Interest Rate Models

key developments in the
Mathematical Theory of Interest Rate Risk Management

presented by

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Chapter 1

Discount bonds and interest rates. Libor and swap rates. Forward prices and forward rates. Short rate and forward short rate. Positive interest conditions. Interest rate derivative structures.

1.1 Discount bonds and interest rates

The formulae involved with interest rate modelling can get complicated. It is important to use an unambiguous scheme of notation that can be carried across a range of different models and at the same time is useful for calculations.

Time 0 denotes the present. Times a, b, c, etc., denote various future times, as do s, t, u, and so on. Alphabetical order will often be used to suggest chronological order. Occasionally, we use an upper case T to draw attention to a particular date (e.g. a termination date).

We use the notation P_{ab} to denote the value at time a of a discount bond maturing at time b. At time b, the bond pays one unit of "currency". We fix a currency throughout here.

In fact, for any class of financial assets we have a corresponding system of discount bonds. Thus, for dollars, P_{ab} denotes the price at time a, in dollars, of a bond that pays one dollar at the maturity b.

Equally, we can speak of a "sterling" discount bond, or even a "gold" discount bond. In the latter case, P_{ab} could denote the price at time a, in ounces of gold, of a contract delivering one ounce of gold at time b.

Occasionally, a comma will be inserted for clarity. Thus $P_{t,x+t}$ denotes the value of a discount bond at time t that matures at time $x + t$.

For any fixed value of t, the system of discount bond prices P_{tT} for $T \in [t, \infty)$ is called the *discount-function* at that time. The present discount function is P_{0T}.

Associated with any discount bond P_{ab} there are various rates that can be quoted.

For example, the *simple* interest rate L_{ab} is defined by:

$$P_{ab} = \frac{1}{1 + (b - a)L_{ab}}. \tag{1.1}$$

The *continuously compounded* rate R_{ab} is defined by:

$$P_{ab} = e^{-(b-a)R_{ab}}. \tag{1.2}$$

The unit of time is one calendar year, and these rates are quoted in an "annualised" basis.

Inverting these relations we find that the simple rate is given by

$$L_{ab} = \frac{1}{b - a}\left(\frac{1}{P_{ab}} - 1\right) \tag{1.3}$$

The corresponding expression for the continuously compounded rate is

$$R_{ab} = -\frac{1}{b - a}\log P_{ab}. \tag{1.4}$$

1.2 Libor and Swap rates

The Libor rate for a given period is usually quoted on a simple annualised basis, so sometimes we call L_{ab} the Libor rate associated with P_{ab}.

Note that although rates can be quoted in various ways, the discount bond price is unique (it is a price!). That is a good reason for focusing on discount bonds. These are the fundamental "assets" of interest rate theory, and it is their behaviour we are trying to model.

Another very important type of rate frequently quoted in the over-the-counter interest rate markets is the *swap rate.*

There are various types of swap rates, and various conventions dealing with day counts, and so on. It is best therefore to give a mathematically concise definition that can be adapted easily to various situations.

The swap rates defined in this way are "pure" in the sense that they are based on the basic discount function, and do not take into account credit, liquidity, and other market factors that may affect "real" swap rates.

Let 0 denote the present, t some date in the future, and $T_1, T_2, \ldots, T_n$ a series of future dates beyond t.

For each such series $(T_1, T_2, \ldots, T_n)$ there is a unique swap rate s_t.

This rate is determined by the condition that if the rate of interest s_t is paid on a unit principal on each of the dates $T_1, T_2, \ldots, T_n$ and if the unit principal is paid at time T_n, then the present value at time t of this cash flow is unity.

More specifically, we have the condition

$$s_t(P_{tT_1} + P_{tT_2} + \cdots + P_{tT_n}) + P_{tT_n} = 1. \tag{1.5}$$

Solving for s_t we have

$$s_t = \frac{1 - P_{tT_n}}{P_{tT_1} + P_{tT_2} + \cdots + P_{tT_n}} \tag{1.6}$$

The sum

$$V_{tT_1 \ldots T_n} = \sum_{i=1}^{n} P_{tT_i} \tag{1.7}$$

is sometimes called the 'basis point value' (bpv) at time t associated with the date system T_1, T_2, …, T_n.

We note that because s_t can always be expressed as a combination of various discount bond values, it makes sense to speak of derivative payoffs based on s_t.

A derivative whose payoff depends on s_t can thus be viewed as a kind of exotic option based on the discount bonds.

There are elements of convention involved in how real swap rates are quoted. For example, if s_t is paid semi-annually (i.e. T_1, T_2, etc., are spaced at half-yearly intervals), then $2s_t$ is the quoted swap rate. This is an artifact of market convention and need not concern us here, but of course it should be born in mind.

1.3 Forward prices and forward rates

The *forward price* of a discount bond will be denoted by P_{tab}.

This is the price contracted at time t for purchase of a discount bond at time a that matures at time b.

A standard arbitrage argument shows that

$$P_{tab} = \frac{P_{tb}}{P_{ta}}. \tag{1.8}$$

The argument runs as follows.

Suppose at time t a 'careless' market maker is willing to sell me a b-maturity bond on a forward basis at time a for a price Q_{tab} that is *less* than P_{tab}.

I would then purchase Q_{tab}/P_{ta} a-maturity bonds at time t, and simultaneously short Q_{tab}/P_{tb} b-maturity bonds.

At the same time I purchase $1/P_{ta}$ b-maturity bonds on a forward basis from the dealer.

At time a, the a-maturity bonds mature, leaving me with Q_{tab}/P_{ta} in cash, which I uses to purchase $1/P_{ta}$ b-maturity bonds (taking advantage of the forward agreement).

Then at time b, the long investment pays off $1/P_{ta}$, whereas I owe Q_{tab}/P_{tb} on the maturing short position.

Since $1/P_{ta} > Q_{tab}/P_{tb}$, I have made a risk free profit.

A similar argument allows me to arbitrage the dealer if a forward price greater than P_{tab} is made.

Thus we see that $P_{tab} = P_{tb}/P_{ta}$ is the correct forward price for a discount bond.

The associated *forward rates* are given by

$$P_{tab} = \frac{1}{1 + (b - a)L_{tab}} \tag{1.9}$$

and

$$P_{tab} = e^{-(b-a)R_{tab}}. \tag{1.10}$$

Here L_{tab} and R_{tab} are the forward rates, quoted at time t, for the period $[a, b]$, on a simple and on a continuously compounded basis, respectively.

We call L_{tab} the forward Libor rate made at time t for the period $[a, b]$.

It also makes sense to speak of a *forward swap rate*.

This is the swap rate s_{ta} contracted at time t for a swap entered into at time a with the payment dates $b_1, b_2, \ldots, b_n$. Then we have

$$s_{ta} = \frac{1 - P_{tab_n}}{P_{tab_1} + P_{tab_2} + \cdots + P_{tab_n}}. \tag{1.11}$$

Clearly we have $s_{tt} = s_t$.

1.4 Short rates and forward short rates.

The rate $r_b = \lim_{a \to b} L_{ab}$ is called the *short rate*.

This is the rate of interest, at time a, on a very short period loan (e.g., "overnight"), expressed on an annualised basis.

If we assume, as seems reasonable, that P_{ab} is differentiable in the maturity date, then a short computation shows that

$$r_a = -\left.\frac{\partial P_{ab}}{\partial b}\right|_{a=b}. \tag{1.12}$$

Over the short term, "compounding" is irrelevant, and thus

$$\lim_{a \to b} L_{ab} = \lim_{a \to b} R_{ab}. \tag{1.13}$$

The *forward short rate* f_{ta} is the rate of interest contracted at time t for a very short period loan at some later time a.

For example, I might agree today to loan you \$1,000,000 for one day, one year from now, at a rate of interest of 6% annualised. Then we would have $f_{01} = 0.06$ ($a = 0$, $b = 1$).

The forward short rate is also called the "instantaneous forward rate" (for example, in Heath, Jarrow & Morton 1992).

We note that the forward short rate is by definition given by the limit

$$f_{ta} = \lim_{b \to a} L_{tba}. \tag{1.14}$$

Thus we have

$$f_{ta} = -\left.\frac{\partial P_{tab}}{\partial b}\right|_{a=b} = -\frac{\partial \ln P_{ta}}{\partial a}. \tag{1.15}$$

The latter relation is often effectively adopted as a *definition* for f_{ta} in the literature, but it is important to see that it is not really a definition: it derives from an underlying economic relation.

The significance of the relation

$$f_{ta} = -\frac{\partial \ln P_{ta}}{\partial a} \tag{1.16}$$

is that it is invertible:

$$P_{tT} = \exp\left(-\int_t^T f_{tu}\, du\right). \tag{1.17}$$

Thus, at any fixed time t, knowledge of the discount function P_{tT} at that time, for maturity T, is equivalent to knowledge of the system of forward short rates f_{tu} determined (i.e. contractable) at that time over the interval $u \in [0, T]$.

Note, incidentally, that (1.17) incorporates the maturity condition $P_{TT} = 1$.

1.5 Positive interest conditions

For many applications we want to build in an *interest rate positivity* condition.

This is not automatic in the HJM framework, but later when we examine the Flesaker-Hughston framework and its extensions we will see how this feature can be incorporated.

For positive interest we require the following two conditions valid for all $0 \leq a \leq b < \infty$:

$$0 < P_{ab} \leq 1, \tag{1.18}$$

$$\frac{\partial P_{ab}}{\partial b} < 0. \tag{1.19}$$

There are various ways of ensuring these conditions are satisfied. For many models they are not. Whether or not this is a material issue depends on the circumstances.

From a fundamental point of view, however, we require nominal interest rates to be strictly positive. This is because if someone offers to loan you money at a negative rate of interest, then you can immediately take advantage of them and effect an arbitrage.

The positive interest conditions are sufficient to ensure that all the commonly encountered rates are positive: Libor rates, swap rates, forward Libor and swap rates, short rate, and forward short rate.

1.6 Interest rate derivative structures

Let us now turn to the consideration of interest-rate related contingent claims.

First, we need to ask what is meant by an "interest rate derivative".

One general mathematical way of defining a European-style interest rate derivative is to say that the payout at time T is any random variable H_T that is $\mathcal{F}_T$-measurable, where $(\mathcal{F}_t)$ is the natural filtration of the multi-dimensional Brownian motion driving the discount-bond system.

In practice, the payout of an interest rate derivative is specified in terms of one or more well-defined *rates* associated with the given contract period.

Equivalently, we let H_T be specified as a function of the values of one or more discount bonds during the interval $[0, T]$. The maturities of these discount bonds may or may not lie in that interval.

For example, the payout

$$\text{(a)} \quad H_T = \max\left(P_{Tb} - K, 0\right) \tag{1.20}$$

defines a call option on a discount bond ($b > T$).

The payout

$$\text{(b)} \quad H_T = X \max\left(L_{Tb} - R, 0\right) \tag{1.21}$$

defines a simple *caplet* on the Libor rate L_{Tb}, where R is the cap rate, and X is the notional paid per interest rate point (e.g., \$1,000,000 per interest rate point above R).

Normally, a caplet is paid "in arrears", meaning the rate is set at some earlier time a, and paid at T, so in that case, the payout is

$$\text{(c)} \quad H_T = X \max\left(L_{aT} - R, 0\right), \tag{1.22}$$

for the rate L_{aT} set earlier at time a.

However, since L_{aT} is known at time a, we can regard the normal caplet as a derivative that pays the discounted value $H_a = P_{aT} H_T$ at the earlier time a, where H_T is the payout defined in (c).

By definition, we have

$$P_{aT} = \frac{1}{1 + (T - a)L_{aT}}. \tag{1.23}$$

It follows, as we noted earlier, that

$$L_{aT} = \frac{1}{T - a}\left(\frac{1}{P_{aT}} - 1\right). \tag{1.24}$$

Therefore, the effective payout H_a at time a is given by the following calculation:

$$\begin{aligned}
H_a &= P_{aT} H_T \\
&= X P_{aT} \max\left(L_{aT} - R, 0\right) \\
&= X P_{aT} \max\left(\frac{1}{T - a}\left(\frac{1}{P_{aT}} - 1\right) - R, 0\right) \\
&= X \max\left(\frac{1}{T - a}\left(1 - P_{aT}\right) - R P_{aT}, 0\right) \\
&= \frac{X}{T - a} \max\left(1 - P_{aT} - (T - a) R P_{aT}, 0\right) \\
&= \frac{X}{T - a}[1 + R(T - a)] \max\left(\frac{1}{1 + R(T - a)} - P_{aT}, 0\right) \\
&= N \max\left(K - P_{aT}, 0\right). \qquad (1.25)
\end{aligned}$$

Here the strike K is given by

$$K = \frac{1}{1 + R(T - a)} \tag{1.26}$$

and the notional N is

$$N = \frac{X[1 + R(T - a)]}{T - a}. \tag{1.27}$$

Thus we see that a position in standard *caplet* is equivalent to a position in N *puts* on the discount bond, where the strike price K on the put is the value of a discount bond with simple yield R.

There are many subtle ways of *transforming* one type of interest rate derivative structure into another with the same effective payoff.

This is important both in the marketing and the risk management of such products.

As another example, suppose we consider the case of a *swaption*, the option to enter into a swap at time t for the dates $(T_1, T_2, \cdots, T_n)$ at a fixed "strike" swap-rate R.

Assuming that the option is to pay the fixed rate R, then the payoff H_t at time t is

$$H_t = V_{tT_1 \ldots T_n} \text{Max}(s_t - R, 0). \tag{1.28}$$

Here $V_{tT_1 \ldots T_n} = \sum_{i=1}^{n} P_{tT_i}$ is the bpv at time t for the coupon dates $(T_1, T_2, \cdots, T_n)$.

Clearly, the option is exercised iff the "actual" swap rate s_t observed at time t is greater than R.

Thus an alternative way of writing the swaption payout H_t is:

$$H_t = \left[1 - P_{tT_n} - R \sum_{i=1}^{n} P_{tT_n} \right]^{+}. \tag{1.29}$$

It should be evident that an alternative interpretation of a swaption is to regard it as an option at time t to acquire (A) a portfolio consisting of a unit of cash and a short position in a T_n-maturity bond, in exchange for (B) a portfolio consisting of R units each of the T_i-maturity bonds for $i = 1, 2, \ldots, n$.

This is the economic interpretation of a swaption in terms of the exchange of actual assets.

The swaption considered above is an option to *pay* the fixed leg of a swap, and is thus called a *payer* swaption. There is an analogous structure which is an option to *receive* the fixed leg of a swap, called a *receiver* swaption.

Chapter 2

Dynamical equations for a non-dividend-paying asset. Money market account and risk premium process. Martingales, supermartingales and submartingales. Martingale relations for a single asset. Transformation to the risk neutral measure. No-arbitrage relation for derivatives. Derivative pricing. Girsanov transformation.

2.1 Dynamical equations for a non-dividend-paying asset

For a single asset with limited liability and price process S_t, the stochastic equation for the dynamics of S_t is:

$$\frac{dS_t}{S_t} = \mu_t \, dt + \sigma_t \, dW_t. \tag{2.1}$$

This equation is defined on a probability space $\Pi = (\Omega, \mathcal{F}, P)$ with filtration $(\mathcal{F}_t)$, with respect to which W_t is a standard Brownian motion.

We assume that μ_t (drift) and σ_t (volatility) are adapted to the filtration $(\mathcal{F}_t)$.

Initially, we consider the simple situation where $(\mathcal{F}_t)$ is generated by W_t. Later, when other basic assets are brought into play, we let the filtration $(\mathcal{F}_t)$ be larger.

We can think of Π as representing the economy, and $(\mathcal{F}_t)$ as representing the market *information flow* up to time t.

For many purposes we can, without serious loss of generality, assume that μ_t and σ_t are *bounded*.

This will be a sufficient technical condition to ensure that the relevant stochastic integrals exist, and the relevant *martingale condition* is satisfied when this is needed. In practice this condition can often be relaxed in various ways.

If μ and σ are constant the solution of S_t is:

$$S_t = S_0 \exp\left(\mu t + \sigma W_t - \tfrac{1}{2}\sigma^2 t\right). \tag{2.2}$$

This is called the *geometric Brownian motion* model for S_t.

The geometric Brownian motion model was introduced by Paul Samuelson, and was used by Fisher Black and Myron Scholes as an assumption in the derivation of their celebrated option pricing formula.

More generally, for path dependent μ_t and σ_t, which for simplicity we may here assume to be adapted and bounded, we have the following solution for the asset price in terms of μ_t and σ_t:

$$S_t = S_0 \exp\left(\int_0^t \mu_s\, ds + \int_0^t \sigma_s\, dW_s - \tfrac{1}{2}\int_0^t \sigma_s^2\, ds\right). \tag{2.3}$$

We regard μ_t and σ_t as being specified *exogenously*.

We can use Ito's lemma to verify that the stochastic equation is satisfied. First, we note that

$$d\log S_t = \frac{dS_t}{S_t} - \frac{1}{2}\frac{(dS_t)^2}{S_t^2}. \tag{2.4}$$

Thus squaring each side we have:

$$(d\log S_t)^2 = \frac{(dS_t)^2}{S_t^2}. \tag{2.5}$$

So putting these two equations together we get:

$$\frac{dS_t}{S_t} = d\log S_t + \tfrac{1}{2}\left(d\log S_t\right)^2 \tag{2.6}$$

But taking the logarithm of (2.3) we have:

$$\log S_t = \log S_0 + \int_0^t \mu_s\, ds + \int_0^t \sigma_s\, dW_s - \tfrac{1}{2}\int_0^t \sigma_s^2\, ds. \tag{2.7}$$

So by taking the stochastic differential we obtain

$$d\log S_t = \mu_t\, dt + \sigma_t\, dW_t - \tfrac{1}{2}\sigma_t^2\, dt. \tag{2.8}$$

Thus by squaring and only keeping the $(dW_t)^2 = dt$ term we also have:

$$(d \log S_t)^2 = \sigma_t^2 \, dt. \tag{2.9}$$

It follows immediately that

$$\frac{dS_t}{S_t} = \mu_t \, dt + \sigma_t \, dW_t. \tag{2.10}$$

2.2 Money market account and risk premium process

To proceed further, we introduce a 'risk-free' asset, the money-market account, with price process B_t, satisfying

$$\frac{dB_t}{B_t} = r_t \, dt, \tag{2.11}$$

Here r_t is the short-term interest rate, which we also assume to be adapted to the market filtration $(\mathcal{F}_t)$.

The solution for the money market account process B_t is

$$B_t = B_0 \exp\left(\int_0^t r_s \, ds \right). \tag{2.12}$$

Now we introduce the *market risk premium process* λ_t, defined for a non-dividend paying asset by

$$\mu_t = r_t + \lambda_t \sigma_t. \tag{2.13}$$

The process λ_t measures, instantaneously, the *extra* rate of return offered by the asset, above the risk-free rate r_t, per unit of volatility σ_t.

Note that in the case of a non-dividend paying asset, and in the absence of risk, the rate of return would be r_t.

In the case of a dividend paying asset, the process for μ_t is given by

$$\mu_t = r_t - \delta_t + \lambda_t \sigma_t, \tag{2.14}$$

where δ_t is the dividend rate.

In the case of a single asset the drift condition (2.13) merely *defines* λ_t.

In the case of multiple assets the relation gets generalised and is equivalent to the condition of *no arbitrage*.

2.3 Martingales, supermartingales and submartingales

Now we derive an important relation that ties together the values of an asset at two different times.

One of the central concepts in the modern theory of finance is the idea of a *martingale*.

The point of the martingale concept is that it gives a mathematical embodiment to the notion of a fair game of chance.

It also helps to clarify in mathematical terms what we mean by a *forecast*.

In what follows we also need to know about the related concepts of supermartingale, and submartingale.

The concept of supermartingale, in particular, plays a special role in interest rate theory.

