks	**Bonds**	**Return**	**Deviation**
1.00		14%	20%
0.80			
0.60			
0.40			
0.20			
0.00		5%	8%

17. **Expected Returns** True or false: If the two stocks have the same expected return of 12 percent, then any portfolio of the two stocks will also have an expected return of 12 percent.

18. **Portfolio Volatility** True or false: If the two stocks have the same standard deviation of 45 percent, then any portfolio of the two stocks will also have a standard deviation of 45 percent.

19. **Portfolio Variance** Suppose two assets have perfect positive correlation. Show that the standard deviation on a portfolio of the two assets is simply

$$\sigma_p = x_A \times \sigma_A + x_B \times \sigma_B$$

(Hint: Look at the expression for the variance of a two-asset portfolio. If the correlation is +1, the expression is a perfect square.)

20. **Portfolio Variance** Suppose two assets have perfect negative correlation. Show that the standard deviation on a portfolio of the two assets is simply

$$\sigma_p = \pm (x_A \times \sigma_A - x_B \times \sigma_B)$$

(Hint: See previous problem.)

21. **Portfolio Variance** Using the result in Problem 20, show that whenever two assets have perfect negative correlation it is possible to find a portfolio with a zero standard deviation. What are the portfolio weights? (Hint: let x be the percentage in the first asset and (1 - x) be the percentage in the second. Set the standard deviation to zero and solve for x).

22. **Portfolio Variance** Suppose two assets have zero correlation and the same standard deviation. What is true about the minimum variance portfolio?

23. **Portfolio Variance** Derive our expression in the chapter for the portfolio weight in the minimum variance portfolio. (Danger! Calculus required!) (Hint: let *x* be the percentage in the first asset and (1 - *x*) the percentage in the second. Take the derivative with respect to *x* and set it to zero. Solve for *x*).

Chapter 17
Diversification and Asset Allocation
Answers and solutions

Answers to Multiple Choice Questions

1. **D**
2. **C**
3. **C**
4. **B**
5. **B**
6. **B**
7. **A**
8. **D**
9. **B**
10. **D**
11. **C**
12. **A**

Answers to Questions and Problems

Core Questions

1. $.2 \times (-.10) + .5 \times (.20) + .3 \times (.30) = 17\%$

2. $.2 \times (-.10 - .17)^2 + .5 \times (.20 - .17)^2 + .3 \times (.30 - .17)^2 = .01159$; taking the square root, $\sigma = 10.7657\%$.

3. $(1/3) \times (-.10) + (1/3) \times (.20) + (1/3) \times (.30) = 13.333\ldots\%$

$(1/3) \times (-.10 - .17)^2 + (1/3) \times (.20 - .17)^2 + (1/3) \times (.30 - .17)^2 = .01111\ldots$; taking the square root, $\sigma = 10.5409\%$.

4.

Calculating Expected Returns

		Roll		Ross	
(1) State of Economy	(2) Probability of State of Economy	(3) Return if State Occurs	(4) Product (2) × (3)	(5) Return if State Occurs	(6) Product (2) × (5)
Bust	.30	-10%	-.03	40%	.12
Boom	.70	50%	.35	10%	.07
		$E(R)$ =	32%	$E(R)$ =	19%

5.

(1) State of Economy	(2) Probability of State of Economy	(3) Return Deviation from Expected Return	(4) Squared Return Deviation	(5) Product (2) × (4)
Roll				
Bust	.30	-.42	.1764	.05292
Boom	.70	.18	.0324	.02268
			σ^2 =	.0756
Ross				
Bust	.30	.21	.0441	.01323
Boom	.70	-.09	.0081	.00567
			σ^2 =	.0189

Taking square roots, the standard deviations are 27.4955% and 13.7477%.

6.

Expected Portfolio Return

(1) State of Economy	(2) Probability of State of Economy	(3) Portfolio Return if State Occurs	(4) Product (2) × (3)
Bust	.30	20%	.06
Boom	.70	26%	.182
		$E(R_p)$ =	24.2%

7.

Calculating Portfolio Variance

(1) State of Economy	(2) Probability of State of Economy	(3) Portfolio Returns if State Occurs	(4) Squared Deviation from Expected Return	(5) Product (2) × (4)
Bust	.30	.05	.053361	.016008
Boom	.70	.38	.009801	.006861
			σ^2_P =	.022869
			σ_P =	15.1225%

8. Based on market history, the average annual standard deviation of return for single, randomly chosen stock is about 50 percent. The average annual standard deviation for an equally-weighted portfolio of many stocks is about 20 percent, or 60 percent less.

9. If the returns on two stocks are highly correlated, they have a strong tendency to move up and down together. If they have no correlation, there is no particular connection between the two. If they are negatively correlated, they tend to move in opposite directions.