A stochastic process M is an $(\mathcal{F}_t)$-*martingale* if

$$\text{(a)} \quad \mathbb{E}\left[|M_t|\right] < \infty, \quad \text{for all } t \geq 0, \tag{2.15}$$

$$\text{(b)} \quad M_s = \mathbb{E}\left[M_t \mid \mathcal{F}_s\right], \quad \text{for all } s < t. \tag{2.16}$$

Part (b) of this definition expresses the idea that the expected value of the process at time t, given information up to time s, is equal to the value of the process at time s.

When there is no ambiguity we sometimes write $\mathbb{E}_t[X] = \mathbb{E}[X|\mathcal{F}_t]$ for conditional expectation with respect to the sigma-algebra $\mathcal{F}_t$.

We can modify the definition above to account for martingales defined only for $t \in [0, T^*]$, where $T^* > 0$ is a fixed time horizon.

A standard Brownian motion W_t is a martingale. So are, for example, the processes given by

$$M_t = \frac{1}{2}(W_t^2 - t), \tag{2.17}$$

$$M_t = \frac{1}{6}(W_t^3 - 3tW_t) \tag{2.18}$$

$$M_t = \frac{1}{24}(W_t^4 - 6tW_t^2 + 3t^2). \tag{2.19}$$

Another example is given by

$$M_t = \exp\left(\sigma W_t - \tfrac{1}{2}\sigma^2 t\right), \tag{2.20}$$

where σ is a constant.

To see that the process $\frac{1}{2}(W_t^2 - t)$ is a martingale, we observe that

$$\begin{aligned} \mathbb{E}_s[W_t^2 - t] &= \mathbb{E}_s[(W_s + (W_t - W_s))^2 - t] \\ &= \mathbb{E}_s[W_s^2] + \mathbb{E}_s[(W_t - W_s)^2] - t \\ &= W_s^2 - s. \end{aligned} \tag{2.21}$$

More generally, let us define the polynomial $H^n(x, y)$ by the generating function

$$\exp\left(\xi x - \tfrac{1}{2}\xi^2 y\right) = \sum_{n=0}^{\infty} \xi^n H^n(x, y). \tag{2.22}$$

Then for each value of n, the process $H^n(W_t, t)$ is a martingale, and the polynomial examples mentioned above arise as the first few values of n.

The polynomials $H^n(x, y)$ are given by

$$H^n(x, y) = \left(\tfrac{1}{2}y\right)^{n/2} h_n(x/\sqrt{2y}), \tag{2.23}$$

where $h_n(u)$ are the standard Hermite polynomials.

Martingales also arise as certain classes of stochastic integrals.

For example, if σ_t is $\mathcal{F}_t$-adapted and bounded, then

$$M_t = M_0 + \int_0^t \sigma_s dW_s \tag{2.24}$$

is a martingale.

So is:

$$M_t = M_0 \exp\left(\int_0^t \sigma_s dW_s - \tfrac{1}{2}\int_0^t \sigma_s^2 ds\right). \tag{2.25}$$

A process X_t is an $(\mathcal{F}_t)$-*supermartingale* if

$$\text{(c)} \quad \mathbb{E}\big[|X_t|\big] < \infty, \quad \text{for all } t \geq 0, \tag{2.26}$$

$$\text{(d)} \quad X_s \geq \mathbb{E}\left[X_t \mid \mathcal{F}_s\right], \quad \text{for all } s < t. \tag{2.27}$$

Similarly, a process X_t is an $(\mathcal{F}_t)$-*submartingale* if

$$\text{(e)} \quad \mathbb{E}\big[|X_t|\big] < \infty, \quad \text{for all } t \geq 0, \tag{2.28}$$

$$\text{(f)} \quad X_s \leq \mathbb{E}\left[X_t \mid \mathcal{F}_s\right], \quad \text{for all } s < t. \tag{2.29}$$

A process is a martingale iff it is both a supermartingale and a submartingale. If X_t is a supermartingale, then $-X_t$ is a submartingale.

Another important way of generating martingales is by taking conditional expectations. Thus if Z is a random variable such that $\mathbb{E}[|Z|] < \infty$, then

$$M_t = \mathbb{E}_t[Z] \tag{2.30}$$

defines a martingale by virtue of the "tower property" of conditional expectation $\mathbb{E}_s\mathbb{E}_t = \mathbb{E}_s$ for $s < t$.

2.4 Martingale relations for a single asset

Returning to the case of a single asset, let us introduce the relationship $\mu_t = r_t + \lambda_t\sigma_t$ into the formula for S_t. We then have

$$\frac{dS_t}{S_t} = r_t\, dt + \sigma_t\left(dW_t + \lambda_t\, dt\right). \tag{2.31}$$

Equivalently, S_t is given by

$$S_t = S_0 \exp\left(\int_0^t r_s\, ds\right) \exp\left(\int_0^t \sigma_s\left(dW_s + \lambda_s\, ds\right) - \tfrac{1}{2}\int_0^t \sigma^2\, ds\right). \tag{2.32}$$

It follows that

$$\frac{S_t}{B_t} = S_0 \exp\left(\int_0^t \sigma_s\left(dW_s + \lambda_s\, ds\right) - \tfrac{1}{2}\int_0^t \sigma^2\, ds\right). \tag{2.33}$$

Now suppose that we define the process Λ_t by

$$\Lambda_t = \exp\left(-\int_0^t \lambda_s\, dW_s - \tfrac{1}{2}\int_0^t \lambda_s^2\, ds\right). \tag{2.34}$$

We call Λ_t the *risk adjustment density* or *risk premium density martingale*. It follows from Itô's lemma that

$$d\Lambda_t = -\Lambda_t\lambda_t\, dW_t. \tag{2.35}$$

Equivalently, by integration of this relation, incorporating the initial condition, we have:

$$\Lambda_t = 1 - \int_0^t \Lambda_s \lambda_s \, dW_s. \tag{2.36}$$

Thus, assuming λ_t is bounded, we have the *martingale* relation

$$\Lambda_s = \mathbb{E}_s \Lambda_t, \quad \text{for all } s \leq t, \quad \text{where } \mathbb{E}_s \Lambda_t := \mathbb{E}\left[\Lambda_t \mid \mathcal{F}_s\right]. \tag{2.37}$$

Now we show the following important result:

$$\frac{\Lambda_t S_t}{B_t} \quad \text{is a martingale.} \tag{2.38}$$

Indeed, a simple computation shows by completing the squares that:

$$\frac{\Lambda_t S_t}{B_t} = \exp\left(\int_0^t (\sigma_s - \lambda_s)\, dW_s - \tfrac{1}{2}\int_0^t (\sigma_s - \lambda_s)^2 ds\right), \tag{2.39}$$

and the desired property follows since σ_t is bounded. The martingale property for $\Lambda_t S_t / B_t$ can be written

$$\Lambda_s \frac{S_s}{B_s} = \mathbb{E}_s \left[\Lambda_t \frac{S_t}{B_t}\right], \quad s < t. \tag{2.40}$$

This is the formula that links past and future values of S_t, and thus can be thought of as a *forecasting* relation.

2.5 Transformation to the risk neutral measure

For any random variable X_t measurable with respect to the sigma-algebra $\mathcal{F}_t$, we define a *new* probability measure P^λ with expectation

$$\mathbb{E}_s^\lambda \left[X_t\right] = \frac{\mathbb{E}_s \left[\Lambda_t X_t\right]}{\Lambda_s}. \tag{2.41}$$

This formula explains why we call Λ_t a “density”.

The new probability measure (i.e. new rule for taking expectations) obtained in this way is called the *risk-neutral measure*.

This terminology is reserved for the measure obtained by use of the density Λ_t associated with the risk premium process λ_t.

Under the risk-neutral measure, we have

$$\frac{S_s}{B_s} = \mathbb{E}_s^\lambda \left[\frac{S_t}{B_t} \right], \quad s < t. \tag{2.42}$$

That is, the *discounted* asset price is a martingale (where the discounting is taken with respect to the money market account).

Another way of putting this is that in the risk neutral measure the value of the asset is a martingale when expressed in units of B_t, i.e., when we use B_t as a *numeraire*.

As we shall see, there are other measures associated with other choices of numeraire.

2.6 No-arbitrage relation for derivatives

Suppose that there is a derivative associated with S_t and its price process is H_t.

We assume that H_t is adapted to the filtration $(\mathcal{F}_t)$ like S_t, and in particular that H_t is fully characterised by an $\mathcal{F}_T$-measurable terminal value H_T, i.e. its payoff.

This means intuitively that H_T can depend in a very general way on the behaviour of W_t (and hence S_t) over the interval $[0, T]$.

Of course, H_T might be relatively simple, like a call option $H_T = \max(S_T - K, 0)$ or a short position in a forward contract $H_T = K - S_T$.

But it might be path-dependent, like a knock-out option, or an Asian option, or an American option (exercisable at some random time $\tau \leq T$, with the proceeds future valued and paid at time T).

For the price dynamics of H_t let us write

$$\frac{dH_t}{H_t} = \mu_t^H \, dt + \sigma_t^H \, dW_t. \tag{2.43}$$

Then a well-known *hedging argument* can be used to establish that

$$\frac{\mu_t^H - r_t}{\sigma_t^H} = \frac{\mu_t - r_t}{\sigma_t}. \tag{2.44}$$

The hedging argument is as follows. Suppose we have a long position in the derivative, and we wish to hedge that position with a short position in the underlying asset.

We form at time t the portfolio with value $H_t - \Delta_t S_t$ where Δ_t is the number of asset units shorted.

We examine the dynamics of the portfolio over the next small interval of time. The change in the value of the portfolio is given by $dH_t - \Delta_t \, dS_t$.

Then if

$$\Delta_t = \frac{H_t \sigma_t^H}{S_t \sigma_t}, \tag{2.45}$$

the "risks" (i.e. the coefficients of dW_t) cancel, and the portfolio offers an instantaneously *definite* rate of return given by

$$\frac{H_t \mu_t^H - \Delta_t S_t \mu_t}{H_t - \Delta_t S_t}. \tag{2.46}$$

We equate this "hedged" rate of return to r_t and insert the correct hedge ratio Δ_t. Then the desired no-arbitrage relation

$$\frac{\mu_t^H - r_t}{\sigma_t^H} = \frac{\mu_t - r_t}{\sigma_t}. \tag{2.47}$$

immediately pops out.

This relation is general, and is applicable in a fully path-dependent context.

2.7 Derivative pricing

We have assumed that (a) both the derivative and the asset price are adapted to the same Brownian motion filtration, (b) there are no dividends, (c) there are no transaction costs, (d) there are no constraints (e.g. limits) on the hedge position, and (e) the hedge portfolio can be adjusted continuously.

Note that if we further assume $H_t = H(S_t, t)$ for some function $H(S, t)$ of two variables, then the relation above becomes a PDE (the Black-Scholes equation) if μ_t, σ_t, r_t and λ_t are all likewise expressible as such functions.

This leads us down the "classical" path of derivative pricing, which can be highly effective when the assumptions indicated apply.

Generally, these assumptions break down if either (a) the derivative is path dependent or

(b) the asset price dynamics are path dependent.

The implication of the no-arbitrage condition (i.e. the general hedging argument) is that *the derivative price and the underlying asset both have the same risk premium λ_t.*

As a consequence, defining Λ_t as before, it follows that $\Lambda_t H_t / B_t$ is a martingale:

$$\frac{\Lambda_s H_s}{B_s} = \mathbb{E}_s \left[\frac{\Lambda_t H_t}{B_t} \right]. \tag{2.48}$$

Equivalently, we have

$$\frac{H_s}{B_s} = \mathbb{E}_s^{\lambda} \left[\frac{H_t}{B_t} \right], \tag{2.49}$$

where $\mathbb{E}^{\lambda}$ denotes expectation in the risk neutral measure. In particular, we have

$$H_0 = \mathbb{E} \left[\frac{\Lambda_T H_T}{B_T} \right]. \tag{2.50}$$

Equivalently:

$$H_0 = \mathbb{E}^{\lambda} \left[\frac{H_T}{B_T} \right]. \tag{2.51}$$

This is the *risk-neutral valuation formula* which says in words that the present value of a derivative is equal to the risk-neutral expectation of its terminal payoff.

For example, if μ, r and σ are constant, and if H_T is a simple call option payoff on S_T, then this reduces to the Black-Scholes formula:

$$H_0 = S_0 \, N \left[\frac{\ln\left(S_0 e^{rT}/K\right) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \right] - e^{-rT} K \, N \left[\frac{\ln\left(S_0 e^{rT}/K\right) - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \right] \tag{2.52}$$

where

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}\xi^2} \, d\xi \tag{2.53}$$

is the standard normal distribution function.

2.8 Girsanov transformation*

These results can be tied together nicely by the use of the Girsanov transformation.

We note that in the case of both the asset and the derivative, as a consequence of the no-arbitrage condition, the term $dW_t + \lambda_t \, dt$ is common to the dynamics:

$$\frac{dS_t}{S_t} = r_t \, dt + \sigma_t \, (dW_t + \lambda_t \, dt) \tag{2.54}$$

$$\frac{dH_t}{H_t} = r_t \, dt + \sigma_t^H \, (dW_t + \lambda_t \, dt) \tag{2.55}$$

Now we define a new process W_t^λ according to the formula

$$W_t^\lambda = W_t + \int_0^t \lambda_s \, ds. \tag{2.56}$$

It follows that $dW_t^\lambda = dW_t + \lambda_t \, dt$.

The essence of the theorem of Girsanov is that if W_t is a Brownian motion with respect to P, then W_t^λ is a Brownian motion with respect to P^λ. Then we say that W_t^λ is a P^λ-Brownian motion. The dynamics of S_t and H_t can be written

$$\frac{dS_t}{S_t} = r_t \, dt + \sigma_t dW_t^\lambda, \tag{2.57}$$

$$\frac{dH_t}{H_t} = r_t \, dt + \sigma_t^H dW_t^\lambda. \tag{2.58}$$

In the risk neutral measure, W_t^λ is a Brownian motion.

Thus we see that, as a consequence of the Girsanov transformation, the risk premium effectively drops out of the dynamics for both the underlying asset as well as the derivative.

With respect to the risk neutral measure both S_t and H_t have a rate of return given by r_t, the rate of return offered on the locally risk-free money-market asset B_t.

A more precise account of Girsanov's theorem is as follows.

Let $(\Omega, \mathcal{F}, P)$ be a probability space equipped with a filtration $(\mathcal{F}_t)$. Suppose that W_t is a n-dimensional $(\mathcal{F}_t)$-Brownian motion defined on this probability space.

Let λ_t^α be a n-dimensional, $(\mathcal{F}_t)$-measurable process satisfying

$$P\left(\int_0^t |\lambda_s|^2 \, ds < \infty\right) = 1. \tag{2.59}$$

Under these assumptions, the process Λ_t given by

$$\Lambda_t = \exp\left(-\frac{1}{2}\int_0^t |\lambda_s|^2 \, ds - \int_0^t \lambda_s \cdot dW_s\right) \tag{2.60}$$

is well defined for all t. We can verify that

$$\Lambda_t = 1 - \int_0^t \Lambda_s \lambda_s \cdot dW_s, \tag{2.61}$$

A sufficient condition for Λ_t to be a martingale is the *Novikov condition*:

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^t |\lambda_s|^2 \, ds\right)\right] < \infty, \tag{2.62}$$

in which case $\mathbb{E}\big[\Lambda_T\big] = 1$. This condition is satisfied, in particular, if λ_t is bounded.

If Λ_t is a martingale, then, given any fixed time $T > 0$, we can define a probability measure Q_T on $(\Omega, \mathcal{F}_T)$ by

$$Q_T(A) = \mathbb{E}\left[\Lambda_T 1_A\right], \quad \text{for all } A \in \mathcal{F}_T. \tag{2.63}$$

The Girsanov theorem states that, given any fixed time $T > 0$, the process W_t^* defined by

$$W_t^* = W_t + \int_0^t \lambda_s \, ds, \quad t \in [0, T] \tag{2.64}$$

is a n-dimensional Brownian motion on $(\Omega, \mathcal{F}_T, Q_T)$.

We can, for example, verify that W_t^* is normally distributed with respect to the measure Q_T by use of the method of characteristic functions.

Given any $t \in [0, T]$, we calculate the characteristic function of the random variable $\tilde{W}_t$.

$$\begin{aligned}
\mathbb{E}^{Q_T}\left[e^{izW_t^*}\right] &= \mathbb{E}^P\left[\Lambda_T e^{izW_t^*}\right] \\
&= \mathbb{E}^P\left[\Lambda_t e^{izW_t^*}\right] \\
&= \mathbb{E}^P\left[\exp\left(-\int_0^t \lambda_s \, dW_s - \tfrac{1}{2}\int_0^t \lambda_s^2 \, ds + izW_t + iz\int_0^t \lambda_s \, ds\right)\right] \\
&= \mathbb{E}^P\left[\exp\left(-\int_0^t (\lambda_s - iz) \, dW_s - \tfrac{1}{2}\int_0^t (\lambda_s - iz)^2 \, ds - \tfrac{1}{2} z^2 t\right)\right] \\
&= \mathbb{E}^P\left[\exp\left(-\int_0^t (\lambda_s - iz) \, dW_s - \tfrac{1}{2}\int_0^t (\lambda_s - iz)^2 \, ds\right)\right] \exp\left(-\tfrac{1}{2} z^2 t\right) \\
&= \exp\left(-\tfrac{1}{2} z^2 t\right).
\end{aligned} \tag{2.65}$$

This shows that the random variable W_t^* is normally distributed, with mean 0 and variance t.

An elaboration of this argument leads to the result that W_t^* is a Q_T-Brownian motion.

Chapter 3

Dynamical equations for multiple assets. Market completeness. Valuation of derivatives in complete multi-asset market. Hedgeable and unhedgeable claims in incomplete markets.

3.1 Dynamical equations for multiple assets

We model the economy by a probability space $(\Omega, \mathcal{F}, P)$ equipped with standard augmented filtration $\{\mathcal{F}_t\}$ generated by a standard n-dimensional Brownian motion W_t^α, $\alpha = 1, 2, \cdots, n$, over the time interval $0 \leq t \leq T^*$, for some terminal date T^*. For some applications we may wish to take $T^* = \infty$.

According to the Ito calculus, we have $dW_t^\alpha dW_t^\beta = \delta^{\alpha\beta} dt$, where $\delta^{\alpha\beta}$ is the identity matrix. Note that the different components of W_t^α are taken to be uncorrelated.

Let us assume we have a system of m non-dividend-paying risky assets with price processes

$$\frac{dS_t^i}{S_t^i} = \mu_t^i dt + \sum_{\alpha=1}^{n} \sigma_t^{i\alpha} dW_t^\alpha. \tag{3.1}$$

Here, S_t^i $(i = 1, 2, \cdots, n)$ represents the price process for asset number i.

The drift process μ_t^i and the volatility process $\sigma_t^{i\alpha}$ are assumed to be bounded and progressively measurable with respect to the filtration $\{\mathcal{F}_t\}$.

Intuitively speaking, the latter condition means that these processes depend on the path of the Brownian motion from 0 up to time t, but otherwise, there is no source of 'extraneous' randomness.

This is essentially a causality condition.

For the moment, we shall not fix the relation between the number of assets m and the number of Brownian motions n.

In the case of a complete market, we normally require that m should be greater than or equal to n.

In other words, for a complete market, there should be at least as many genuinely 'independent' assets as there are 'sources of randomness'.

Otherwise, there may be more sources of randomness than there are independent means of hedging away this randomness! That would mean an 'incomplete' market.

At time t, the relative magnitude of the price fluctuation of asset i due to Brownian motion number α is given by $\sigma_t^{i\alpha}$, which we call the *volatility matrix*.