10. An efficient portfolio is one that has the highest return for its level of risk.

11. Notice that we have historical information here, so we calculate the sample average and sample standard deviation (using n - 1) just like we did in Chapter 1. Notice also that the portfolio has less risk than either asset.

Annual Returns on Stocks A and B

Year	Stock A	Stock B	Portfolio AB
1995	15%	55%	39%
1996	35	-40	-10
1997	-15	45	21
1998	20	0	8
1999	0	10	6
Avg returns	11%	14%	12.8%
Std deviations	19.17%	37.98%	18.32%

Intermediate Questions

12. *Portfolio Returns and Volatilities*. Given the following information, calculate the expected return and standard deviation for portfolio that has 40 percent invested in Stock A, 30 percent in Stock B, and the balance in Stock C.

State of Economy	Probability of State of Economy	Returns			
		Stock A	Stock B	Stock C	Portfolio
Boom	.40	15%	18%	20%	17.4%
Bust	.60	5	0	-5	.5%

$E(R_P) = .4 \times (.174) + .6 \times (.005) = 7.26\%$

$\sigma^2_P = .4 \times (.174 - .0726)^2 + .6 \times (.005 - .0726)^2 = .006855$; taking the square root, $\sigma_P = 8.2793\%$.

13. $E(R_P) = .4 \times (.30) + .6 \times (.26) = 27.6\%$
$\sigma^2_P = .4^2 \times .65^2 + .6^2 \times .45^2 + 2 \times .4 \times .6 \times .65 \times .45 \times .3 = .18626$; $\sigma_P = 42.73\%$.

14. $\sigma^2_P = .4^2 \times .65^2 + .6^2 \times .45^2 + 2 \times .4 \times .6 \times .65 \times .45 \times 1 = .2809$; $\sigma_P = 53\%$.
$\sigma^2_P = .4^2 \times .65^2 + .6^2 \times .45^2 + 2 \times .4 \times .6 \times .65 \times .45 \times 0 = .1405$; $\sigma_P = 37.48\%$.
$\sigma^2_P = .4^2 \times .65^2 + .6^2 \times .45^2 + 2 \times .4 \times .6 \times .65 \times .45 \times (-1) = .0001$; $\sigma_P = 1\%$.

15. $(.45^2 - .65 \times .45 \times .3)/(.45^2 + .65^2 - 2 \times .65 \times .45 \times .3) = .255$
$E(R_P) = .255 \times (.30) + .745 \times (.26) = 27.02\%$
$\sigma^2_P = .255^2 \times .65^2 + .745^2 \times .45^2 + 2 \times .255 \times .745 \times .65 \times .45 \times .3 = .1732$; $\sigma_P = 41.6\%$.

16.

Risk and Return with Stocks and Bonds

Portfolio Weights		Expected	Standard
Stocks	Bonds	Return	Deviation
1.00	0.00	14%	20%
0.80	0.20	12.2%	15.92%
0.60	0.40	10.4%	12.11%
0.40	0.60	8.6%	8.91%
0.20	0.80	6.8%	7.2%
0.00	1.00	5%	8%

17. True.

18. False.

19. Look at σ_p^2 :

$\sigma_p^2 = (x_A \times \sigma_A + x_B \times \sigma_B)^2$
$= x_A^2 \times \sigma_A^2 + x_B^2 \times \sigma_B^2 + 2 \times x_A \times x_B \times \sigma_A \times \sigma_B \times 1$, which is precisely the expression for the variance on a two-asset portfolio when the correlation is +1.

20. Look at σ_p^2 :

$\sigma_p^2 = (x_A \times \sigma_A - x_B \times \sigma_B)^2$
$= x_A^2 \times \sigma_A^2 + x_B^2 \times \sigma_B^2 + 2 \times x_A \times x_B \times \sigma_A \times \sigma_B \times (-1)$, which is precisely the expression for the variance on a two-asset portfolio when the correlation is -1.

21. From the previous question, with a correlation of -1:

$\sigma_p = x_A \times \sigma_A - x_B \times \sigma_B = x \times \sigma_A - (1 - x_) \times \sigma_B$

Set this to equal zero and solve for *x* to get:

$0 = x \times \sigma_A - (1 - x_) \times \sigma_B$

$x = \sigma_B / (\sigma_A + \sigma_B)$

This is the weight on the first asset.

22. If two assets have zero correlation and the same standard deviation, then evaluating the general expression for the minimum variance portfolio shows that $x = ½$; in other words, an equally-weighted portfolio is minimum variance.

23. Let ρ stand for the correlation, then:

$\sigma_p^2 = x_A^2 \times \sigma_A^2 + x_B^2 \times \sigma_B^2 + 2 \times x_A \times x_B \times \sigma_A \times \sigma_B \times \rho$

$= x^2 \times \sigma_A^2 + (1 - x)^2 \times \sigma_B^2 + 2 \times x \times (1 - x) \times \sigma_A \times \sigma_B \times \rho$

Take the derivative with respect to x and set equal to zero:

$$\frac{ds_P^2}{dx} = 2xs^2_A - 2(1-x)s^2_B + 2s_A s_B r - 4xs_A s_B r = 0$$

Solve for x to get the expression in the text.

Table 17.7 Portfolio standard deviations

Standard deviations of annual portfolio returns

(1) Number of Stocks in Portfolio	(2) Average Standard Deviation of Annual Portfolio Returns	(3) Ratio of Portfolio Standard Deviation to Standard Deviation of a Single Stock
1	49.24%	1.00
2	37.36	0.76
4	29.69	0.60
6	26.64	0.54
8	24.98	0.51
10	23.93	0.49
20	21.68	0.44
30	20.87	0.42
40	20.46	0.42
50	20.20	0.41
100	19.69	0.40
200	19.42	0.39
300	19.34	0.39
400	19.29	0.39
500	19.27	0.39
1,000	19.21	0.39

These figures are from Table 1 in Meir Statman, "How Many Stocks Make a Diversified Portfolio?" *Journal of Financial and Quantitative Analysis* 22 (September 1987), pp. 353–64. They were derived from E. J. Elton and M. J. Gruber, "Risk Reduction and Portfolio Size: An Analytic Solution," *Journal of Business* 50 (October 1977), pp. 415–37.

Figure 17.1 Portfolio diversification

Portfolio diversification

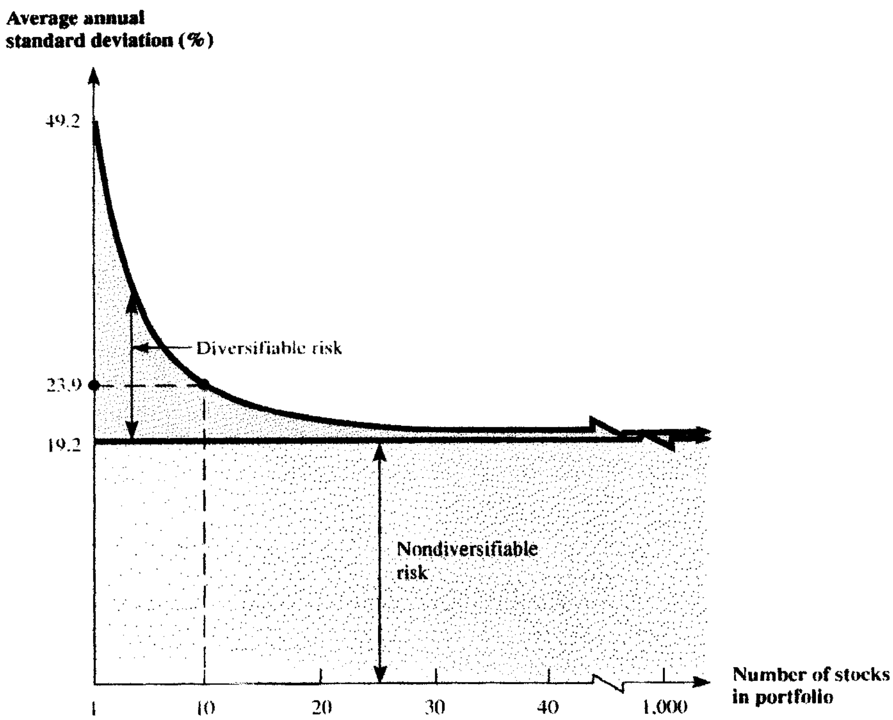
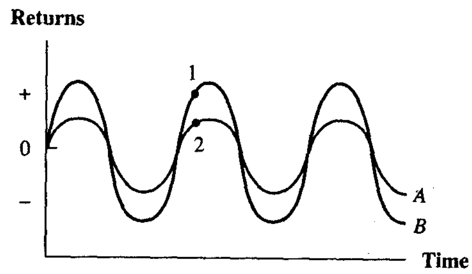


Figure 17.2 Correlations

Examples of different correlation coefficients

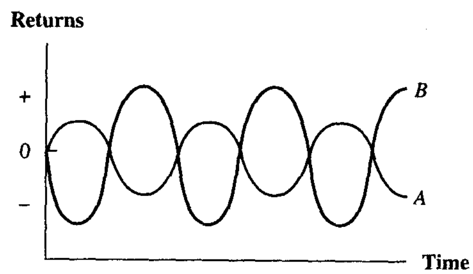
The graphs on the left-hand side of the figure plot the separate returns on the two securities through time. Each point on the graphs on the right-hand side represents the returns for both A and B over a particular time period.