The exogenous specification of μ_t^i and $\sigma_t^{i\alpha}$ determines the asset price processes S_t^i, once initial prices have been given, according to the formula

$$S_t^i = S_0^i \exp\left(\int_0^t \left(\mu_s^i - \tfrac{1}{2}\sigma_s^{i2}\right) ds + \int_0^t \sigma_s^i dW_s\right). \tag{3.2}$$

Here we use the compact notation

$$\sigma_s^i dW_s = \sum_{\alpha=1}^{n} \sigma_s^{i\alpha} dW_s^\alpha \tag{3.3}$$

and

$$\sigma_s^{i2} = \sum_\alpha \sigma_s^{i\alpha}\sigma_s^{i\alpha}. \tag{3.4}$$

For each fixed value of i, we think of σ_s^i as a vector volatility process with n components, one for each of the n independent Brownian motion.

3.2 Market completeness

For some considerations we impose a condition of *market completeness*. For market completeness we require first that the $m \times n$ rectangular matrix $\sigma_t^{i\alpha}$ should be of rank n.

The interpretation of this condition is that any fluctuation in the Brownian motion is necessarily realised by at least one of the assets in the form of a corresponding price fluctuation.

More precisely, $\sigma_t^{i\alpha}$ is of maximal rank n at time t if, for any nonzero vector $\eta^\alpha = (\eta^1, \eta^2, \cdots, \eta^n)$ we have

$$\sum_{\alpha=1}^{n} \eta^\alpha \sigma_t^{i\alpha} \neq 0. \tag{3.5}$$

If this holds for all $\eta^\alpha \neq 0$, then any fluctuation dW_t^α in the Brownian motion results in a nontrivial asset price fluctuation dS_t^i.

This is evident from the basic dynamical equations.

Additionally, we will sometimes require to impose a condition on the volatility structure, sufficient to keep it from getting to 'close' to degeneracy.

This can be imposed by requiring that the symmetric matrix

$$\rho_t^{\alpha\beta} = \sum_{i=1}^{m} \sigma_t^{i\alpha} \sigma_t^{i\beta} \tag{3.6}$$

satisfies the condition that there exists a number ϵ such that

$$\rho_t^{\alpha\beta} > \epsilon \delta^{\alpha\beta}. \tag{3.7}$$

In other words

$$\sum_{\alpha,\beta} \left(\rho_t^{\alpha\beta} - \epsilon \delta^{\alpha\beta} \right) \eta^\alpha \eta^\beta > 0 \tag{3.8}$$

for any nonvanishing vector η^α. This ensures that the eigenvalues of $\rho_t^{\alpha\beta}$ are bounded from below by ϵ.

3.3 Absence of arbitrage in a multi-asset context

Now let us consider the principle of no arbitrage.

This principle implies in the case of an asset that pays no dividend that the drift is of the form

$$\mu_t^i = r_t + \sum_{\alpha=1}^{n} \lambda_t^\alpha \sigma_t^{i\alpha}, \tag{3.9}$$

for some progressively measurable vector process λ_t, independent of the value of i.

This is the market risk premium vector, which has the interpretation of being the extra rate of return, above the interest rate, per unit of volatility in the factor α.

Hence, the no-arbitrage condition tells us that the given family of assets shares a common risk premium process λ_t^α.

Once we deduce the existence of a market risk premium process, we obtain the following stochastic equation for the asset dynamics:

$$\frac{dS_t^i}{S_t^i} = r_t dt + \sum_{\alpha=1}^{n} \sigma_t^{i\alpha}(dW_t^\alpha + \lambda_t^\alpha dt). \tag{3.10}$$

We note the important fact that, in a complete market, the risk premium vector is *uniquely determined* by the given stochastic system.

This follows from the observation that, if (3.9) were satisfied for any other choice of risk premium vector, say, $\lambda_t^\alpha + \eta_t^\alpha$, then the market completeness would imply $\eta_t^\alpha = 0$.

In an incomplete market we can then ask whether it is appropriate to regard λ_t^α as being *exogenously* specified.

3.4 Valuation of derivatives in complete multi-asset markets

Consider now the valuation of derivatives in a complete market.

Many aspects of the present analysis have analogues in the case of a single asset, but there are some new twists as well that carry over to interest rate theory.

First, we need to introduce the unit initialised money market account process:

$$B_t = \exp\left(\int_0^t r_s ds\right) \tag{3.11}$$

In a complete market with risk premium vector λ_t, the asset price processes are

$$S_t^i = S_0^i \exp\left(\int_0^t r_s ds + \int_0^t \sigma_s^i(dW_s + \lambda_s ds) - \tfrac{1}{2}\int_0^t (\sigma_s^i)^2 ds\right). \tag{3.12}$$

As a consequence, we see that the ratios of S_t^i to B_t (discounted asset prices) are given by

$$\frac{S_t^i}{B_t} = S_0^i \exp\left(\int_0^t \sigma_s^i (dW_s + \lambda_s ds) - \tfrac{1}{2}\int_0^t (\sigma_s^i)^2 ds\right). \tag{3.13}$$

The combination $dW_t + \lambda_t dt$ appearing here suggests that, with a change of measure, the discounted asset prices will be martingales.

To see this, we form the density martingale

$$\Lambda_t = \exp\left(-\int_0^t \lambda_s dW_s - \tfrac{1}{2}\int_0^t \lambda_s^2 ds\right). \tag{3.14}$$

A short calculation shows that the ratio

$$\frac{\Lambda_t S_t^i}{B_t} = S_0^i \exp\left(-\int_0^t (\sigma_s^i - \lambda_s) dW_s - \tfrac{1}{2}\int_0^t (\sigma_s^i - \lambda_s)^2 ds\right) \tag{3.15}$$

is a martingale:

$$\frac{\Lambda_s S_s^i}{B_s} = \mathbb{E}_s\left[\frac{\Lambda_t S_t^i}{B_t}\right]. \tag{3.16}$$

This relation has to hold among all the given assets subject to a no arbitrage condition.

We may therefore consider the situation where one or more of these assets is a *derivative.*

Let H_T denote the payoff of such a derivative, and let H_t denote the price process for the derivative at earlier times.

It follows that the value of the derivative is given by:

$$H_t = \frac{B_t}{\Lambda_t}\mathbb{E}_t\left[\frac{\Lambda_T}{B_T} H_T\right]. \tag{3.17}$$

For the present value we then obtain the risk neutral valuation formula:

$$H_0 = \mathbb{E}^{\lambda}\left[\frac{H_T}{B_T}\right]. \tag{3.18}$$

If dividends are paid, then we need to modify these formulae slightly.

In the dynamics for S_t^i we replace r_t with $r_t - \delta_t^i$ where δ_t^i is the dividend rate, and we find that

$$M_t^i = \frac{\Lambda_t S_t^i}{B_t} + \int_0^t \frac{\Lambda_u \delta_u^i S_u^i}{B_u} du \tag{3.19}$$

is a martingale.

Then we can develop pricing formulae where both the assets and the derivatives pay continuous dividend.

3.5 Natural numeraire and state-price density

There is an interesting economic interpretation of the basic derivatives pricing formula (3.17).

We note that the process Λ_t is "dimensionless", whereas B_t is an asset price. Thus, the ratio B_t/Λ_t is also an asset price.

Writing $\xi_t = B_t/\Lambda_t$ we deduce that the dynamical equation for ξ_t is

$$\frac{d\xi_t}{\xi_t} = (r_t + \lambda_t^2)dt + \lambda_t dW_t. \tag{3.20}$$

We think of the process ξ_t as the value process for a special portfolio in the money market account and the basic risky assets with the value process ξ_t.

Sometimes the value process ξ_t is referred to as the "natural numeraire portfolio".

The present value of any other asset, when valued in units of the numeraire portfolio, acts as an *unbiased forecast* for the future value of that asset, when expressed in units of the numeraire portfolio at that time. In other words,

$$\frac{S_s^i}{\xi_s} = \mathbb{E}_s\left[\frac{S_t^i}{\xi_t}\right]. \tag{3.21}$$

Another useful way of thinking about ξ_t is to define the related process

$$V_t = \frac{1}{\xi_t}. \tag{3.22}$$

This is called the *state-price density.*

The state price is the value of one unit of cash in units of the natural numeraire.

For any non-dividend-paying asset S_t we have

$$S_t = \mathbb{E}_t\left[\frac{V_T}{V_t}S_T\right]. \tag{3.23}$$

Now suppose that S_t is a 'derivative' that pays one unit of cash at time T.

Then S_t is the price process P_{tT} of a *discount bond* with maturity T. Thus:

$$P_{tT} = \mathbb{E}_t\left[\frac{V_T}{V_t}\right]. \tag{3.24}$$

3.6 Incomplete markets

We now consider more generally the case where the market is not complete.

In practice, it is common to encounter derivatives that cannot be completely hedged.

Nevertheless, we may consider a 'decomposition' of a given product into a 'hedgeable' and 'unhedgeable' parts.

If the market in incomplete, then typically then volatility matrix $\sigma_t^{i\alpha}$ is degenerate (i.e. it has one or more zero eigenvalues).

This implies that the risk premium vector λ_t^α that satisfies the no arbitrage condition (3.9) is not uniquely determined by the specification of the asset price processes.

Nevertheless, we may consider the subspace of $\mathbb{R}^n$ spanned by the nondegenerate components of the volatility matrix $\sigma_t^{i\alpha}$, and construct a decomposition of the form

$$\lambda_t^\alpha = \psi_t^\alpha + \varphi_t^\alpha \tag{3.25}$$

Here, ψ_t^α is the vector λ_t^α with minimum length that satisfies the condition

$$\mu_t^i = r_t + \sum_{\alpha=1}^{n} \lambda_t^\alpha \sigma_t^{i\alpha}, \tag{3.26}$$

whereas φ_t^α satisfies

$$\sum_{\alpha=1}^{n} \varphi_t^\alpha \sigma_t^{\imath\alpha} = 0. \tag{3.27}$$

We now define the process ξ_t by the dynamics

$$\frac{d\xi_t}{\xi_t} = \left(r_t + \psi_t^2\right) dt + \psi_t \cdot dW_t \tag{3.28}$$

This is the unique natural numeraire process corresponding to the hedgeable part of the portfolio.

In other words, ξ_t is the unique *attainable* numeraire process.

In a complete market, the derivative price process is given by

$$G_t = \mathbb{E}_t\left[\frac{H_T}{\xi_T}\right]. \tag{3.29}$$

However, in an incomplete market, the derivative payout H_T contains unhedgeable components.

Therefore, we consider the decomposition

$$H_T = J_T + K_T. \tag{3.30}$$

Here, J_T corresponds to the hedgeable part of the derivative.

This is obtained by taking the conditional expectation $\mathbb{E}_t[H_T/\xi_T]$, and projecting the resulting martingale into the subspace spanned by the volatility vectors $\sigma_t^{i\alpha}$.

Then we let $t \to T$ and multiply by ξ_T to obtain J_T.

For the remaining unhedgeable part K_T, its expectation is given by

$$\mathbb{E}_t\left[\frac{K_T}{\xi_T}\right] = 0. \tag{3.31}$$

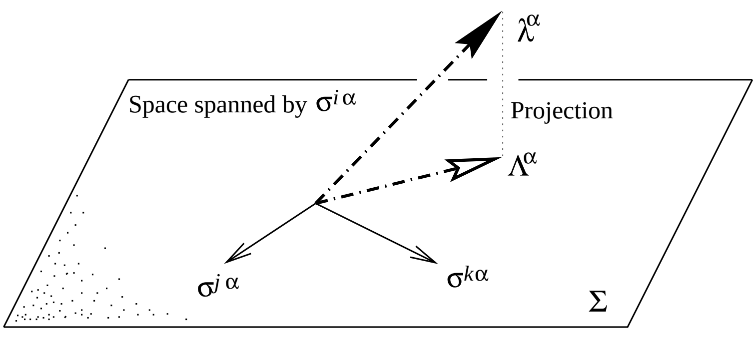

Figure 3.1: *The decomposition of the risk premium vector.*

Hence the hedgeable part of the product can be priced in essentially the conventional manner, while the unhedgeable part can, say, be transferred to a specialist desk to deal with the residual risk.

Chapter 4

Discount bond dynamics. Interest rate volatility and correlation. Short rate and instantaneous forward rate processes. Heath-Jarrow-Morton (HJM) framework. Valuation and hedging of interest rate derivatives.

4.1 Price processes for discount bonds

Now we turn to the modelling of interest rate dynamics.

The key idea here is to keep the *discount bonds* in the centre of the stage.

The short rate, forward short rates, Libor rates, forward Libor rates, and swap rates are all *subsidiary* processes.

If one focuses on *discount bonds*, then the theory of interest rates assumes a unified, coherent shape, and also fits in nicely with the consideration of other asset classes, e.g., foreign currencies, credit-risky bonds, inflation-linked bonds, equities and so on.

As indicated earlier, we write P_{ab} for the value at time a of a discount bond that matures at time b to deliver one unit of currency. The initial discount function is given by P_{0b}, and we have the maturity condition $P_{aa} = 1$ for all a.

For any given value of b we regard P_{ab} as a stochastic process in the a variable over the interval $0 \leq a \leq b$. Thus we have a one-parameter family of assets for which the price processes are given by P_{ab}.

We call a the "process index" and b the "maturity index".

We can infer from context whether P_{ab} refers to "the value at time a of a bond that matures at b", or "the whole process for fixed b", or "the values at a fixed time a for a range of maturity dates b", or "the whole system of processes".

We shall assume here that the market is driven by a multi-dimensional family of independent Brownian motions W_t^α.

The "factor index" α can be understood (as before) as labelling the basis for a finite dimensional vector space, or as an "abstract" index representing a Hilbert space element in the infinite dimensional case.

The discount bond dynamics are then given by the stochastic equation

$$\frac{dP_{ab}}{P_{ab}} = \mu_{ab}\, da + \Omega_{ab}\, dW_a \tag{4.1}$$

Here μ_{ab} is the drift process for the b-maturity bond. Ω_{ab} is the corresponding vector volatility process. Both are assumed to be adapted to the filtration $(\mathcal{F}_t)$ generated by W_t^α.

We require that $\Omega_{aa} = 0$, corresponding to the fact that *a maturing bond has zero volatility*.

We also make the technically useful assumption that the process Ω_{ab} is differentiable in the maturity index.

More specifically, we assume that there exists a process σ_{as} such that

$$\Omega_{ab} = -\int_a^b \sigma_{as}\, ds, \tag{4.2}$$

where the minus sign appears as a matter of convention.

This relation enforces the constraint $\Omega_{aa} = 0$.

Note that in the term $\Omega_{ab}\, dW_a$ there is, as indicated earlier, an implied summation over the suppressed vector indices.

Now we impose the no-arbitrage condition. By the same line of argument as in the multi-asset case this ensures the existence of a risk premium vector λ_a^α such that the drift μ_{ab} is given by

$$\mu_{ab} = r_a + \sum_{\alpha=1}^{n} \lambda_a^\alpha \Omega_{ab}^\alpha. \tag{4.3}$$

Suppressing vector indices, we write this as $\mu_{ab} = r_a + \lambda_a \Omega_{ab}$.

Here r_a is the short rate, i.e. the rate of return on an instantaneously maturing discount bond.

In the discussion on multi-asset dynamics, we regarded r_a as an *exogenously specified* process. However, in the present consideration, the short rate is given by

$$r_a = - \left. \frac{\partial P_{ab}}{\partial b} \right|_{a=b} . \tag{4.4}$$

In fact, we shall show that r_a can be effectively *eliminated* as a fundamental variable, and an expression for the discount bonds can be derived entirely in terms of λ_a^α and Ω_{ab}^α.

Alternatively, we can eliminate Ω_{ab}^α, and an expression for the discount bond can be derived in terms of the martingale Λ_t (which incorporates λ_a^α) and the short rate process r_t (which can be specified arbitrarily). This will be shown later.

These diverse but ultimately *equivalent* ways of characterising interest rate dynamics are at the root of the various *apparently* diverse approaches to modelling that have been developed.

Inserting the expression for the drift (4.3) into the dynamics (4.1) of the bond prices, we get

$$\frac{dP_{ab}}{P_{ab}} = r_a \, da + \Omega_{ab} \left(dW_a + \lambda_a \, da \right) . \tag{4.5}$$

Basic interest rate models usually assume the interest rate market is *complete*.

This means, in particular, that the process Ω_{ab}^α has to satisfy a nondegeneracy sufficient to ensure that there does not exist at any time a vector η^α such that $\sum_\alpha \Omega_{ab}^\alpha \eta^\alpha = 0$ for all $b > a$.

In essence, this is equivalent to assuming that any interest rate derivative can be hedged with a suitable self-financing portfolio of discount bonds, together with the money market account.

Because the system of discount bonds is infinite, but each individual bond has a finite life, there are various ways in which the completeness condition can be met.

It is important to recognise that completeness is a rather strong assumption, and therefore may not be realised in practice.

Even if the discount bond market is *not* complete, there are circumstances in which it is appropriate to regard a definite choice of λ_a as being specified exogenously.

Note that the risk premium vector in the discount bond dynamics (4.5) combines suggestively with the Brownian motion so as to indicate a change of measure. We shall return to this point when we consider the valuation of interest rate derivatives.

4.2 Discount bond volatility and correlation

Let us now consider some "local" properties of the discount bond dynamics.

The dynamical equations under the assumption of no arbitrage are

$$\frac{dP_{ab}}{P_{ab}} = r_a\,da + \Omega_{ab}\cdot(dW_a + \lambda_a\,da)\,. \tag{4.6}$$

It follows on account of the Ito relations

$$dW_t^\alpha\,dW_t^\beta = \delta^{\alpha\beta}\,dt, \quad dW_t^\alpha\,dt = 0, \quad (dt)^2 = 0, \tag{4.7}$$

that

$$\left(\frac{dP_{ab}}{P_{ab}}\right)^2 = |\Omega_{ab}|^2\;da. \tag{4.8}$$

Here $|\Omega_{ab}|^2 = \sum_{\alpha=1}^n \Omega_{ab}^\alpha\Omega_{ab}^\alpha$ is the squared magnitude of the volatility vector for the bond with maturity b.

We refer to $|\Omega_{ab}|$ as the *local volatility* of the b-maturity discount bond.

If we consider bonds of two different maturities, say b and c, then the instantaneous or *local correlation* for their price dynamics is given by the process

$$\rho_{tbc} = \frac{\Omega_{tb}\cdot\Omega_{tc}}{|\Omega_{tb}|\,|\Omega_{tc}|}. \tag{4.9}$$

Clearly we have $-1 \le \rho_{tbc} \le 1$.

To work out the dynamics of P_{ab}, we need to know the vector processes Ω_{tb}^α and λ_t^α.

However, to work out the probability laws for P_{ab}, we only require the scalar combinations $|\Omega_{ab}|$, ρ_{tbc}, $|\lambda_t|$, and $\lambda_t\cdot\Omega_{tb}$.

4.3 Solution for the discount bond processes

The dynamical equation for the bond price involves the bond volatility, the relative risk, and the short rate.

However, we shall show now that the short rate can be eliminated, to give a representation of the bond price process in which the exogenous variables are the volatility process and the relative risk process.

The solution of the bond dynamics can be expressed in the form

$$P_{ab} = P_{0b} B_a \exp\left(\int_0^a \Omega_{sb} \left(dW_s + \lambda_s \, ds \right) - \tfrac{1}{2} \int_0^a |\Omega_{sb}|^2 \, ds \right). \tag{4.10}$$

Here B_a is the unit-initialised money market account process, given as usual by

$$B_a = \exp\left(\int_0^a r_s \, ds \right). \tag{4.11}$$

We observe, on the other hand, that the maturity condition $P_{aa} = 1$ allows us to solve for B_a in (4.10).

In particular, if we set $a = b$, we get:

$$B_a = (P_{0a})^{-1} \exp\left(- \int_0^a \Omega_{sa} \left(dW_s + \lambda_s \, ds \right) + \tfrac{1}{2} \int_0^a |\Omega_{sa}|^2 \, ds \right). \tag{4.12}$$

This shows how the short rate can be expressed in terms of the risk premium vector and the discount bond volatility.

More explicitly, by taking logarithms in (4.12), differentiating with respect to a, and using the relation $\Omega_{aa} = 0$, we get the following formula for r_a:

$$r_a = -\partial_a \ln P_{0a} + \int_0^a \Omega_{sa} \partial_a \Omega_{sa} \, ds - \int_0^a \partial_a \Omega_{sa} \left(dW_s + \lambda_s \, ds \right). \tag{4.13}$$

Here ∂_a denotes differentiation with respect to a. Thus we have solved for r_a in terms of λ_a^α and Ω_{ab}^α.

In obtaining these expressions, suitable technical conditions are required to be satisfied by the discount bond drift and volatility. We shall return later to address this issue more explicitly.

Inserting formula (4.12) for the money market account into (4.10) for the discount bonds, we then obtain the following general *quotient formula* for the discount bonds:

$$P_{ab} = P_{0ab} \frac{\exp\left(\int_0^a \Omega_{sb} \left(dW_s + \lambda_s \, ds \right) - \frac{1}{2} \int_0^a |\Omega_{sb}|^2 \, ds \right)}{\exp\left(\int_0^a \Omega_{sa} \left(dW_s + \lambda_s \, ds \right) - \frac{1}{2} \int_0^a |\Omega_{sa}|^2 \, ds \right)}. \tag{4.14}$$

Here $P_{0ab} = P_{0b}/P_{0a}$ denotes the forward value of a b-maturity bond, i.e., the value negotiated today for purchase at time a of a b-maturity discount bond.

In the quotient formula (4.14) note that the numerator and the denominator are essentially similar in structure, except the b in the numerator gets replaced by an a is the denominator.

The quotient formula is the desired explicit expression for the bond prices in terms of the two exogenous variables, the volatility vector and the relative risk vector, with the elimination of the short rate.

4.4 HJM dynamics for the forward short rate

The forward short rate process is given by

$$f_{ab} = -\partial_b \ln P_{ab}. \tag{4.15}$$

From the quotient formula it follows by differentiation that these rates can be expressed as follows:

$$f_{ab} = -\partial_b \ln P_{0b} + \int_0^a \Omega_{sb} \partial_b \Omega_{sb}\, ds + \int_0^a \partial_b \Omega_{sb} \left(dW_s + \lambda_s\, ds \right). \tag{4.16}$$

Heath, Jarrow & Morton (1992) take a general Itô process for the forward short rates as the starting point, and impose appropriate no-arbitrage and market completeness conditions to obtain an expression of the form (4.16).

We write $\sigma_{ab} = -\partial_b \Omega_{ab}$ for the forward short rate (i.e. instantaneous forward rate) volatility.

It follows that

$$\Omega_{ab} = -\int_a^b \sigma_{au}\, du. \tag{4.17}$$

This builds in the constraint $\Omega_{aa} = 0$, as we noted earlier.

Then for the forward short rate processes in terms of σ_{ab} we obtain:

$$f_{ab} = f_{0b} + \int_0^a \sigma_{sb} \left(\int_s^b \sigma_{su}\, du \right) ds + \int_0^a \sigma_{sb} \left(dW_s + \lambda_s\, ds \right). \tag{4.18}$$

Taking the stochastic differential of this expression on the process index we get

$$df_{ab} = \sigma_{ab} \left(\int_a^b \sigma_{au}\, du \right) da + \sigma_{ab} \cdot \left(dW_a + \lambda_a\, da \right). \tag{4.19}$$

These are the dynamics of the forward short rate, sometimes called the *HJM dynamics*.

It should be clear that the arbitrage-free dynamics of the discount bond system and the HJM forward short rate dynamics are for most practical purposes *entirely equivalent*.

Given the solution of the stochastic equation for f_{ab}, we can use the relation

$$P_{ab} = \exp\left(-\int_a^b f_{au}\, du\right) \tag{4.20}$$

to find the bond price.

The forward short rate processes are important from a conceptual point of view, but it should be noted that practical applications invariably refer back to the bond price process P_{ab} and the short rate process r_a.

4.5 Risk neutral valuation of interest rate derivatives

For the value of H_t at time t of a hedgeable interest rate derivative that pays H_T at time T, we have the forecasting relation

$$\frac{\Lambda_t H_t}{B_t} = \mathbb{E}_t\left[\frac{\Lambda_T H_T}{B_T}\right]. \tag{4.21}$$

Equivalently, we can write this in the form

$$H_t = B_t \mathbb{E}_t^{\lambda}\left[\frac{H_T}{B_T}\right]. \tag{4.22}$$

Here $\mathbb{E}_t^{\lambda}$ denotes conditional expectation in the risk-neutral measure induced by Λ_t which is defined by the exponential martingale

$$\Lambda_t = \exp\left(-\int_0^t \lambda_s \cdot dW_s - \tfrac{1}{2}\int_0^t |\lambda_s|^2\, ds\right). \tag{4.23}$$

One of the important features of interest rate theory is that the discount bonds themselves can be viewed as a species of "derivative".

The bond that matures at time T has a payoff of *unity* at that time.

As a consequence, if we set $H_T = 1$ in (4.21), we have the following *risk-neutral valuation formula* for the discount bond price process:

$$P_{tT} = \frac{B_t}{\Lambda_t}\mathbb{E}_t\left[\frac{\Lambda_T}{B_T}\right]. \tag{4.24}$$

An expression of this form was derived by Vasicek (1977).

In the risk neutral measure this can be written as follows:

$$\begin{aligned} P_{tT} &= B_t\mathbb{E}_t^{\lambda}\left[\frac{1}{B_T}\right] \\ &= \mathbb{E}_t^{\lambda}\left[\exp\left(-\int_t^T r_s\, ds\right)\right]. \end{aligned} \tag{4.25}$$

These formulae are often used as a *starting point* for interest rate modelling.

This is because it is possible to specify λ_t and r_t *exogenously*, without any *a priori* relation holding between them.

In particular, it follows from the risk-neutral valuation formula that $P_{tt} = 1$, and that for *any* choice of the process r_t and risk premium density Λ_t, the ratio

$$\frac{\Lambda_t P_{tT}}{B_t} \tag{4.26}$$

is a martingale.

This implies that the bond-price system P_{tT} satisfies the no-arbitrage condition, and thus qualifies as a bona-fide interest-rate model.

Thus summing up, we see that there are two apparently distinct but nevertheless entirely equivalent ways of "covering" the entire category of interest rate models:

(a) by specifying the relative risk process and vector volatility processes,

(b) by specifying the relative risk density together with the short rate process.

We shall return later to investigate in more detail the problem of how to characterise a general interest rate model, but let us first consider some specific interest rate models.

4.6 Market Models*

A good example of an important spin-off of the HJM approach, which has enjoyed considerable popularity as a basis for applications, is the so-called "market model" methodology.

There are a number of different variations on this approach–too many to attempt to survey here–according to which the forward Libor rates and/or swap rates associated the discount bond system are regarded as the "fundamental" dynamical entities.

In its simplest form, the idea of the market model is as follows. The forward Libor rates L_{tab} are defined in a standard way by the relation

$$P_{tab} = \frac{1}{1 + (b - a)L_{tab}}, \tag{4.27}$$

where $P_{tab} = P_{tb}/P_{ta}$ denotes the forward price made at time t for purchase of a b-maturity discount bond at time a.

For convenience we introduce a "tenor" parameter $\delta = b - a$, and write $L^{\delta}_{ta} = L_{tab}$.

It is then a straightforward exercise in Ito calculus to work out the dynamics of L^{δ}_{ta}, starting from the bond price dynamics given by

$$\frac{dP_{tT}}{P_{tT}} = r_t dt + \Omega_{tT}(dW_t + \lambda_t dt). \tag{4.28}$$

The result is a relation of the following form:

$$\frac{dL^{\delta}_{ta}}{L^{\delta}_{ta}} = \frac{1 + \delta L^{\delta}_{ta}}{\delta L^{\delta}_{ta}}(\Omega_{t,a} - \Omega_{t,a+\delta})(dW_t + \lambda_t dt - \Omega_{t,a+\delta}). \tag{4.29}$$

The key observation that follows is that if $\omega_{t,a}$ is a prescribed deterministic volatility process for a given fixed tenor then we can solve the equation

$$\frac{1 + \delta L^{\delta}_{ta}}{\delta L^{\delta}_{ta}}(\Omega_{t,a} - \Omega_{t,a+\delta}) = \omega_{t,a} \tag{4.30}$$

for the bond volatility in terms of the forward Libor rates and $\omega_{t,a}$.

This shows that there exists an HJM model with the prescribed deterministic volatility for the given forward Libor rate.

The next step is to change measure so as to eliminate the drift, which can clearly be carried out since now we know the bond volatility process.

As a consequence, we are left with a log-normal process for the forward Libor rate in the new measure.

It is generally recognised that the market model framework has probably been the single most influential development in interest rate theory in the post-1992 years following the advent of the HJM approach.

Many authors have contributed, in one way or another, and to varying degrees, to its origination and promulgation, and it would be impossible here to attempt with any success an objective account of the development of the market models and their various extensions, with all the relevant attributions.

Chapter 5

General theory of short rate diffusion models. Diffusion processes. The Feynman-Kac formula. Derivation of the discount bond pricing equation.

5.1 General theory of short rate diffusion models

An interesting and important class of interest rate models can be obtained by assuming:

(a) the short rate is a diffusion process,

(b) the discount bonds depend on r_t as a state variable.

Such models are often called "short rate" models.

Many of the most well-known interest rate models fall into this category, including for example the Vasicek model, the CIR model, the Black-Karazinski model, the Black-Derman-Toy model, the Hull-White model, and the rational lognormal model.

More specifically, we consider a family of discount bond price processes P_{tT} such that

$$P_{tT} = P(t, r_t, T). \tag{5.1}$$

Here $P(t, r, T)$ is a function of three variables. Thus the short rate acts as a "state variable" for this family of models.

The short rate process r_t $(t \geq 0)$ is assumed to satisfy a stochastic differential equation of the form

$$dr_t = \mu(t, r_t)\, dt + \sigma(t, r_t)\, dW_t. \tag{5.2}$$

Each of $\mu(t, r)$ and $\sigma(t, r)$ is a function of two variables.

The process W_t is a standard one-dimensional Brownian motion with respect to the natural probability measure.

Our goal is to derive a partial differential equation satisfied by the function $P(t, r, T)$ that arises as a consequence of the dynamical equations for P_{tT}.

This is called the 'bond pricing equation'.

5.2 Diffusion processes

Before embarking on a derivation of the bond-pricing equation, we digress briefly to say a few words about diffusions.

This is a topic of great interest in its own right, with many applications.

A process X_t satisfying a stochastic differential equation of the form

$$dX_t = a(t, X_t)\, dt + b(t, X_t)\, dW_t \tag{5.3}$$

where $a(t, x)$ and $b(t, x)$ are deterministic functions is called a *time inhomogeneous diffusion*.

If $a(t, x)$ and $b(t, x)$ do not depend explicitly on t, then X_t is a *time homogeneous diffusion*.

Generalisations of (5.3) can also be considered for which X_t and W_t are both multi-dimensional.

Early interest rate models (e.g., the Vasicek and CIR models) were based on homogeneous diffusions, but later it was recognised that inhomogeneous diffusions added flexibility to the models for fitting initial data, in particular initial yield curve and implied volatility data.

Homogeneous diffusions are more appropriate to equilibrium models, but these are not so useful in a banking context.

If $f(t, x)$ is a smooth function of two variables, then by Ito's formula we have

$$df(t, X_t) = \left(\frac{\partial f}{\partial t} + \frac{1}{2} b^2 \frac{\partial^2 f}{\partial x^2} + a \frac{\partial f}{\partial x} \right) dt + b \frac{\partial f}{\partial x} dW_t. \tag{5.4}$$

Here, of course, the derivatives $\partial f / \partial t$, $\partial f / \partial x$ and $\partial^2 f / \partial x^2$ are valued at $x = X_t$.

The second order differential operator

$$\mathcal{L} = \frac{1}{2} b^2 \frac{\partial^2}{\partial x^2} + a \frac{\partial}{\partial x} \tag{5.5}$$

is called the *generator* of the diffusion.

The generators of diffusions arise naturally in connection with elliptic and parabolic partial differential equations, and in certain cases there are natural probabilistic interpretations of the solutions of these equations.

These results form the basis of the importance of PDE methods in finance, and have numerous practical applications. We give a few examples.

(a) Consider the parabolic equation

$$\frac{\partial \psi}{\partial t} = \mathcal{L}\psi, \tag{5.6}$$

subject to the initial condition $\psi(0, x) = f(x)$, for some prescribed continuous function f.

Now let $\mathbb{E}^x$ denote the expectation operator. Here, the superscript x indicates that we assume that initially $X_0 = x$.

Under appropriate technical assumptions, the solution of (5.6) is given by

$$\psi(t, x) = \mathbb{E}^x[f(X_t)]. \tag{5.7}$$

(b) A more general result is the following. Consider the partial differential equation

$$\frac{\partial \psi}{\partial t} = \mathcal{L}\psi - g\psi + h, \tag{5.8}$$

subject to the initial condition $\psi(0, x) = f(x)$.

Here, $g(x) \geq 0$, $h(t, x)$ and $f(x)$ are prescribed continuous functions.

Then, under suitable technical assumptions, the solution of (5.8) can be expressed in the form

$$\psi(t, x) = \mathbb{E}^x \left[\int_0^t \exp\left(- \int_0^u g(X_s)\, ds \right) h(u, X_u)\, du + \exp\left(- \int_0^t g(X_s)\, ds \right) f(X_t) \right]. \tag{5.9}$$

This result is known as the *Feynman-Kac formula*.

(c) We now turn to a different kind of result, involving stopping times. Let $f(x)$ be a smooth function, and let τ be a stopping time such that $\mathbb{E}^x[\tau] < \infty$.

We recall that τ is a stopping time relative to the filtration $\{\mathcal{F}_t\}$ if for every t the event

$\{\tau \leq t\}$ is $\mathcal{F}_t$-measurable.

Intuitively, this means that once we reach any given time t, we can determine whether the even has occurred or not.

Then *Dynkin's formula* says that

$$\mathbb{E}^x[f(X_\tau)] = f(x) + \mathbb{E}^x\left[\int_0^\tau \mathcal{L}f(X_s)\,ds\right]. \tag{5.10}$$

(d) Another very useful result is the so-called *Kolmogorov forward equation*, also known as the *Fokker-Planck equation*.

This is a partial differential equation for the probability density function $\rho(t,x)$ of X_t:

$$\frac{\partial \rho}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial x^2}(b^2\rho) - \frac{\partial}{\partial x}(a\rho). \tag{5.11}$$

Given an initial distribution $\rho(0,x)$ for X_0, we can work out the distribution of X_t at later times t by solving this equation.

We see, for example, that if $a = 0$ and $b = 1$ is a constant, then the diffusion is a Brownian motion and the Fokker-Planck equation reduces to the *heat equation*.

These results have various multi-dimensional generalisations.

5.3 Derivation of the bond-pricing equation

Let us return to the bond-price process $P_{tT} = P(t, r_t, T)$, $t \in [0,T]$.

For each fixed value of T, we can use Itô's lemma to obtain

$$dP(t,r_t,T) = \left(\frac{\partial P}{\partial t} + \mu\frac{\partial P}{\partial r} + \tfrac{1}{2}\sigma^2\frac{\partial^2 P}{\partial r^2}\right)dt + \left(\sigma\frac{\partial P}{\partial r}\right)dW_t. \tag{5.12}$$

The bracketed expressions are valued at (t, r_t).

In the case of an interest rate system driven by a single Brownian motion, the no-arbitrage dynamics of P_{tT} are given by

$$dP_{tT} = (r_t + \lambda_t\Omega_{tT})P_{tT}\,dt + \Omega_{tT}P_{tT}\,dW_t. \tag{5.13}$$

Equating these two relations we see that the volatility process is given by

$$\Omega_{tT} = \sigma(t, r_t) \frac{1}{P(t, r_t, T)} \frac{\partial P(t, r_t, T)}{\partial r}. \tag{5.14}$$

Comparison of the drift terms appearing in (5.12) and (5.13) then allows us to deduce that the risk premium process λ_t must be of the form

$$\lambda_t = \lambda(t, r_t), \tag{5.15}$$

for some function $\lambda(t, r)$ of two variables.

In the light of these observations, we can equate the drifts in (5.12) and (5.13) to obtain the following PDE:

$$\frac{\partial P}{\partial t} + \mu \frac{\partial P}{\partial r} + \tfrac{1}{2}\sigma^2 \frac{\partial^2 P}{\partial r^2} = rP + \lambda\sigma \frac{\partial P}{\partial r}. \tag{5.16}$$

This is the *bond-pricing equation* for the short-rate state variable models.

We require a solution of this PDE subject to the terminal condition $P_{TT} = 1$, or equivalently, $P(T, r, T) = 1$ for all r.

5.4 Solution and calibration of the bond pricing equation

To obtain an interest rate model by this technique, we first need to specify the functions $\mu(t, r)$, $\sigma(t, r)$ and $\lambda(t, r)$.

Then we solve the stochastic differential equation (5.2) for the process r_t, $t \geq 0$.

Ideally, we want diffusions such that $r_t > 0$, for all $t \geq 0$, but some models, particularly older models such as the Vasicek model, do not necessarily have this property.

We then solve the partial differential equation (5.16) for $P(t, r, T)$ subject to the *terminal maturity condition* $P(T, r, T) = 1$, which must hold for all values of the variable r.

Finally, we may also wish to impose an initial condition $P(0, r_0, T) = P_{0T}$, where

$$r_0 = -\left(\partial_T P_{0T}\right)\big|_{T=0}. \tag{5.17}$$

This condition incorporates the *initial discount function* P_{0T} into the dynamics.

The "terminal" condition $P(T, r, T) = 1$ is typical of the sort of information one needs to obtain a *unique* solution of a parabolic equation such as (5.16), if we are told the functions μ, σ and λ.

This is related to the fact that, subject to some technical considerations, the ordinary heat equation

$$\frac{\partial \phi}{\partial t} = \frac{1}{2}\frac{\partial^2 \phi}{\partial x^2} \tag{5.18}$$

has a unique solution $\phi(t, x)$ if we supply the initial condition $\phi(0, x) = f(x)$.

It will thus not always be possible to impose an *initial* condition such as (5.17) as well.

This point is illustrated, for example, in the so-called "equilibrium" or "stationary" models, for which μ, σ and λ do not depend on the variable t, and are functions of r alone.

In this case r_t, $t \geq 0$, is a stationary diffusion process.

Some well-known examples of models of this type are:

(a) the Vasicek model, for which $\mu(r) = k(\theta - r)$, $\sigma(r) = \sigma$, and $\lambda(r) = \lambda$ where k, θ, σ and λ are constants;

(b) the CIR model, for which $\mu(r) = k(\theta - r)$, $\sigma(r) = \sigma\sqrt{r}$ and $\lambda(r) = \xi\sqrt{r}/\sigma$, where k, θ, σ and ξ are constants.

In both cases, k and θ are taken to be positive.

In the CIR model we also require $k\theta > \frac{1}{2}\sigma^2$, which ensures that $r_t > 0$, for all $t \geq 0$, if we assume that $r_0 > 0$.

It should be noted that the CIR volatility parameter has different "dimensions" from the Vasicek volatility parameter.

In these examples, the specification of the parameters k, θ, σ, λ, ξ and r_0 is sufficient to completely determine the initial discount function.

In other words, in a *stationary short rate model we cannot expect to be able to incorporate an arbitrary initial yield curve.*

Historically, this is one of the reasons why the "extended" models were developed. Originally, the goal of interest rate modelling was to determine, by equilibrium conditions, a finite dimensional 'class' of yield curves to which the actual yield curve would have to belong.