Perfect positive correlation
Corr $(R_A, R_B)=1$



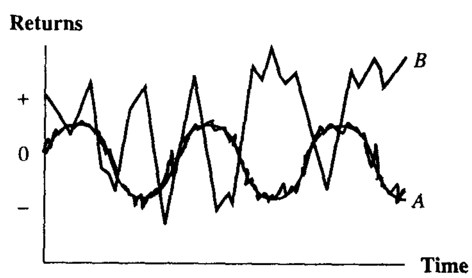
Both the return on Security *A* and the return on Security *B* are higher than average at the same time. Both the return on Security *A* and the return on Security *B* are lower than average at the same time.

Perfect negative correlation
Corr $(R_A, R_B)=-1$



Security *A* has a higher-than-average return when Security *B* has a lower-than-average return, and vice versa.

Zero correlation
Corr $(R_A, R_B)=0$

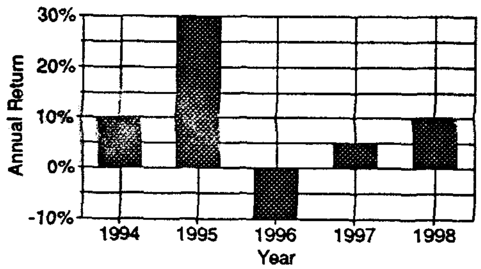


The return on Security *A* is completely unrelated to the return on Security *B*.