Thus, one way to incorporate *initial conditions* is to regard the functions μ, σ, λ as only *partially* specified. For example, if we set

$$\mu(t,r) = k(t)(\theta(t) - r) \quad \text{and} \quad \sigma(t,r) = \sigma(t), \tag{5.19}$$

then we obtain the so-called *extended Vasicek model*, due to Hull and White, and to Jamshidian.

In this case, the disposable functions $k(t)$, $\theta(t)$ and $\sigma(t)$ are chosen so that the initial condition is satisfied for the discount function.

That only "uses up" one of the three functions (θ as it happens), so it is possible (in principle) to fix some *other* initial conditions as well, e.g. implied volatility data for certain classes of interest rate options.

If sufficient initial "market" data is specified to fix all three functions, then we say that the model has been *calibrated*.

There is no known general *a priori* principle that dictates which initial data should be incorporated into an interest rate model.

The banking industry is still experimenting with this issue.

A more useful point of view, perhaps, is based on the idea of conditioning.

That is to say, the interest rate model is always conditioned on the data available, and likewise the pricing of derivatives is always conditional on the information supplied regarding the prices of other derivatives.

Of course, we might try to calibrate an *equilibrium* model to the initial yield curve, e.g. by choosing the parameters k, θ, σ, λ and r_0 in the case of Vasicek or CIR to fit the initial term structure.

In practice, one expects this to be difficult since the initial (i.e. current) yield curve can exhibit a good measure of highly tuned microstructure (e.g. with some rates specified down to a basis point).

Also, due to spreads and variations between deals there is necessarily some 'fuzziness' in the specification.

It should be clear that a short-rate model is *a fortiori* a discount bond model, and is therefore an HJM model.

Thus, mathematically speaking, short rate models constitute a *subset* of the HJM class, not a *distinct* class.

Sometimes it is said that ‘all’ HJM models are short rate models. This is true, but only if one gives a rather different interpretation to the meaning of ‘short rate’ model. Usually it is clear from context.

Chapter 6

Theory of affine term structure models, including the Vasicek model and the Cox-Ingersoll-Ross (CIR) model.

6.1 The Vasicek model

Let us consider in more detail the model of Vasicek (1977).

The short rate is assumed to follow a dynamical relation of the form

$$dr_t = k(\theta - r_t)\,dt + \sigma\,dW_t. \tag{6.1}$$

The constants k, θ and σ are taken to be *positive*, and have the following interpretation:

σ is the *absolute volatility* of the rate r_t,

θ is the *mean reversion level*,

k is the *mean reversion rate*.

Clearly, r_t is a diffusion process.

The dynamics of r_t are exactly solvable in the case of the Vasicek model. The result is called an Ornstein-Uhlenbeck process, and the solution for r_t is as follows:

$$r_t = \theta + (r_0 - \theta)e^{-kt} + \sigma \int_0^t e^{k(s-t)}\,dW_s. \tag{6.2}$$

One easily checks that (6.2) satisfies (6.1), subject to the initial condition r_0.

The technique we use to solve (6.1) is to multiply each side of the equation by the integrating factor e^{kt}, and the result drops out quickly.

The theory of this process is described in a well-known article by Doob.

From the formula for r_t we can read off a number of qualitative features of the process.

For example, since $\mathbb{E}\left[\int_0^t e^{ks}\, dW_s\right] = 0$, we see that

$$\mathbb{E}\left[r_t\right] = \theta + (r_0 - \theta)e^{-kt}. \tag{6.3}$$

This result shows that the mean of r_t starts at r_0, and then over time reverts to θ.

The 'speed' of this movement is governed by the constant k.

Also, we have

$$\mathrm{var}\left[r_t\right] = \frac{\sigma^2}{2k}\left(1 - e^{-2kt}\right). \tag{6.4}$$

Thus the variance of r_t is initially zero, and it increases to a maximum level given by $\frac{1}{2}\sigma^2/k$. The calculation of the variance involves a simple application of the Ito isometry.

In particular, if $f(s,t)$ is deterministic, then

$$\mathbb{E}\left[\left(\int_0^t f(s,t) dW_s\right)^2\right] = \int_0^t f^2(s,t) ds. \tag{6.5}$$

There is a *characteristic time-scale* $1/k$ associated with the mean-reversion rate. This determines the time scale over which r_t moves from r_0 towards θ.

The bond pricing equation is exactly solvable in the Vasicek model if we assume the relative risk λ is a constant.

The solution can be written as follows:

$$P(t, r_t, T) = \exp\left(\frac{1}{k}\left(1 - e^{-k(T-t)}\right)(R^\infty - r_t) - (T-t)R^\infty - \frac{\sigma^2}{4k^3}\left(1 - e^{-k(T-t)}\right)^2\right). \tag{6.6}$$

The constant R^∞ here is defined by

$$R^\infty = \theta - \frac{\lambda\sigma}{k} - \frac{\sigma^2}{2k^2}. \tag{6.7}$$

The significance of R^∞ is that it represents the continuously compounded rate of interest (yield) on a bond of very long maturity. This is seen as follows.

We recall that, given the bond price P_{tT}, the continuously compounded yield R_{tT} is

$$P_{tT} = \exp\left(-(T-t)R_{tT}\right). \tag{6.8}$$

Inverting this relation we have

$$R_{tT} = -\frac{1}{T-t}\ln P_{tT}. \tag{6.9}$$

Hence in the present example we have

$$R_{tT} = R + (r_t - R)\frac{1}{k(T-t)}\left(1 - e^{-k(T-t)}\right) + \frac{\sigma^2}{4k^3(T-t)}\left(1 - e^{-k(T-t)}\right)^2. \tag{6.10}$$

Thus for fixed t, we have $R_{tT} \to R$ as $T \to \infty$.

On the other hand, we can also check that $R_{tt} = r_t$. In other words, the yield on a very short maturity bond is the short rate.

6.2 Affine models

The expression for the bond price in the Vasicek model can be simplified if we introduce the variable $u = T - t$ for the *time to maturity*, and set

$$B_{tu} = P_{t,t+u} = P(t, r_t, t+u). \tag{6.11}$$

Here B_{tu} represents the price at time t of a bond with u years until maturity.

We call u the *tenor* of the bond.

For convenience, let us define the function

$$f(u) = \frac{1}{k}\left(1 - e^{-ku}\right). \tag{6.12}$$

Then for the Vasicek bond price we have

$$B_{tu} = \exp\left(-f(u)r_t + (f(u) - u)R - \frac{\sigma^2}{2k}f^2(u)\right). \tag{6.13}$$

We note that the exponent is a linear function of r_t, and the coefficients are functions of u.

The class of interest rate models for which $B_{tu} = P_{t,t+u}$ can be put into the form

$$B_{tu} = e^{-f(u)r_t - g(u)} \tag{6.14}$$

is of special interest. These are called the *stationary affine models*.

A short rate model generates a stationary discount bond system if r_t is stationary and B_{tu} can be expressed in the form $B_{tu} = B(r_t, u)$, where $B(r, u)$ is a function of two variables.

If the bond price takes the more general form

$$P_{tT} = e^{-F(t,T)r_t - G(t,T)} \tag{6.15}$$

for deterministic functions $F(t, T)$ and $G(t, T)$, then we have an *extended* affine model.

For example, it is an interesting exercise to show that in the case of an *extended Vasicek model*, for which r_t follows a process of the form

$$dr_t = k(t)(\theta(t) - r_t)\,dt + \sigma(t)\,dW_t, \tag{6.16}$$

where $k(t)$, $\theta(t)$ and $\sigma(t)$ are positive, deterministic functions, then the bond price system is of the extended affine type.

In the extended Vasicek model (also sometimes known as the Hull-White model), it is a straightforward exercise to show that

$$r_t = r_0 e^{-\beta(t)} + e^{-\beta(t)} \int_0^t e^{\beta(s)} \theta(s)\,ds + e^{-\beta(t)} \int_0^t e^{\beta(s)} \sigma(s)\,dW_s, \tag{6.17}$$

where

$$\beta(t) := \int_0^t k(s)\,ds \tag{6.18}$$

The relevant calculations follow the arguments given earlier.

In particular, for constant parameters, this reduces to the previous expression given for the short rate process r_t.

6.3 The CIR model

In this important and rather more complicated model (Cox, Ingersoll & Ross 1985), the short rate is assumed to be a *mean reverting square-root process*, for which the dynamics are given by:

$$dr_t = k(\theta - r_t)\,dt + \sigma\sqrt{r_t}\,dW_t. \tag{6.19}$$

The solution of this equation is not as easy as in the Vasicek model.

Nevertheless, its essential features can be revealed by writing the dynamics in integral form

$$r_t = \theta + (r_0 - \theta)e^{-kt} + \sigma \int_0^t e^{-k(t-s)} \sqrt{r_s}\, dW_s. \tag{6.20}$$

The mean reverting property of r_t is apparent.

We immediately infer the mean of r_t:

$$\mathbb{E}[r_t] = \theta + (r_0 - \theta)e^{-kt}. \tag{6.21}$$

By use of the Itô isometry we then obtain

$$\begin{aligned} \text{var}[r_t] &= \mathbb{E}\left[(r_t - \mathbb{E}[r_t])^2\right] \\ &= \sigma^2 e^{-2kt} \mathbb{E}\left[\int_0^t e^{2ks} r_s ds\right] \end{aligned} \tag{6.22}$$

Substituting the expression for $\mathbb{E}[r_s]$ into this and integrating, we get:

$$\text{var}[r_t] = \frac{\sigma^2 \theta}{2k}\left(1 - e^{-kt}\right)^2 + \frac{\sigma^2}{k} r_0 e^{-kt}\left(1 - e^{-kt}\right). \tag{6.23}$$

Thus for small t we have $\text{var}[r_t] \simeq \sigma^2 t r_0$.

Whereas, for large t we have $\text{var}[r_t] \to \sigma^2\theta/2k$.

It is a subtle result due to Feller that:

(a) if $r_0 > 0$, then $r_t \geq 0$, and

(b) if $r_0 > 0$ and $k\theta > \frac{1}{2}\sigma^2$, then $r_t > 0$ (strictly positive interest rates).

Now so far we have not yet considered the market price of risk.

To obtain a solution for the bond pricing equation it turns out that we need to assume that the relative risk process is of the special form

$$\lambda_t = \frac{\xi\sqrt{r_t}}{\sigma} \tag{6.24}$$

where ξ is a constant.

Then we can solve for P_{tT} as a function $P(t, r_t, T)$.

If we write P_{tT} in the affine form

$$P_{tT} = e^{-F(t,T)r_t - G(t,T)} \tag{6.25}$$

then the solution for $F(t, T)$ and $G(t, T)$ can be found in terms of k, θ, σ and ξ.

In particular, writing

$$P(t, r, T) = e^{-F(t,T)r - G(t,T)}, \tag{6.26}$$

we see that the bond-pricing equation reduces to two conditions, namely

$$1 + \frac{\partial F}{\partial t} = (k + \xi)F + \tfrac{1}{2}\sigma^2 F^2 \tag{6.27}$$

and

$$\frac{\partial G}{\partial t} = -k\theta F. \tag{6.28}$$

We need to solve these subject to the boundary conditions

$$F(T, T) = 0, \qquad G(T, T) = 0, \quad \text{and} \quad \left.\frac{\partial F}{\partial t}\right|_{t=T} = 1. \tag{6.29}$$

For convenience, we define the constants

$$\nu := k + \xi, \qquad \gamma := \sqrt{\nu^2 + 2\sigma^2} \tag{6.30}$$

Then the solution is given by

$$F(t, T) = \frac{2(e^{\gamma x} - 1)}{(\gamma + \nu)(e^{\gamma x} - 1) + 2\gamma} \tag{6.31}$$

$$e^{-G(t,T)} = \left[\frac{2\gamma e^{(\gamma+\nu)x}}{(\gamma + \nu)(e^{\gamma x} - 1) + 2\gamma}\right]^{\frac{2k\theta}{\sigma^2}} \tag{6.32}$$

where $x = T - t$.

Chapter 7

Overview of term structure frameworks. Admissible term structures and term structure comparison. Dynamics of the term structure density. Positive interest HJM volatility structure.

7.1 Overview of term structure frameworks

Dynamical models for interest rates suffer from the fact that it is difficult to isolate the *independent* degrees of freedom in the evolution of the term structure.

The question is, which ingredients in the determination of an interest rate model can and should be specified independently and exogenously?

We shall consider briefly two examples of this can be done for general interest rate models, indicating as well the associated drawbacks.

Example 1. Dynamic models for the short rate. The independent degrees of freedom are given by:

(a) the specification of the short rate r_t as an essentially arbitrary Ito process, and

(b) a market risk premium λ_t^α $(\alpha = 1, 2, \cdots, n)$.

The model for the discount bonds is

$$P_{tT} = \frac{1}{\Lambda_t}\mathbb{E}_t\left[\Lambda_T \exp\left(-\int_t^T r_s ds\right)\right]. \tag{7.1}$$

Here $\mathbb{E}_t$ denotes conditional expectation with respect to the filtration $\mathcal{F}_t$. The density martingale Λ_t is defined by

$$\Lambda_t = \exp\left(-\int_0^t \lambda_s dW_s - \tfrac{1}{2}\int_0^t \lambda_s^2 ds\right). \tag{7.2}$$

where

$$\lambda_s dW_s = \sum_{\alpha=1}^{n} \lambda_s^\alpha dW_s^\alpha \tag{7.3}$$

An advantage of this general model is that r_t and λ_t^α can be specified exogenously, and for interest rate positivity it suffices to let the process r_t be positive.

There are two disadvantages to this approach.

Firstly, the model is specified implicitly: the conditional expectation is generally difficult to calculate. Secondly, the initial term structure is not fed in directly.

A further simplification can be achieved by introducing the state price density:

$$V_t = \Lambda_t \exp\left(-\int_0^t r_s ds\right). \tag{7.4}$$

It follows that

$$P_{tT} = \frac{\mathbb{E}_t[V_T]}{V_t}. \tag{7.5}$$

Then it is sufficient to specify the state price density V_t alone, and we recover r_t and λ_t^α from the relation

$$\frac{dV_t}{V_t} = -r_t dt - \lambda_t dW_t. \tag{7.6}$$

Example 2. The Heath-Jarrow-Morton framework. In this case the independent dynamical degrees of freedom consist of:

(a) the initial term structure P_{0T},

(b) the market risk premium process λ_t^α, and

(c) the forward short rate volatility process σ_{tT}^α for each maturity T.

The model for the discount bonds is

$$P_{tT} = \exp\left(\int_t^T f_{ts} ds\right). \tag{7.7}$$

The forward short rates are

$$f_{tT} = -\frac{\partial}{\partial T}\ln P_{0T} - \int_{s=0}^{t} \sigma_{sT}\Omega_{sT} ds + \int_{s=0}^{t} \sigma_{sT}(dW_s + \lambda_s ds), \tag{7.8}$$

where

$$\Omega^{\alpha}_{tT} = -\int_{u=t}^{T} \sigma^{\alpha}_{tu} du. \tag{7.9}$$

The advantage of the HJM framework is that it allows a direct input of the initial term structure, as well as control over the volatility structure of the discount bonds.

A disadvantage of the HJM approach is that there is no guarantee of interest rate positivity, and it is not easy to impose a condition on σ^{α}_{tT} to achieve this.

Now we consider an alternative framework for isolating the independent degrees of freedom in interest rate dynamics that has the virtue of retaining the desirable features of both examples cited above, while eliminating the undesirable features.

The key idea is the introduction of a term structure density process $\rho_t(x)$ defined by

$$\rho_t(x) = -\frac{\partial}{\partial x} B_{tx}. \tag{7.10}$$

Here B_{tx} denotes the system of bond prices at time t when we parameterise the bonds by the tenor variable $x = T - t$ (Musiela parameterisation), so

$$B_{tx} = P_{t,t+x}. \tag{7.11}$$

We make the assumption that $B_{tx} \to 0$ for large x.

It is then a straightforward exercise to verify that the interest rate positivity conditions

$$0 < B_{tx} \leq 1, \quad \text{and} \quad \frac{\partial}{\partial x} B_{tx} < 0 \tag{7.12}$$

are equivalent to the following relations on $\rho_t(x)$:

$$\rho_t(x) > 0, \quad \text{and} \quad \int_0^{\infty} \rho_t(x) dx = 1. \tag{7.13}$$

We therefore conclude that any positive interest rate model can be regarded as a random process on the space of density functions on the positive real line.

The idea is that we treat the yield curve as a mathematical object in its own right, identified as a "point" ρ lying in the space $\mathfrak{M}$ of all possible yield curves.

With the specification of an initial yield curve ρ_0 we model the resulting dynamics as a random trajectory ρ_t in $\mathfrak{M}$. By bringing the structure of $\mathfrak{M}$ into play it is possible both to clarify the status of existing interest models, and also to devise new interest rate models.

7.2 Admissible term structures and term structure comparison.

There is a natural 'information geometry' associated with the space of yield curves.

Let $t = 0$ denote the present, and P_{0x} a family of discount bond prices satisfying $P_{00} = 1$, where x is the tenor ($0 \leq x < \infty$).

We impose the condition that interest rates should always be positive with the following criterion:

Definition. *A term structure is said to be admissible if the discount function* P_{0x} *is of class* C^∞ *and satisfies* $0 < P_{0x} \leq 1$, $\partial_x P_{0x} < 0$, *and* $\lim_{x\to\infty} P_{0x} = 0$.

An admissible discount function can be viewed as a complementary probability distribution. In other words, we can think of the tenor date as an abstract random variable X, and for its distribution write

$$\Pr[X < x] = 1 - P_{0x}. \tag{7.14}$$

The associated density function $\rho(x) = -\partial_x P_{0x}$ satisfies $\rho(x) > 0$ for all x, and

$$\int_x^\infty \rho(u)du = P_{0x}.$$

We say that a density function is *smooth* if it is of class C^∞ on the positive half-line $\mathbb{R}^1_+ = [0, \infty)$.

Proposition 1. *The system of admissible term structures is isomorphic to the convex space* $\mathcal{D}(\mathbb{R}^1_+)$ *of everywhere positive smooth density functions on the positive real line.*

The requirement that P_{0x} should be of class C^∞ can be weakened, but in practice any term structure can be approximated arbitrarily closely by a 'nearby' term structure with a smooth density.

It is reasonable to insist that the forward short rate curve $f_{0x} = -\partial_x \ln P_{0x}$ is piecewise continuous and nonvanishing for all $x < \infty$.

Given a pair of term structure densities $\rho_1(x)$ and $\rho_2(x)$ we can define a distance function ϕ_{12} on $\mathfrak{M}$ by

$$\phi_{12} = \cos^{-1} \int_0^\infty \xi_1(x)\xi_2(x)dx, \tag{7.15}$$

where $\xi_i(x) = \sqrt{\rho_i(x)}$. We call this angle the Bhattacharyya distance between the given yield curves.

The geometrical interpretation of ϕ_{12} arises from the fact that the map $\rho(x) \to \xi(x)$ associates to each point of $\mathfrak{M}$ a point in the positive orthant $\mathcal{S}^+$ of the unit sphere in the Hilbert space $L^2(\mathbb{R}^1_+)$, and ϕ_{12} is the resulting spherical angle on $\mathcal{S}^+$.

Note that $0 \leq \phi < \frac{1}{2}\pi$ and that orthogonality can never be achieved if forward rates are nonvanishing.

As a simple illustration we consider the family of discount bonds given by

$$P_{0x} = \left(1 + \frac{Rx}{\kappa}\right)^{-\kappa}, \tag{7.16}$$

where R and κ are constants.

In this case we have a flat term structure, with a constant annualised rate of interest R assuming compounding at the frequency κ over the life of each bond.

For $\kappa = 1$ this reduces to the case of a flat rate on the basis of a simple yield, and in the limit $\kappa \to \infty$ we recover the case of a flat rate on the basis of continuous compounding.