Figure 17.3 Impact of Diversification

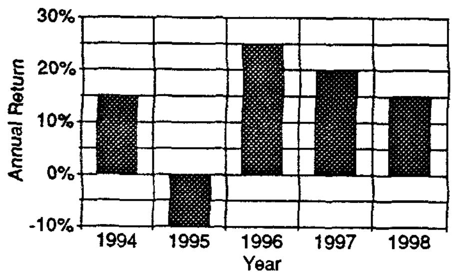
Stock A Annual Returns

1994 - 1998



Stock B Annual Returns

1994 - 1998



Portfolio AB Annual Returns

1994 - 1998

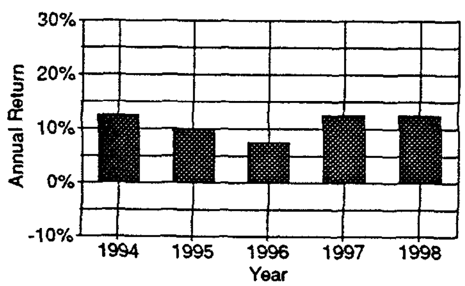


Figure 17.4

Risk and Return with Stocks and Bonds

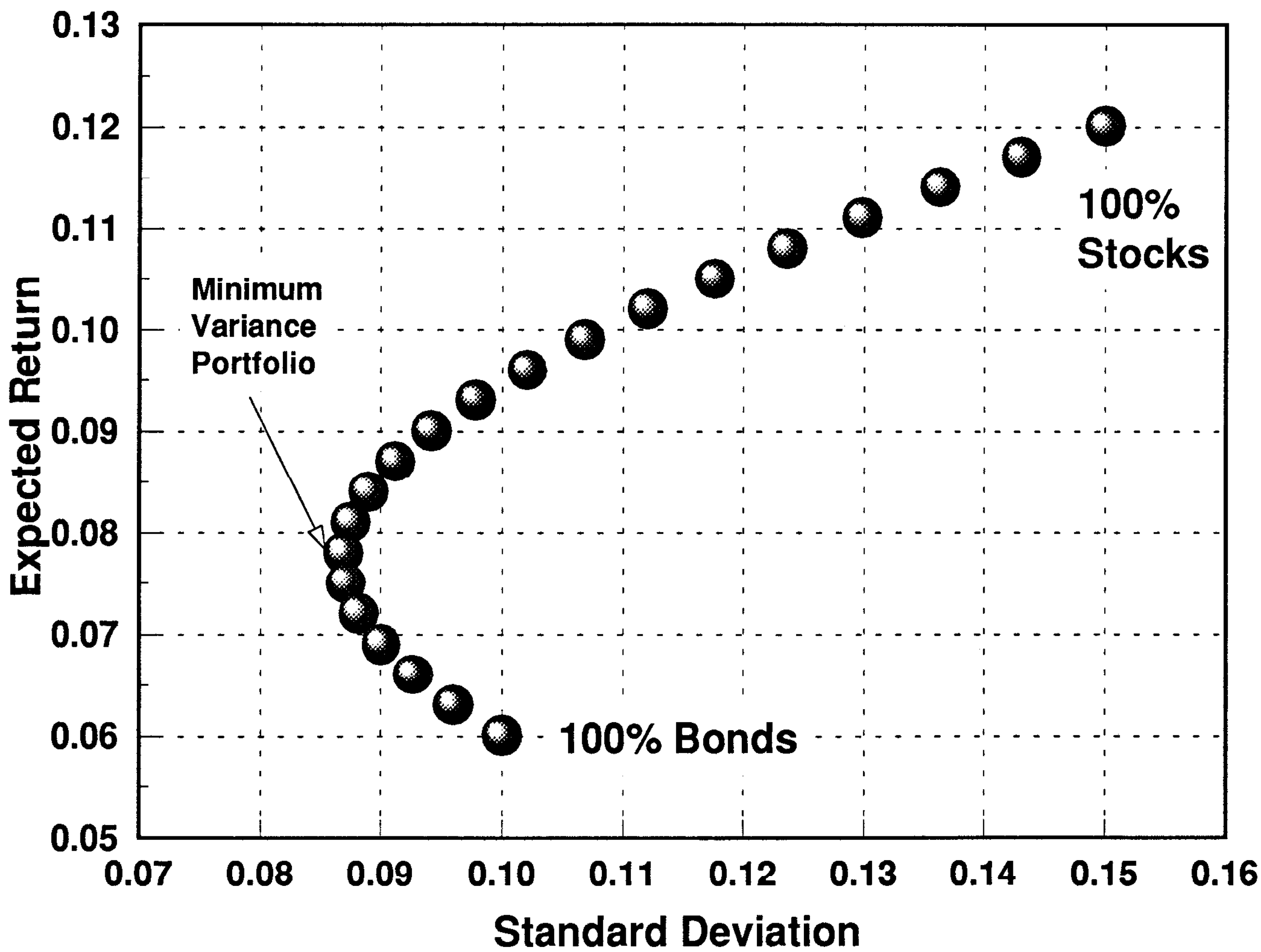


Figure 17.5

Risk and Return with Two Assets

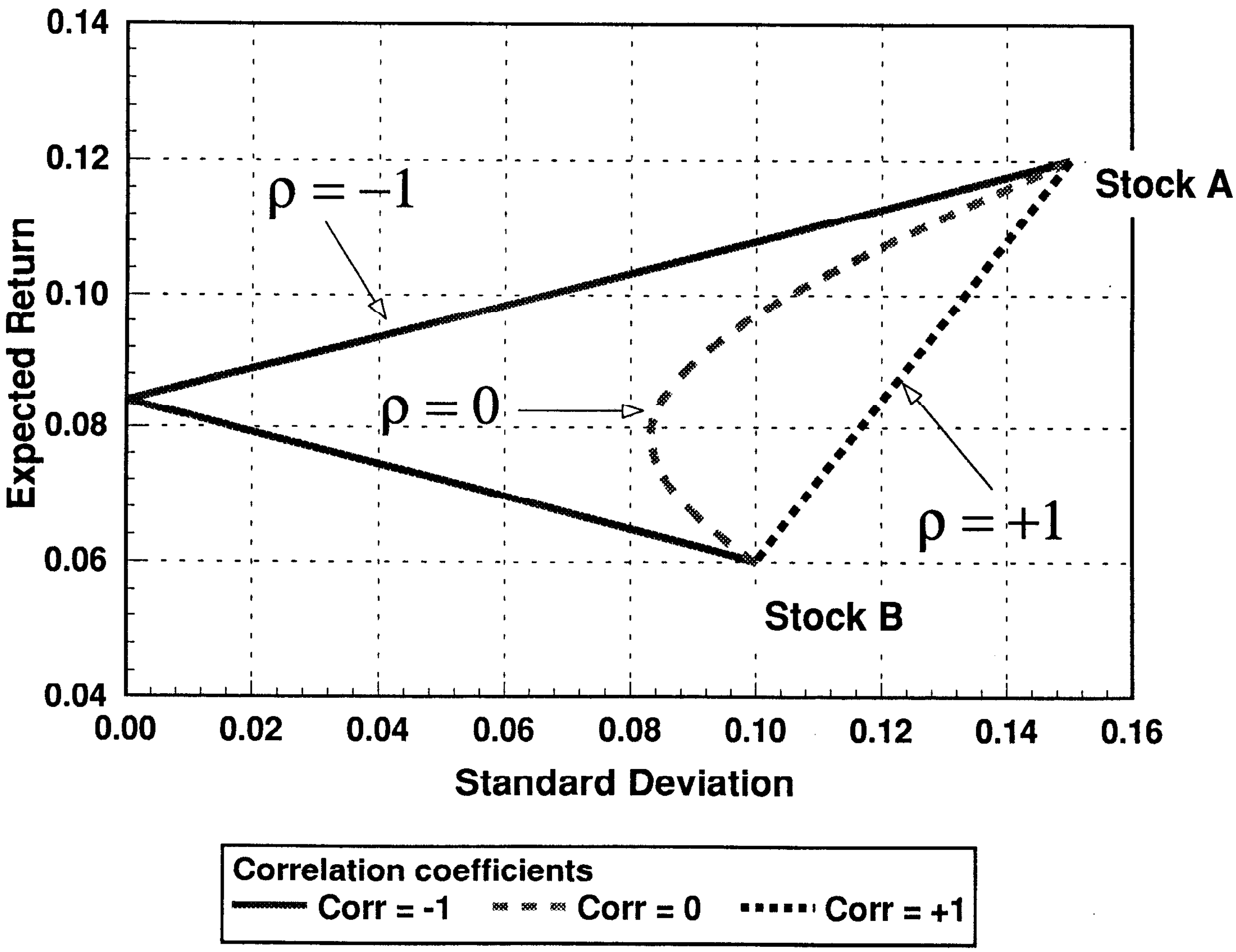
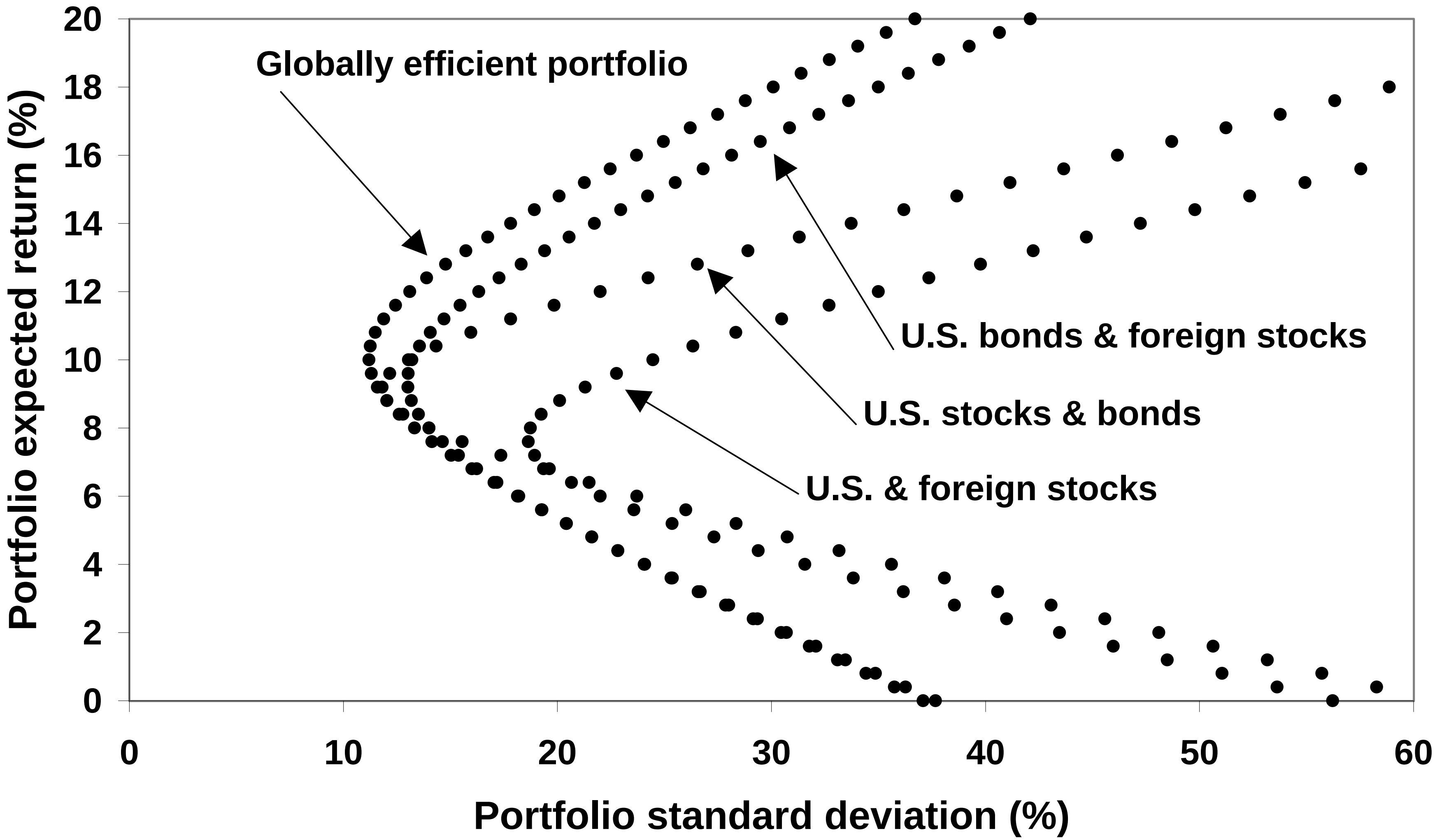


Figure 17.6 Globally efficient portfolio



Egg basket analysis

What's the best way to diversify your fund portfolio? Some new software from Value Line helps you find out.

By Thomas Easton

DIVERSIFY. Don't put all your eggs in one basket. We have all heard the wisdom, but few of us heed it in picking mutual funds. People who own the Fidelity Contrafund, say, may branch out to Fidelity Growth & Income with their next fund purchase, thinking they're diversifying. They aren't.

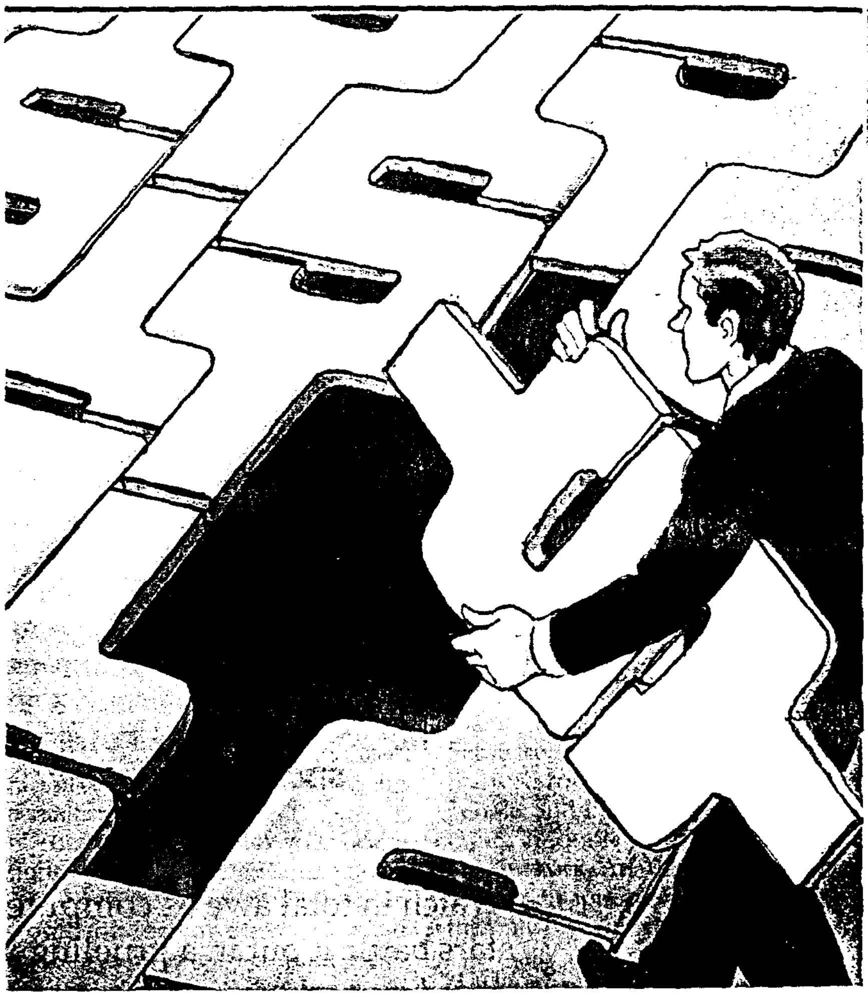
In the statistical scheme of things, these are two peas in a pod. They may both prove to be good funds, but the likelihood is that in a month when one goes up, the other will, too, and when one goes down, so will the other.

So, too, with almost any pair of big-company stock funds—Fidelity Magellan and Vanguard Index 500, Fidelity Destiny I and Vanguard Windsor. They tend to track pretty closely. You don't accomplish a lot by dividing your eggs between those two baskets. You might as well put all your money in the one blue-chip fund you like the most.

So, too, any two long-term domestic bond funds are likely to match each other's ups and downs pretty well, even if they have seemingly different objectives—one junk bonds and the other Ginnie Mae, one a Treasury bond fund and the other a corporate fund.

Surprisingly, most bond funds have in recent years become similar to most stock funds in the general timing of their up and down movements. Look at what has happened on Wall Street recently. Last year interest rates went down, sending bond funds up and stock funds with them. In early April the process reversed. They crashed in unison.

If you want true diversity, therefore, make sure you spread your money among funds that do not move in lockstep with each other. The Benham Treasury Note Fund and the Lexington Strategic Silver Fund are one such pair. When inflation perks up, so, most of the time, do silver prices and interest rates. That makes the T note fund do badly and the Silver fund do well. The reverse is also true.


JEAN FRANÇOIS ALLAUX

Heartland Value Fund and MFS World Governments-A bond fund are another intriguing pair. When one zigs, the other zags. These two funds are not mirror opposites, to be sure, but they really do travel on different wavelengths.

Here's yet another pair of funds that tend to veer off in different directions: GAM International Fund and Vanguard GNMA Fund.

How did we find these diversifications? With a computer. We fed it monthly returns since 1986 for 560 funds and asked it to calculate, for each pair, what statisticians call the correlation coefficient. That's a measure of how closely two data series track each other.

If you want to apply correlation analysis to your fund portfolio, try Fund Analyzer, recently out from Value Line. You feed the program two or more fund names. It tells you the risk and return of each fund over the past ten years, as well as the risk and return of a blend *(see graphs)*.