For the density function $\rho(x) = -\partial_x P_{0x}$ associated with (7.16) we obtain

$$\rho(x) = R\left(1 + \frac{Rx}{\kappa}\right)^{-(\kappa+1)}. \tag{7.17}$$

Let us write $\rho_i(x)$ for the density corresponding to $R = R_i$ $(i = 1, 2)$ for a fixed value of κ. A direct calculation of the integral (7.15) for $\kappa = 1$ gives

$$\phi_{12} = \cos^{-1}\left(\frac{\sqrt{R_1 R_2}}{R_1 - R_2} \log \frac{R_1}{R_2}\right). \tag{7.18}$$

In the limit $\kappa \to \infty$ (continuous compounding) we find that

$$\phi_{12} = \cos^{-1}\left(\frac{2\sqrt{R_1 R_2}}{R_1 + R_2}\right). \tag{7.19}$$

Note that the bracketed term in (7.19) is the ratio of the geometric and arithmetic means of the two rates. In this limit we have $\rho(x) \to R\mathrm{e}^{-Rx}$.

7.3 Dynamics of the term structure density

Now let us consider the evolution of the term structure density.

We write P_{tT} for the random value at time t of a discount bond that matures at time T, where $T \in \mathbb{R}^1_+$ and $0 \le t \le T$, and assume, for each T, that P_{tT} is an Ito process on the interval $t \in [0, T]$:

$$dP_{tT} = m_{tT}dt + \Sigma_{tT}dW_t. \tag{7.20}$$

The absolute drift m_{tT} and the absolute volatility process Σ_{tT} are assumed to satisfy regularity conditions sufficient to ensure that $\partial_T P_{tT}$ is also an Ito process.

For interest rate positivity we require $0 < P_{tT} \le 1$ and $\partial_T P_{tT} < 0$.

Additionally we impose the asymptotic conditions $\lim_{T\to\infty} P_{tT} = 0$, and $\lim_{T\to\infty} \partial_T P_{tT} = 0$.

Because P_{tT} is positive, the forward short rate process f_{tT} is an Ito process iff $-\partial_T P_{tT}$ is an Ito process.

For no arbitrage we require the existence of an exogenous market risk premium process λ_t such that

$$m_{tT} = r_t P_{tT} + \lambda_t \Sigma_{tT}. \tag{7.21}$$

We do not assume the bond market is complete. If the bond market is complete, however, then λ_t is determined endogenously by the bond price system.

We introduce the Musiela parameterisation $x = T - t$, and write $B_{tx} = P_{t,t+x}$ for the price at time t of a bond for which the time to maturity is x.

We have the following dynamics for B_{tx}:

$$dB_{tx} = (r_t - f_{t,t+x})B_{tx}dt + \Sigma_{t,t+x}(dW_t + \lambda_t dt). \tag{7.22}$$

Now consider the time dependent term structure density $\rho_t(x)$ defined by (7.10), for which we have the normalisation condition

$$\int_{x=0}^{\infty} \rho_t(x)dx = 1, \tag{7.23}$$

or equivalently

$$\int_{u=t}^{\infty} \rho_t(u-t)du = 1. \tag{7.24}$$

The relation

$$\rho_t(T-t) = f_{tT}P_{tT} \tag{7.25}$$

allows us to deduce an interpretation of the normalisation condition. In particular, the formula

$$\int_t^{\infty} P_{tu}f_{tu}du = 1 \tag{7.26}$$

says that the value at time t of a continuous cash flow in perpetuity that pays the small amount $f_{tu}du$ at time u is always unity.

Thus we can think of f_{tu} as defining the 'convenience yield' associated with a position in cash.

An analogous calculation shows that

$$\int_t^{\infty} P_{tu}^{\kappa}f_{tu}du = \frac{1}{\kappa} \tag{7.27}$$

for any positive value of the exponent κ.

This relation can be interpreted by saying that if we 'fix' the convenience yield (e.g., by swapping the unit of cash for the corresponding future cash flow), and then rescale all the interest rates R_{tu} by the same factor κ, so $R_{tu} \to \kappa R_{tu}$ for all $u \geq t$, then the value of the promised cash flow scales inversely with respect to κ.

Returning now to the evolutionary equation we write

$$\omega_{tx} = -\partial_x \Sigma_{t,t+x}. \tag{7.28}$$

Then we obtain the following dynamics for $\rho_t(x)$:

$$d\rho_t(x) = (r_t\rho_t(x) + \partial_x\rho_t(x))dt + \omega_{tx}(dW_t + \lambda_t dt). \tag{7.29}$$

The process ω_{tx} is subject to the constraint $\int_0^{\infty}\omega_{tx}dx = 0$, which implies that ω_{tx} is of the form

$$\omega_{tx} = \rho_t(x)(\nu_t(x) - \bar{\nu}_t), \tag{7.30}$$

where $\nu_t(x)$ is unconstrained, and

$$\bar{\nu}_t = \int_0^{\infty} \rho_t(u)\nu_t(u)du. \tag{7.31}$$

It follows from equation (7.28) that the absolute discount bond volatility Σ_{tT} is given in the Musiela parameterisation by

$$\begin{aligned}\Sigma_{t,t+x} &= \int_{u=x}^{\infty} \omega_{tu} du \\ &= \int_{u=x}^{\infty} \rho_t(u)\nu_t(u)du - \bar{\nu}_t B_{tx}.\end{aligned} \tag{7.32}$$

This relation has an interesting probabilistic interpretation. Suppose, in particular, we write $I_x(u)$ for the indicator function

$$I_x(u) = \chi(u \geq x), \tag{7.33}$$

where $\chi(A)$ is unity if A is true and vanishes otherwise.

Then the bond price B_{tx} can be written in the form of an abstract 'expectation':

$$B_{tx} = \int_{u=0}^{\infty} \rho_t(u) I_x(u) du = \mathbb{M}_t \left[I_x \right], \tag{7.34}$$

where

$$\mathbb{M}_t[g] = \int_{u=0}^{\infty} \rho_t(u) g(u) du \tag{7.35}$$

for any function $g(x)$.

The absolute discount bond volatility can then be expressed as an abstract covariance of the form

$$\Sigma_{t,t+x} = \mathbb{M}_t \left[I_x \nu_t \right] - \mathbb{M}_t \left[I_x \right] \mathbb{M}_t[\nu_t]. \tag{7.36}$$

We see that the bond volatility structure $\Sigma_{t,t+x}$ is invariant under the transformation $\nu_t(x) \to \nu_t(x) + \alpha_t$, where α_t is independent of x.

This 'gauge' freedom can be used to set $\lambda_t = -\bar{\nu}_t$. Then λ_t and Σ_{tx} are both determined by $\nu_t(x)$.

Proposition 2. *The general admissible term structure evolution based on the filtration generated by a Brownian motion W_t on $\mathcal{H}$ is a measure valued process $\rho_t(x)$ on $\mathcal{D}(\mathbb{R}^1_+)$ that satisfies*

$$d\rho_t(x) = \left(r_t \rho_t(x) + \partial_x \rho_t(x) \right) dt + \rho_t(x) \left(\nu_t(x) - \bar{\nu}_t \right) \left(dW_t - \bar{\nu}_t dt \right), \tag{7.37}$$

where $\bar\nu_t = \int_0^\infty \rho_t(u)\nu_t(u)du$. *The volatility structure* $\nu_t(x)$ *can be specified exogenously along with the initial term structure density* $\rho_0(x)$. *The associated short rate process* $r_t = \rho_t(0)$ *satisfies*

$$dr_t = \left(r_t^2 + \partial_x\rho_t(x)|_{x=0}\right)dt + r_t(\nu_t(0) - \bar\nu_t)(dW_t - \bar\nu_t dt). \tag{7.38}$$

The dynamical equation for the term structure density can be solved exactly as follows:

Proposition 3. *The solution of the dynamical equation for* $\rho_t(x)$ *in terms of the volatility structure* $\nu_t(x)$ *and the initial term structure density* $\rho_0(x)$ *is*

$$\rho_t(T-t) = \rho_0(T)\frac{\exp\left(\int_{s=0}^t V_{sT}dW_s - \frac{1}{2}\int_{s=0}^t V_{sT}^2 ds\right)}{\int_{u=t}^\infty \rho_0(u)\exp\left(\int_{s=0}^t V_{su}dW_s - \frac{1}{2}\int_{s=0}^t V_{su}^2 ds\right)du}, \tag{7.39}$$

where

$$V_{tu} = \nu_t(u-t). \tag{7.40}$$

*Proof**. The second term in the drift on the right of (7.37) can be eliminated by setting $x = T - t$, which gives us

$$d\rho_t(T-t) = r_t\rho_t(T-t)dt + \rho_t(T-t)\left(\nu_t(T-t) - \bar\nu_t\right)\left(dW_t - \bar\nu_t dt\right). \tag{7.41}$$

Integrating this relation and separating out the terms involving $\bar\nu_t$ we obtain

$$\rho_t(T-t) = \rho_0(T)\frac{\exp\left(\int_{s=0}^t r_s ds + \int_{s=0}^t \nu_s(T-s)dW_s - \frac{1}{2}\int_{s=0}^t \nu_s^2(T-s)ds\right)}{\exp\left(\int_{s=0}^t \bar\nu_s dW_s - \frac{1}{2}\int_{s=0}^t \bar\nu_s^2 ds\right)}. \tag{7.42}$$

It follows by use of the definition (7.40) that

$$\rho_t(T-t) = \rho_0(T)\frac{\exp\left(\int_{s=0}^t r_s ds + \int_{s=0}^t V_{sT}dW_s - \frac{1}{2}\int_{s=0}^t V_{sT}^2 ds\right)}{\exp\left(\int_{s=0}^t \bar\nu_s dW_s - \frac{1}{2}\int_{s=0}^t \bar\nu_s^2 ds\right)}. \tag{7.43}$$

Then with an application of the normalisation condition (7.24) we deduce as a consequence of (7.43) that

$$\begin{aligned}&\exp\left(-\int_{s=0}^t r_s ds + \int_{s=0}^t \bar\nu_s dW_s - \tfrac{1}{2}\int_{s=0}^t \bar\nu_s^2 ds\right)\\ &\quad = \int_{u=t}^\infty \rho_0(u)\exp\left(\int_{s=0}^t V_{su}dW_s - \tfrac{1}{2}\int_{s=0}^t V_{su}^2 ds\right)du.\end{aligned} \tag{7.44}$$

When this relation is inserted in the denominator of (7.43), we immediately obtain (7.39). ♢

7.4 Positive interest HJM volatility structure.

It is interesting in this connection to note, by setting $T = t$ in (7.39), that the short rate process is given by

$$r_t = \rho_0(t) \frac{\exp\left(\int_{s=0}^t V_{st} dW_s - \frac{1}{2}\int_{s=0}^t V_{st}^2 ds\right)}{\int_{u=t}^\infty \rho_0(u) \exp\left(\int_{s=0}^t V_{su} dW_s - \frac{1}{2}\int_{s=0}^t V_{su}^2 ds\right) du}. \tag{7.45}$$

We observe, in particular, that in a deterministic model, with $V_{st} = 0$, this formula reduces to $r_t = \rho_0(t)/\int_t^\infty \rho_0(u)du$, or, in other words, $r_t = f_{0t}$.

For the market risk premium process it follows from (7.31) together with the relation $\lambda_t = -\bar{\nu}_t$ that

$$\lambda_t^\alpha = -\frac{\int_{u=t}^\infty \rho_0(u) V_{tu}^\alpha \exp\left(\int_{s=0}^t V_{su} dW_s - \frac{1}{2}\int_{s=0}^t V_{su}^2 ds\right) du}{\int_{u=t}^\infty \rho_0(u) \exp\left(\int_{s=0}^t V_{su} dW_s - \frac{1}{2}\int_{s=0}^t V_{su}^2 ds\right) du}. \tag{7.46}$$

These formulae show that, given the initial term structure density $\rho_0(x)$ and the volatility structure $\nu_t(x)$, we can reconstruct the short rate process and the market risk premium processes.

We deduce from (7.39) that the corresponding formula for the bond price process is

$$P_{tT} = \frac{\int_{u=T}^\infty \rho_0(u) \exp\left(\int_{s=0}^t V_{su} dW_s - \frac{1}{2}\int_{s=0}^t V_{su}^2 ds\right) du}{\int_{u=t}^\infty \rho_0(u) \exp\left(\int_{s=0}^t V_{su} dW_s - \frac{1}{2}\int_{s=0}^t V_{su}^2 ds\right) du}. \tag{7.47}$$

For the unit-initialised money market account B_t, satisfying $dB_t = r_t B_t dt$ and $B_0 = 1$, we have

$$B_t = \frac{\exp\left(\int_{s=0}^t \bar{\nu}_s dW_s - \frac{1}{2}\int_{s=0}^t \bar{\nu}_s^2 ds\right)}{\int_{u=t}^\infty \rho_0(u) \exp\left(\int_{s=0}^t V_{su} dW_s - \frac{1}{2}\int_{s=0}^t V_{su}^2 ds\right) du}, \tag{7.48}$$

which follows directly from (7.44).

The density martingale Λ_t is given by

$$\Lambda_t = \exp\left(\int_{s=0}^t \bar{\nu}_s dW_s - \frac{1}{2}\int_{s=0}^t \bar{\nu}_s^2 ds\right), \tag{7.49}$$

For the state price density we have

$$Z_t = \int_{u=t}^{\infty} \rho_0(u) \exp\left(\int_{s=0}^{t} V_{su} dW_s - \tfrac{1}{2} \int_{s=0}^{t} V_{su}^2 ds \right) du. \tag{7.50}$$

As a consequence we can then check that $Z_t = \Lambda_t / B_t$.

If we divide (7.39) by (7.47) we are led to a recipe for constructing the general positive interest HJM forward short rate system f_{tT} in terms of freely specified data:

$$f_{tT} = \rho_0(T) \frac{\exp\left(\int_{s=0}^{t} V_{sT} dW_s - \frac{1}{2} \int_{s=0}^{t} V_{sT}^2 ds \right)}{\int_{u=T}^{\infty} \rho_0(u) \exp\left(\int_{s=0}^{t} V_{su} dW_s - \frac{1}{2} \int_{s=0}^{t} V_{su}^2 ds \right) du}. \tag{7.51}$$

Note that when $T = t$ this expression reduces to formula (7.45). A short calculation then allows us to deduce the following result:

Proposition 4. *The general positive interest HJM forward short rate volatility structure is*

$$\sigma_{tT} = f_{tT}(V_{tT} - U_{tT}) \tag{7.52}$$

where f_{tT} is given by (7.51), *and*

$$U_{tT} = \frac{\int_{u=T}^{\infty} \rho_0(u) V_{tu} \exp\left(\int_{s=0}^{t} V_{su} dW_s - \frac{1}{2} \int_{s=0}^{t} V_{su}^2 ds \right) du}{\int_{u=T}^{\infty} \rho_0(u) \exp\left(\int_{s=0}^{t} V_{su} dW_s - \frac{1}{2} \int_{s=0}^{t} V_{su}^2 ds \right) du}. \tag{7.53}$$

The initial term structure density $\rho_0(x)$ and the volatility structure V_{tu} $(u \geq t)$ are freely specifiable.

In other words, in the HJM theory the forward short rate volatility is not freely specifiable if the interest rates are to be positive.

Instead it must be of the form (7.52) where V_{tT} is freely specifiable, along with the initial term structure.

This result establishes a connection between the present approach and Example 2, and resolves the outstanding difficulty associated with that example.

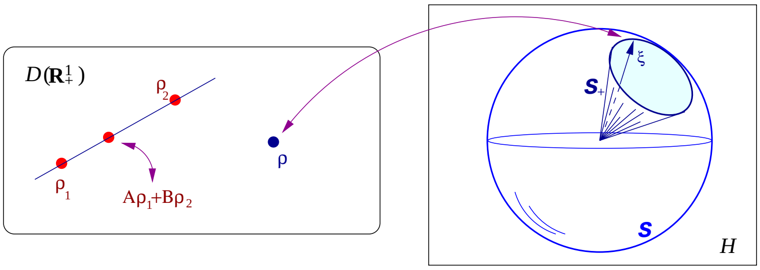

Figure 7.1: *The system of admissible term structures.* A smooth positive interest term structure can be regarded as a point in $\mathcal{D}(\mathbb{R}^1_+)$, the convex space of smooth and everywhere positive density functions on the positive half-line $\mathbb{R}^1_+$. Associated with each point $\rho \in \mathcal{D}(\mathbb{R}^1_+)$ there is a ray ξ lying in the positive orthant $\mathcal{S}_+$ of the unit sphere $\mathcal{S}$ in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^1_+)$. A dynamical trajectory on $\mathcal{D}(\mathbb{R}^1_+)$ can then be mapped to a corresponding trajectory in $\mathcal{S}_+$.

Chapter 8

Construction of admissible models. Moment analysis and the role of perpetual annuity. The information content of the term structure. Entropic calibration. Canonical term structures.

8.1 Construction of admissible models

As a consequence of Proposition 3 we see that the general term structure density can also be expressed in the form

$$\rho_t(x) = \frac{\rho_0(t+x)M_{t,t+x}}{\int_{u=0}^{\infty} \rho_0(t+u)M_{t,t+u}du}, \tag{8.1}$$

or equivalently

$$\rho_t(T-t) = \frac{\rho_0(T)M_{tT}}{\int_{u=t}^{\infty} \rho_0(u)M_{tu}du}, \tag{8.2}$$

where for each T the process M_{tT} is a martingale ($0 \le t \le T < \infty$) such that $M_{tT} > 0$ and $M_{0T} = 1$.

The process M_{tT} is the exponential martingale associated with V_{tT}.

This expression for $\rho_t(T-t)$ arises also in the Flesaker and Hughston framework, in which the discount bond system has the representation

$$P_{tT} = \frac{\int_{u=T}^{\infty} \rho_0(u)M_{tu}du}{\int_{u=t}^{\infty} \rho_0(u)M_{tu}du}. \tag{8.3}$$

Quasi-lognormal models. An interesting class of specific models is obtained if we restrict the Brownian motion to be one-dimensional and let the volatility structure $V_{tu} = \nu_t(u-t)$ appearing in (7.39) be *deterministic*.

Then V_{tu} is a function of two variables defined on the region $0 \leq t \leq u < \infty$. The resulting term structure model has a good deal of tractability and exhibits some desirable features.

In particular, the function V_{tu} has the right structure for allowing a calibration of the model to a family of implied caplet volatilities for a fixed strike (e.g., at-the-money).

If the dimensionality of the Brownian motion is increased then other strikes can be incorporated as well.

*Semi-linear models**. Another interesting special case can be obtained if we write

$$\rho_0(u) = \int_0^\infty \mathrm{e}^{-uR}\phi(R)dR, \tag{8.4}$$

for the initial term structure density, where $\phi(R)$ is the inverse Laplace transform of $\rho_0(u)$.

Then for certain choices of the martingale family M_{tu} the integration in (8.3) can be carried out explicitly.

An example can be obtained as follows. Let M_t be a martingale $(0 \leq t < \infty)$ and Q_t the associated quadratic variation satisfying $(dM_t)^2 = dQ_t$, and set

$$M_{tT} = \exp\left((\alpha+\beta T)M_t - \tfrac{1}{2}(\alpha+\beta T)^2 Q_t\right). \tag{8.5}$$

This model arises if we put

$$\nu_t(T-t) = (\alpha+\beta T)\sigma_t \tag{8.6}$$

in Proposition 2, where the process σ_t is defined by $dM_t = \sigma_t dW_t$. Then the u-integration can be carried out explicitly in the expressions for $\rho_t(x)$ and P_{tT}, and the results can be expressed in closed form:

$$P_{tT} = \frac{\int_{R=0}^\infty \phi(R)\left(\int_{u=T}^\infty \mathrm{e}^{-uR}M_{tu}du\right)dR}{\int_{R=0}^\infty \phi(R)\left(\int_{u=t}^\infty \mathrm{e}^{-uR}M_{tu}du\right)dR}. \tag{8.7}$$

Here the bracketed expression in the integrand in the numerator is given by:

$$\begin{aligned}\int_{u=T}^\infty \mathrm{e}^{-uR}M_{tu}du &= \frac{1}{|\beta|\sqrt{Q_t}}\exp\left(\frac{(M_t-R/\beta)^2}{2Q_t}+\alpha R/\beta\right)\\ &\quad\times\mathcal{N}\left(\pm\frac{M_t-R/\beta}{\sqrt{Q_t}}\mp(\alpha+\beta T)\sqrt{Q_t}\right),\end{aligned} \tag{8.8}$$

where

$$\mathcal{N}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\tfrac{1}{2}\xi^2\right) d\xi \tag{8.9}$$

is the normal distribution function, and the $\pm$ sign is chosen in accordance with the sign of β.