The Value Line software assumes that you, like any warm-blooded investor, want two things from your portfolio: high return and low risk. Alas, you usually have to trade one of these objectives off against the other. A stock fund will have high risk (high volatility) and a high expected return. A short-term bond fund will be low on both scores. A 50/50 blend will be in the middle on both measures.

You don't need a fancy computer program to tell you how large a position you need in the short bond fund to blend with your stock fund. It's a matter of taste for risk. The short fund, being less vulnerable to interest rate changes, is the less volatile. So throw enough of it into your mix to enable you to sleep well at night. You can do that in your head or on the back of an old charge slip.

But that won't give you true diversity. For that you need funds that have a low degree of correlation with each other. Then diversification really pays. The return of a 50/50 blend lies right on the midpoint of the two funds' returns, but their combined risk level is a bit *lower* than an average of the two funds' risk levels.

Look at the Heartland Value/MFS World pair in the lower-right graph. The returns for the funds over the past ten years have averaged 14.3% and 11.8%, respectively. A blend would have earned you 13%, right in the middle. But the risk of the blend is below the average of the two funds' risk measures.

This, graphically, is what you gain by not putting all your eggs in one basket.

The Value Line software cannot, at present, take a given fund and search through its database for funds that have low correlations with that fund. For now, you have to hunt and peck for a good combination. But Value Line says it will offer the searching feature in a later version of the software.

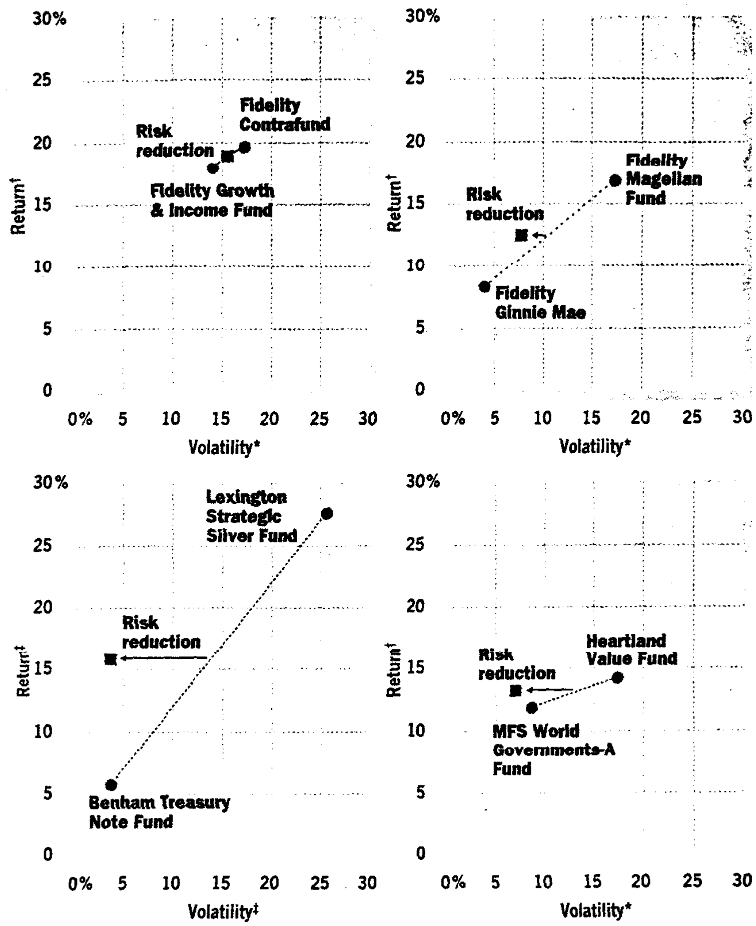
The software, with data for 5,000 funds, costs $295; it's $149 for an edition covering 1,900 no- and low-load funds.

Morningstar, the fund rating company, says it's working on a similar fund diversification package. Ibbotson Associates publishes a yearbook providing data on optimal groups of asset categories such as precious metals and large-capitalization stocks, but not individual funds.

If you are going to use this software, understand its limits. It can give you a pretty good idea of which funds tend to move in lockstep and which ones don't. But it can't tell you which funds are going to go up the most.

So don't add Twentieth Century Ultra to your portfolio just because it went up a lot in the past. Add it if it complements another fund. ■

Risk and reward



*Annualized standard deviation calculated over ten years.
†Compound annual return over ten years. ‡Three years.
■Fifty-fifty blend of two funds.

Source: Value Line Publishing, Inc.

Balancing Funds. Start with upper left and go clockwise. FidelityContrafund + Fidelity Growth & Income: Two eggs, one basket. Less return, less risk. Gain from diversification? Zilch. Fidelity Magellan + Fidelity Ginnie Mae: Just a small gain from diversification. Heartland Value + MFS World Governments-A: Give up some return, but see the risk dramatically decline. Lexington Strategic Silver + Benham Treasury Note: Dramatic diversification. When one wiggles, the other waggles.