For example, in the case of an initial term structure with a constant continuously compounding rate r, corresponding to the choice $\phi(R) = \delta(R - r)$, we obtain

$$P_{tT} = \frac{\mathcal{N}\left(\pm\frac{M_t - r/\beta}{\sqrt{Q_t}} \mp (\alpha + \beta T)\sqrt{Q_t}\right)}{\mathcal{N}\left(\pm\frac{M_t - r/\beta}{\sqrt{Q_t}} \mp (\alpha + \beta t)\sqrt{Q_t}\right)}. \tag{8.10}$$

8.2 Moment analysis and the role of the perpetual annuity

Some interesting aspects of the term structure dynamics are captured in the properties of the moments of $\rho_t(x)$, defined by

$$\bar{x}_t = \int_0^\infty x\rho_t(x)dx, \quad \bar{x}_t^{(n)} = \int_0^\infty (x - \bar{x}_t)^n \rho_t(x)dx \tag{8.11}$$

where $n \geq 2$.

For example, in the case of a continuously compounded flat yield curve given at $t = 0$ by the density function $\rho_0(x) = R\mathrm{e}^{-Rx}$, we have $\bar{x}_0 = R^{-1}$, $\bar{x}_0^{(2)} = R^{-2}$, $\bar{x}_0^{(3)} = 3R^{-3}$ and $\bar{x}_0^{(4)} = 9R^{-4}$.

The first four moments, if they exist, are the mean, variance, skewness and kurtosis of the distribution of the 'abstract' random variable X characterising the term structure.

The mean $\bar{x}_t$ is a characteristic time-scale associated with the yield curve, and its inverse $1/\bar{x}_t$ is an associated characteristic yield. The financial significance of $\bar{x}_t$ will be discussed shortly.

For simplicity we introduce the following notation for the variance process:

$$v_t = \int_0^\infty x^2 \rho_t(x)dx - (\bar{x}_t)^2. \tag{8.12}$$

We assume that $\rho_t(x)$ and the discount bond volatility $\Sigma_{t,t+x}$ fall off sufficiently rapidly to ensure that $\lim_{x\to\infty} x^n \rho_t(x) = 0$ and $\lim_{x\to\infty} x^n \Sigma_{t,t+x} = 0$ for $n = 1, 2$, and that the integrals $\int_0^\infty x^n \rho_t(x)dx$ and $\int_0^\infty x^{n-1}\Sigma_{t,t+x}dx$ exist for $n = 1, 2$.

Proposition 5. *The mean $\bar{x}_t$ of an admissible arbitrage-free term structure satisfies*

$$d\bar{x}_t = (r_t\bar{x}_t - 1)dt + \bar{\Sigma}_t(dW_t + \lambda_t dt), \tag{8.13}$$

where $\bar{\Sigma}_t = \int_0^\infty \Sigma_{t,t+x}dx$.

There is a critical value $\bar{x}_t^*$ for the first moment given by

$$\bar{x}_t^* = \frac{1}{r_t}(1 - \lambda_t\bar{\Sigma}_t), \tag{8.14}$$

such that when $\bar{x}_t > \bar{x}_t^*$ the drift of $\bar{x}_t$ is positive, and the drift increases as $\bar{x}$ increases.

When $\bar{x}_t < \bar{x}_t^*$, the drift of $\bar{x}_t$ is negative, and the drift decreases further as $\bar{x}_t$ decreases.

The first moment $\bar{x}_t$ has the natural financial interpretation of being the value at time t of a perpetual annuity paid on a continuous basis.

In particular, an integration by parts shows that

$$\bar{x}_t = \int_0^\infty B_{tx}dx, \tag{8.15}$$

corresponding to an annuity of one unit of cash per year paid continuously in perpetuity.

Higher moments of the term structure density can then be interpreted in terms of the duration, convexity, etc., of the annuity—in other words, as a measure of the sensitivity of the value of the annuity to an overall change in interest rate levels.

For example, let us write

$$B_{tx} = \mathrm{e}^{-xr_t(x)}, \tag{8.16}$$

where $r_t(x)$ is the continuously compounded rate at time t for tenor x, then under a small parallel shift Δr in the yield curve given by

$$r_t(x) \longrightarrow r_t(x) + \Delta r, \tag{8.17}$$

we have, to first order,

$$B_{tx} \longrightarrow (1 - x\Delta r)B_{tx}. \tag{8.18}$$

Therefore, to first order the value of the annuity changes by the amount

$$\bar{x}_t \longrightarrow \bar{x}_t - \tfrac{1}{2}\Delta r \int_0^\infty x^2\rho_t(x)dx, \tag{8.19}$$

where in obtaining the second term we use an integration by parts.

Proposition 6. *Under a parallel shift in the yield curve the change $\Delta\bar{x}_t$ in the value of the perpetual is*

$$\Delta\bar{x}_t = -D_t\bar{x}_t\Delta r, \tag{8.20}$$

where the duration D_t of the perpetual annuity is given by

$$D_t = \frac{\frac{1}{2}\int_0^\infty x^2\rho_t(x)dx}{\int_0^\infty x\rho_t(x)dx}. \tag{8.21}$$

8.3 The information content of the term structure

Now we introduce another important example of a functional of the term structure, the Shannon entropy of the density function $\rho_t(x)$. This is defined by

$$S_t[\rho] = -\int_0^\infty \rho_t(x)\ln\rho_t(x)dx. \tag{8.22}$$

Because $\rho_t(x)$ has dimensions of inverse time, $S_t[\rho]$ is defined only up to an overall additive constant. The difference of the entropies associated with two yield curves therefore has an invariant significance.

One can think of $S_t[\rho]$ as being a measure of the 'information content' of the term structure at time t. In particular, the higher the value of $S_t[\rho]$, the lower the information content.

Since $\rho_t(x)$ is subject to a dynamical law, we can infer a corresponding dynamics for the entropy.

Proposition 7. *The entropy associated with an admissible arbitrage-free term structure dynamics obeys the evolutionary law*

$$dS_t = \left(r_t(S_t + \ln r_t - 1) + \tfrac{1}{2}\Gamma_t\right)dt + \left(\int_0^\infty \nu_t(x)s_t(x)dx - \bar{\nu}_t S_t\right)dW_t^* \tag{8.23}$$

where $dW^* = dW_t + \lambda_t dt$, $s_t(x) = -\rho_t(x)\ln\rho_t(x)$ is the entropy density, and the process Γ_t is defined by

$$\Gamma_t = \int_0^\infty (\nu_t(x) - \bar{\nu}_t)^2\rho_t(x)dx. \tag{8.24}$$

8.4 Entropic calibration

The principle of entropy maximisation can be used as the basis for a new yield curve calibration methodology.

In particular, given a set of data points on a yield curve, the 'least biased' term structure can be determined by maximising the Shannon entropy subject to the given data constraints.

The general idea behind the maximisation of entropy under constraints can be sketched as follows.

Suppose that, given a function $H(X)$ of a random variable X, we are told that the expectation of $H(X)$ with respect to an unknown distribution with density $\rho(x)$ is U, i.e.,

$$\int_0^\infty H(x)\rho(x)dx = U. \tag{8.25}$$

The aim then is to find the density $\rho(x)$ that is least biased and yet consistent with the information (8.25).

In other words, we wish to eliminate any superfluous information in $\rho(x)$.

We also have the normalisation condition

$$\int_0^\infty \rho(x)dx = 1. \tag{8.26}$$

Subject to the constraints (8.25) and (8.26) we then determine the density $\rho(x)$ that maximises the entropy.

This is carried out by introducing Lagrange multipliers, and considering the variational relation

$$\frac{\delta}{\delta\rho}\left(-\rho\ln\rho - \lambda\rho H - \nu\rho\right) = 0. \tag{8.27}$$

The solution is

$$\rho(x) = \exp\left(-\lambda H(x) - \nu - 1\right), \tag{8.28}$$

where λ and ν are determined implicitly.

Let us illustrate the idea by considering the situation in which we are given a set of data points on the yield curve together with the value of a perpetual annuity.

The problem is to calibrate the initial term structure to the given data.

This example is interesting because if we are given only the value $\bar{x}_0$ of the perpetual annuity, then the maximum entropy term structure is

$$\rho_0(x) = R\mathrm{e}^{-Rx}, \tag{8.29}$$

where $R = 1/\bar{x}_0$, and thus $P_{0x} = \mathrm{e}^{-Rx}$ for the discount function.

Therefore, we see that it is the annuity constraint that leads to the desired exponential 'die-off' of the discount function.

This feature is preserved in the more elaborate examples we discuss below, where bond data-points are introduced as well.

In the more general situation, the bond prices with a given set of tenors x_i $(i = 1, 2, \cdots, r)$ are observed to be $B_{0x_i} = \eta_i$.

In addition, we have the initial value $\bar{x}_0 = \xi$ of the perpetual annuity.

Subject to these constraints, the maximum entropy term structure is determined by the variational principle

$$\frac{\delta}{\delta\rho}\left(-\rho(x)\ln\rho(x) - \lambda\rho(x)x - \sum_{i=1}^{r}\mu^i\rho(x)I_{x_i}(x) - \nu\rho(x)\right) = 0, \tag{8.30}$$

where $I_{x_i}(x) = 1$ for $x \geq x_i$ and vanishes otherwise.

The parameters λ, μ^i and ν are determined by the normalisation condition and data constraints

$$\int_{x=0}^{\infty} x\rho(x)dx = \xi, \quad \text{and} \quad \int_{x=0}^{\infty} I_{x_i}(x)\rho(x)dx = \eta_i. \tag{8.31}$$

The solution is

$$\rho(x) = \frac{1}{Z(\lambda,\mu)}\exp\left(-\lambda x - \sum_{i=1}^{r}\mu^i I_{x_i}(x)\right), \tag{8.32}$$

where

$$Z(\lambda,\mu) = \int_0^{\infty}\exp\left(-\lambda x - \sum_{i=1}^{r}\mu^i I_{x_i}(x)\right)dx. \tag{8.33}$$

The Lagrange multipliers are then determined implicitly by

$$-\frac{\partial \ln Z}{\partial \lambda} = \xi \quad \text{and} \quad -\frac{\partial \ln Z}{\partial \mu^i} = \eta_i. \tag{8.34}$$

As a consequence of (8.32) we see that pointwise calibration to the discount bond prices, along with the information of the price of the annuity, gives a piecewise exponential term structure density function.

If there is further information at our disposal, then that can also be included in the system of constraints so that all available information is used efficiently in the calibration procedure.

We now consider in more detail the simple case where the observed data consist of two pieces of information—the bond price P_{0T_1} for a fixed maturity date T_1, and the value $\xi = \bar{x}_0$ of the perpetual annuity.

This is a rather artificial example; nevertheless it serves to illuminate the main points of the procedure.

The variational problem implies the existence of three rates r_0, r_1, and R such that the term structure density is

$$\rho(x) = \begin{cases} r_0 \mathrm{e}^{-Rx} & \text{for} \quad 0 \leq x < T_1 \\ r_1 \mathrm{e}^{-Rx} & \text{for} \quad T_1 \leq x < \infty. \end{cases} \tag{8.35}$$

The constraints are given by:

$$\int_0^{T_1} \rho(x)dx = 1 - P_{0T_1} \tag{8.36}$$

for the bond price;

$$\int_0^{\infty} \rho(x)dx = 1 \tag{8.37}$$

for the normalisation; and

$$\int_0^{\infty} x\rho(x)dx = \xi \tag{8.38}$$

for the perpetual annuity.

A short calculation shows that these relations reduce to:

$$1 - \frac{r_0}{R}\left(1 - \mathrm{e}^{-RT_1}\right) = P_{0T_1}, \tag{8.39}$$

$$\frac{r_1 - r_0}{R} \mathrm{e}^{-RT_1} + \frac{r_0}{R} = 1, \tag{8.40}$$

$$\frac{r_1 - r_0}{R} \mathrm{e}^{-RT_1} \left(T_1 + \frac{1}{R} \right) + \frac{r_0}{R^2} = \xi. \tag{8.41}$$

Clearly, given P_{0T_1} and ξ, we can proceed to infer values of r_0, r_1, and R.

In particular, equation (8.39) allows us to deduce the bond price P_{0T_1} if we are given r_0 and R, whereas we can use (8.40) to eliminate r_1 in (8.41) to obtain

$$\frac{1}{R} + T_1 \left(1 - \frac{r_0}{R} \right) = \xi \tag{8.42}$$

for the value of the perpetual in terms of r_0 and R.

Alternatively, given the initial short rate r_0 and the value of the perpetual ξ we have

$$R = \frac{1 - r_0 T_1}{\xi - T_1}. \tag{8.43}$$

This value of R can then be inserted in (8.39) to determine the bond price.

The scale factor r_1 is given by

$$r_1 = R P_{0T_1} \mathrm{e}^{RT_1}. \tag{8.44}$$

Thus we obtain

$$\rho(x) = \begin{cases} r_0 \mathrm{e}^{-Rx} & (0 \leq x < T_1) \\ R P_{0T_1} \mathrm{e}^{-R(x-T_1)} & (T_1 \leq x < \infty), \end{cases} \tag{8.45}$$

for the term structure density, and

$$P_{0x} = \begin{cases} 1 - \frac{r_0}{R} \left(1 - \mathrm{e}^{-Rx} \right) & (0 \leq x < T_1) \\ P_{0T_1} \mathrm{e}^{-R(x-T_1)} & (T_1 \leq x < \infty), \end{cases} \tag{8.46}$$

for the discount function, from which yield curve R_{0x} can be constructed via the standard prescription

$$R_{0x} = -\frac{\ln P_{0x}}{x}, \tag{8.47}$$

and it should be evident by inspection that R_{0x} is continuous in x.

In this example we can alternatively regard the short rate r_0 and the bond price P_{0T_1} as the actual 'independent' data. Then (8.39) can be used to deduce R, which allows us to infer the annuity price ξ by use of (8.42).

This illustrates the point that, although we assume from the outset the *existence* of a perpetual, we can infer an implied value of that instrument by the use of other market data (e.g., the short rate).

The same idea carries forward in the case where we have multiple data points for the bond prices, for a given set of n maturity dates T_j $(j = 1, 2, \cdots, n)$, and we are led to a simple iterative algorithm for determining the term structure in terms of the short rate and the specified bond data points.

Proposition 8. *Given a set of bond prices P_{0T_j} $(j = 1, 2, \cdots, n)$ and the existence of the value of the perpetual annuity, the maximum entropy term structure density function is*

$$\rho(x) = \sum_{k=0}^{n} I_{T_k T_{k+1}}(x) r_k \mathrm{e}^{-Rx}. \tag{8.48}$$

Here $T_0 = 0$, $T_{n+1} = \infty$, $I_{T_k T_{k+1}}(x) = 1$ if $x \in [T_k, T_{k+1})$ and vanishes otherwise, r_0 is the short rate, and

$$r_k = R \frac{P_{0T_k} - P_{0T_{k+1}}}{\mathrm{e}^{-RT_k} - \mathrm{e}^{-RT_{k+1}}}. \tag{8.49}$$

The value of R is determined from equation (8.39).

The corresponding discount function P_{0x} is given by

$$P_{0x} = P_{0T_k} - \frac{r_k}{R}(\mathrm{e}^{-RT_k} - \mathrm{e}^{-Rx}) \tag{8.50}$$

for $x \in [T_k, T_{k+1})$.

Proof. To see this, we insert the piecewise exponential density function (8.48) into a series of constraints of the form (8.36) for the bond prices, together with the normalisation constraint (8.37) and the perpetual constraint (8.38).

Then the bond price constraints give rise to a set of relations of the form

$$\frac{r_k}{R}(\mathrm{e}^{-RT_k} - \mathrm{e}^{-RT_{k+1}}) = P_{0T_k} - P_{0T_{k+1}}, \tag{8.51}$$

for $k = 0, 1, \cdots, n-1$.

In particular, for $k = 0$, we recover (8.39), which can be used to solve for R in terms of the short rate r_0 and the bond price P_{0T_1}.

Then, by substitution of this in (8.51) for general k, and the use of further bond price data, we obtain the other rates r_k $(k \neq n)$.

As for r_n, we note that if we divide the integration range in (8.36) into two regions $[0, T_n]$ and $[T_n, \infty]$, then the normalisation condition becomes

$$\frac{r_n}{R}\mathrm{e}^{-RT_n} = P_{0T_n}, \tag{8.52}$$

which determines r_n in terms of R and P_{0T_n}.

Substitution of these results in the perpetual constraint

$$\frac{1}{R}\sum_{k=1}^{n}(r_k - r_{k-1})\mathrm{e}^{-RT_k}\left(T_k + \frac{1}{R}\right) + \frac{r_0}{R^2} = \xi \tag{8.53}$$

allows the implied value ξ of the perpetual annuity to be determined from the short rate r_0 and the bond price data P_{0T_j}.

The discount function can be determined by use of the fact that

$$\begin{aligned} 1 - P_{0x} &= \int_0^x \rho(u)du \\ &= \int_0^{T_k} \rho(u)du + r_k \int_{T_k}^x \mathrm{e}^{-Ru}du \\ &= 1 - P_{0T_k} + r_k \int_{T_k}^x \mathrm{e}^{-Ru}du, \end{aligned} \tag{8.54}$$

when $x \in [T_k, T_{k+1})$. ◇

Next, we turn to the problem: Given an existing term structure $\rho_2(x)$ and a set of new data points, how does one determine the new term structure that is 'closest' to the previous one?

This can be addressed by use of the statistical J-divergence:

$$J(\rho_1, \rho_2) = S\left(\tfrac{1}{2}(\rho_1 + \rho_2)\right) - \tfrac{1}{2}\left(S(\rho_1) + S(\rho_2)\right). \tag{8.55}$$

Here, as before, the entropy is defined by

$$S(\rho) = -\int_0^\infty \rho(x)\ln\rho(x)dx. \tag{8.56}$$

Statistical J-divergence defines the 'separation' between ρ_1 and ρ_2.

The solution to the problem here is given by the ρ_1 that *minimises* the J-divergence, subject to the constraints:

$$\int_0^\infty \rho_1(x) = 1 \tag{8.57}$$

and

$$\int_{T_i}^\infty \rho_1(x)dx = P_{0T_i}. \tag{8.58}$$

Introducing Lagrange multipliers μ_i, we must solve for

$$\frac{\delta}{\delta \rho_1}\left(J(\rho_1, \rho_2) - \sum_{i=0}^{n} \mu_i \int_0^\infty I_{T_i}(x)\rho_1(x)dx \right) = 0. \tag{8.59}$$

The solution can be written as:

$$\rho_1(x) = \sum_{k=0}^{n} I_{T_k T_{k+1}}(x) \frac{1}{2\exp(\delta_k) - 1} \rho_2(x). \tag{8.60}$$

Here, $T_0 = 0$, $T_{n+1} = \infty$, and

$$\delta_k = -2 \sum_{i=0}^{k} \mu_i \tag{8.61}$$

Let us write

$$\int_{T_i}^\infty \rho_2(x)dx = Q_{0T_i}, \tag{8.62}$$

This is just the bond prices in the 'old' term structure.

To eliminate Lagrange multipliers, we note that

$$\int_0^{T_i} \rho_1(x)dx = 1 - P_{0T_i}, \tag{8.63}$$

which implies

$$\begin{aligned} 1 - P_{0T_i} &= \sum_{j=0}^{i-1} \int_{T_j}^{T_{j+1}} \rho_1(x)dx \\ &= \sum_{j=0}^{i-1} \frac{1}{2\exp(\delta_j) - 1} \int_{T_j}^{T_{j+1}} \rho_2(x)dx. \end{aligned} \tag{8.64}$$

If we also recall that

$$\int_{T_j}^{T_{j+1}} \rho_2(x)dx = Q_{0T_j} - Q_{0T_{j+1}} \tag{8.65}$$

then the solution for this calibration problem can be summarised in the following form.

Proposition 9. *Given a set of bond prices P_{0T_j} $(j = 1, 2, \cdots, n)$ and an existing term structure density $\hat{\rho}(x)$, the minimum J-divergence term structure density function is*

$$\rho(x) = \left(\sum_{k=0}^{n} I_{T_k T_{k+1}}(x) \Delta_k \right) \hat{\rho}(x) \tag{8.66}$$

Here $T_0 = 0$, $T_{n+1} = \infty$, $I_{T_k T_{k+1}}(x) = 1$ if $x \in [T_k, T_{k+1})$ and 0 otherwise, and

$$\Delta_k = \frac{P_{0T_k} - P_{0T_{k+1}}}{Q_{0T_k} - Q_{0T_{k+1}}} \tag{8.67}$$

The corresponding discount function P_{0x} is

$$P_{0x} = P_{0T_k} - \frac{P_{0T_k} - P_{0T_{k+1}}}{Q_{0T_k} - Q_{0T_{k+1}}} (Q_{0T_k} - Q_{0x}) \tag{8.68}$$

for $x \in [T_k, T_{k+1})$.

8.5 Canonical term structures*

As an interesting example of a class of models that arises as a consequence of the maximisation of an entropy functional under constraints, we let the term structure density be of the form

$$\rho_t(T - t) = \frac{\exp(-g_{tT} - \theta_t h_{tT})}{\int_{u=t}^{\infty} \exp(-g_{tu} - \theta_t h_{tu})\, du}, \tag{8.69}$$

where θ_t is a one-dimensional Ito process, and the functions g_{tT} and h_{tT} are deterministic, defined over the range $0 \leq t \leq T < \infty$.

At each time t the term structure density thus defined belongs to an exponential family parameterised by the value of θ_t. If we set

$$Z(\theta) = \int_{u=t}^{\infty} \exp(-g_{tu} - \theta_t h_{tu})\, du, \tag{8.70}$$

then we find that all the moments of the function h_{tT} can be determined from the generating function $Z(\theta)$ by formal differentiation.

For example, for the first moment of h_{tT} we have

$$\int_{u=t}^{\infty} h_{tu}\rho_t(u-t)du = -\frac{\partial \ln Z(\theta)}{\partial\theta}. \tag{8.71}$$

The corresponding bond price system can then be written in the Flesaker-Hughston form

$$P_{tT} = \frac{\int_{u=T}^{\infty} N_{tu}du}{\int_{u=t}^{\infty} N_{tu}du}, \tag{8.72}$$

where $N_{tT} = \exp(-g_{tT} - \theta_t h_{tT})$. By Ito's lemma, it follows that N_{tT} satisfies

$$\frac{dN_{tT}}{N_{tT}} = -\left(\dot{g}_{tT} + \theta_t \dot{h}_{tT}\right) dt - h_{tT}d\theta_t + \tfrac{1}{2}h_{tT}^2(d\theta_t)^2, \tag{8.73}$$

where the dot indicates partial differentiation with respect to t, so $\dot{g}_{tT} = \partial_t g_{tT}$ and $\dot{h}_{tT} = \partial_t h_{tT}$.

We assume that the trajectory θ_t of the canonical parameter satisfies a stochastic equation of the form

$$d\theta_t = \alpha_t dt + \beta_t dW_t. \tag{8.74}$$

The no-arbitrage condition implies that N_{tT} is a positive martingale. Therefore, the drift of N_{tT} vanishes for all T:

$$\dot{g}_{tT} + \dot{h}_{tT}\theta_t + \alpha_t h_{tT} = \tfrac{1}{2}\beta_t^2 h_{tT}^2. \tag{8.75}$$

This relation implies that the processes α_t and β_t determining the dynamics of θ_t are of the form

$$\alpha_t = A_t\theta_t + B_t, \quad \text{and} \quad \tfrac{1}{2}\beta_t^2 = C_t\theta_t + D_t \tag{8.76}$$

where the functions A_t, B_t, C_t and D_t are deterministic.

It follows that θ_t is a square-root process. Substitution of these equations into (8.75) gives a set of Bernoulli equations of the form

$$\dot{h}_{tT} + A_t h_{tT} - C_t h_{tT}^2 = 0 \tag{8.77}$$

for h_{tT} and

$$\dot{g}_{tT} + B_t h_{tT} - D_t h_{tT}^2 = 0 \tag{8.78}$$

for g_{tT}.

The general solution of (8.77) is

$$\frac{1}{h_{tT}} = -\exp\left(\int_0^t A_u du\right)\left(\int_0^t \frac{C_u}{\exp(\int_0^u A_v dv)} du + E_T\right), \tag{8.79}$$

where E_T is an function of T, determined by the initial term structure.

To proceed further, let us consider the special case where $D_t = 0$ and θ_t is positive and mean-reverting. Then B_t and C_t are both positive and A_t is negative, and for g_{tT} we have

$$g_{tT} = -\int_0^t B_u h_{uT} du + F_T, \tag{8.80}$$

where F_T is another arbitrary function. In the elementary case where A_t, B_t and C_t are constants, the functions h_{tT} and g_{tT} are given by

$$h_{tT} = \frac{A}{C - G_T \mathrm{e}^{At}} \tag{8.81}$$

and

$$g_{tT} = \frac{B}{C} \ln\left(\frac{G_T - C\mathrm{e}^{-At}}{G_T - C}\right) - F_T, \tag{8.82}$$

where $G_T = AE_T + C$. The condition that h_{tT} should be positive ensures that G_T is of the form $G_T = CH_T\mathrm{e}^{-AT}$ where the function H_T satisfies $H_T > 1$ but is otherwise arbitrary.

For N_{tT} we then obtain:

$$N_{tT} = \left(\frac{H_T - \mathrm{e}^{AT}}{H_T - \mathrm{e}^{A(T-t)}}\right)^{\frac{B}{C}} \exp\left(\frac{A\theta_t}{C(H_T\mathrm{e}^{-A(T-t)} - 1)} - F_T\right). \tag{8.83}$$

The function F_T is then determined by the specification of the initial term structure for $t = 0$.

In particular, because $N_{0T} = \rho_0(T)$, we obtain

$$N_{tT} = \rho_0(T)\left(\frac{H_T - \mathrm{e}^{AT}}{H_T - \mathrm{e}^{A(T-t)}}\right)^{\frac{B}{C}} \exp\left(\frac{A\theta_t}{C(H_T\mathrm{e}^{-A(T-t)} - 1)} - \frac{A\theta_0}{C(H_T\mathrm{e}^{-AT} - 1)}\right). \tag{8.84}$$

Chapter 9

Review of the Flesaker-Hughston framework. Integral formulae for discount bonds. Supermartingales and potentials. Rational log-normal model.

9.1 Risk-adjusted discount bond volatility

We return now to the general theory of interest rate dynamics, and establish another expression for the discount bonds, which we call the *integral representation.*

This representation has the advantage of bringing out the positive interest condition.

Recall that for the general arbitrage-free dynamics of a discount bond system we have the following dynamics:

$$\frac{dP_{ab}}{P_{ab}} = r_a\, da + \Omega_{ab}\,(dW_a + \lambda_a\, da). \tag{9.1}$$

Here P_{ab} is the value of a discount bond at time a that matures at time b, r_a is the short rate, Ω_{ab} is the bond volatility vector, and λ_a is the relative risk vector.

The economy is modelled by a probability space $(\Omega, \mathcal{F}, P)$ with filtration $(\mathcal{F}_t)$. We assume that $(\mathcal{F}_t)$ is generated in a standard way by a multi-dimensional Brownian motion.

We recall the fact that under suitable technical conditions the solution to the dynamical equation is

$$P_{ab} = P_{0b} B_a \exp\left(\int_0^a \Omega_{sb}\,(dW_s + \lambda_s\, ds) - \tfrac{1}{2}\int_0^a \Omega_{sb}^2\, ds\right) \tag{9.2}$$

Here B_a is the unit-initialised money market account process.

The solution for B_a, obtained by setting $P_{aa} = 1$, is

$$B_a = (P_{0a})^{-1} \exp\left(-\int_0^a \Omega_{sa}\,(dW_s + \lambda_s\, ds) + \tfrac{1}{2}\int_0^a \Omega_{sa}^2\, ds\right). \tag{9.3}$$

For the short-term interest rate r_a, we have

$$r_a = -\partial_a \ln P_{0a} + \int_0^a \Omega_{sa} \partial_a \Omega_{sa} \, ds - \int_0^a \partial_a \Omega_{sa} \, (dW_s + \lambda_s ds). \tag{9.4}$$

Putting these ingredients together (inserting (9.3) into (9.2)) we have the formula

$$P_{ab} = P_{0ab} \frac{\exp\left(\int_0^a \Omega_{sb} \, (dW_s + \lambda_s \, ds) - \frac{1}{2} \int_0^a \Omega_{sb}^2 \, ds\right)}{\exp\left(\int_0^a \Omega_{sa} \, (dW_s + \lambda_s \, ds) - \frac{1}{2} \int_0^a \Omega_{sa}^2 ds\right)}. \tag{9.5}$$

Here, $P_{0ab} = P_{0b}/P_{0a}$ denotes the *forward* value of a b-maturity bond.

Recall that P_{0ab} is the value negotiated today for purchase at time a of a b-maturity bond.

It will be useful to build an analogy with the single asset situation.

In that case we recall that for the dynamics of a non-dividend paying asset S_t we have

$$S_t = S_0 B_t \exp\left(\int_0^t \sigma_s \, (dW_s + \lambda_s \, ds) - \frac{1}{2} \int_0^t \sigma_s^2 \, ds\right). \tag{9.6}$$

Then, introducing the density martingale, we deduce, under suitable technical conditions, that the following ratio is a martingale:

$$\frac{\Lambda_t S_t}{B_t} = \exp\left(\int_0^t (\sigma_s - \lambda_s) \, dW_s - \frac{1}{2} \int_0^t (\sigma_s - \lambda_s)^2 \, ds\right). \tag{9.7}$$

In the case of interest rate dynamics, σ_s gets replaced by Ω_{sb}, and a result similar to (9.7) holds for each discount bond.

More specifically, we have

$$\frac{\Lambda_a P_{ab}}{B_a} = P_{0b} \exp\left(\int_0^a V_{sb} \, dW_s - \frac{1}{2} \int_0^a V_{sb}^2 \, ds\right) \tag{9.8}$$

where

$$V_{ab} := \Omega_{ab} - \lambda_a. \tag{9.9}$$

The quantity V_{ab}, which we call "risk-adjusted volatility", plays a useful role in the theory of interest rates.

Note that V_{ab} contains the information of both the discount bond volatility and the interest rate market price of risk.

This is because of the constraint $\Omega_{aa} = 0$ (a maturing bond has zero volatility), which implies that $\lambda_a = -V_{aa}$ and that $\Omega_{ab} = V_{ab} - V_{aa}$.

Note that "risk premium" and "volatility" have the same units (inverse square-root time), so it makes sense to combine them additively.

Now, setting $P_{aa} = 1$, we obtain a formula for Λ_a/B_a, and for the discount bonds we get

$$P_{ab} = \frac{P_{0b} \exp\left(\int_0^a V_{sb}\, dW_s - \frac{1}{2}\int_0^a V_{sb}^2\, ds\right)}{P_{0a} \exp\left(\int_0^a V_{sa}\, dW_s - \frac{1}{2}\int_0^a V_{sa}^2\, ds\right)}. \tag{9.10}$$

In this expression we note that, for each fixed value of b, the numerator is an exponential martingale.

9.2 Integral representation for discount bonds

We have thus represented P_{ab} as a quotient of the form

$$P_{ab} = \frac{\Delta_{ab}}{\Delta_{aa}}, \tag{9.11}$$

where Δ_{ab} is a one-parameter family of positive martingales.

Here the martingale property holds with respect to the "natural" probability measure P.

We make technical assumptions sufficient to ensure that the bond price goes to zero for large values of the maturity, and that the martingale property of Δ_{ab} is preserved under differentiation with respect to the maturity parameter.

We find then that Δ_{ab} can be expressed in the form

$$\Delta_{ab} = \int_b^\infty (-\partial_s P_{0s}) M_{as}\, ds. \tag{9.12}$$

Here M_{as} is a one-parameter family of martingales, initialised to unity at time zero ($M_{0s} = 1$) to ensure satisfaction of the initial condition $\Delta_{0b} = P_{0b}$.

The argument that establishes the integral representation is as follows.

Since $\lim_{b\to\infty} P_{ab} = 0$ by assumption, we have $\lim_{b\to\infty} \Delta_{ab} = 0$, and thus

$$\Delta_{ab} = -\int_b^\infty \partial_s \Delta_{as}\, ds. \tag{9.13}$$

By assumption, $\partial_s \Delta_{as}$ is a martingale.

Since $\Delta_{0b} = P_{0b}$, it follows that

$$M_{ab} = \frac{\partial_s \Delta_{as}}{\partial_s P_{0s}} \tag{9.14}$$

is a unit-initialised martingale, and thus we obtain (9.12).

In particular, for a *positive* interest rate model it is necessary and sufficient that *initial* interest rates are positive, and that the martingale family M_{as} should be positive.

With these ingredients in place, we see that the discount bond process can be written in the form:

$$P_{ab} = \frac{\int_b^\infty (-\partial_s P_{0s}) M_{as}\, ds}{\int_a^\infty (-\partial_s P_{0s}) M_{as}\, ds}. \tag{9.15}$$

This is the "positive interest" integral representation for the general interest rate model.

By a "Flesaker-Hughston" model we usually mean any representation of the discount bonds in the form (9.15) for some choice of the martingale family M_{as}.

One can verify by inspection that if *initial* interest rates satisfy the *positivity conditions*

$$0 < P_{0b} \leq 1 \quad \text{and} \quad \partial_b P_{0b} < 0. \tag{9.16}$$

If the martingale family M_{as} is positive, then the positive interest conditions

$$0 < P_{ab} \leq 1 \quad \text{and} \quad \partial_b P_{ab} < 0 \tag{9.17}$$

are satisfied for future valuation dates, and for bonds of all maturities.

9.3 Integral representations in the risk-neutral measure

A representation of the form (9.15) exists for any measure equivalent to the natural measure.

That is to say, in the absence of arbitrage, and with some technical conditions, given a probability measure $\hat{P}$ equivalent to the natural economic measure P, there exists a martingale family M_{as} such that the discount bonds P_{ab} are given by an integral representation of the form (9.15).

In the case of a complete market the representation thus obtained is unique.

We note that if M_{as} represents the martingale family with respect to the natural probability measure P, then

$$\hat{M}_{as} = \frac{M_{as}}{\Lambda_a} \tag{9.18}$$

is the appropriate new martingale family, with respect to a new measure $\hat{P}$, where Λ_a is the change-of-measure density martingale.

This is because if M_t is any martingale with respect to P, then M_t/Λ_t is a martingale with respect to $\hat{P}$, where $\hat{P}$ is defined in terms of conditional expectation by

$$\hat{\mathbb{E}}_a[X_b] = \frac{\mathbb{E}_a[\Lambda_b X_b]}{\Lambda_a} \tag{9.19}$$

for any random variable which is measurable with respect to $\mathcal{F}_b$.

Now by use of the risk neutral measure we have the bond valuation formula

$$P_{ab} = B_a \hat{\mathbb{E}}_a \left[\frac{1}{B_b} \right]. \tag{9.20}$$

Here $B_a = \exp\left(\int_0^a r_s \, ds\right)$ is the money market account.

By inspection we evidently have

$$\Delta_{ab} = \hat{\mathbb{E}}_a \left[\frac{1}{B_b} \right]. \tag{9.21}$$

Therefore we deduce that the martingale family for the risk-neutral measure is

$$\hat{M}_{as} = \hat{\mathbb{E}}_a \left[\frac{r_s}{(-\partial_s P_{0s})\, B_s} \right]. \tag{9.22}$$

As a consequence, we see that for the natural measure we have

$$\begin{aligned} M_{as} &= \Lambda_a \hat{M}_{as} \\ &= \Lambda_a \hat{\mathbb{E}}_a \left[\frac{r_s}{(-\partial_s P_{0s})\, B_s} \right] \\ &= \mathbb{E}_a \left[\frac{\Lambda_s r_s}{(-\partial_s P_{0s})\, B_s} \right] \end{aligned} \tag{9.23}$$

This gives us a construction for the martingale family M_{as}, given r_s and Λ_s together with initial bond data.

9.4 Potentials and positive supermartingales

Let us return now to the representation of the discount bonds given by

$$P_{ab} = \frac{\Delta_{ab}}{\Delta_{aa}}. \tag{9.24}$$

For Δ_{ab} here, we have

$$\Delta_{ab} = P_{0b} \exp\left(\int_0^a V_{sb}\, dW_s - \tfrac{1}{2} \int_0^a V_{sb}^2\, ds \right), \tag{9.25}$$

where

$$V_{sb} = \Omega_{sb} - \lambda_s. \tag{9.26}$$

For Δ_{aa} we have

$$\Delta_{aa} = \frac{\Lambda_a}{B_a}. \tag{9.27}$$

We note that $\Delta_{ab} = \mathbb{E}_a[\Delta_{bb}]$.

The quantity $V_t = \Delta_{tt}$ is the *state-price density*.

The state-price density satisfies the following differential equation:

$$\frac{dV_t}{V_t} = -r_t\, dt - \lambda_t dW_t \tag{9.28}$$

Thus we see that if r_t is positive, then V_t is a *positive supermartingale*.

Now as a consequence of (9.24) we have

$$P_{0t} = \frac{\mathbb{E}[V_t]}{V_0}. \tag{9.29}$$

Thus to ensure that the initial discount function vanishes asymptotically, we require

$$\lim_{t\to\infty} \mathbb{E}[V_t] = 0. \tag{9.30}$$

A positive supermartingale V_t that satisfies (9.30) is called a *potential*.

We see therefore that the concept of a potential is mathematically very natural as a basis for interest rate theory.

In the potential method we represent the bond price by a formula of the following form:

$$P_{ab} = \frac{\mathbb{E}_a[V_b]}{V_a}. \tag{9.31}$$

The potential method can be used to generate a number of new and potentially interesting interest rate models.

There are several ways of representing potentials.

One method is to introduce strictly increasing adapted process A_t defined for all time $0 \leq t \leq \infty$, and write

$$V_t = \mathbb{E}_t[A_\infty] - A_t \tag{9.32}$$

If we write A_t in the form

$$A_t = \int_0^t \eta_s ds \tag{9.33}$$

where η_s is positive, then clearly

$$\begin{aligned} V_t &= \mathbb{E}_t\left[\int_0^\infty \eta_s ds\right] - \int_0^t \eta_s ds \\ &= \mathbb{E}_t\left[\int_t^\infty \eta_s ds\right] \\ &= \int_t^\infty \mathbb{E}_t[\eta_s] ds. \end{aligned} \tag{9.34}$$

Now, if we define P_{0t} according to (9.29), clearly we have

$$P_{0t} = \frac{\int_t^\infty \mathbb{E}[\eta_s] ds}{\int_0^\infty \mathbb{E}[\eta_s] ds}, \tag{9.35}$$

Therefore, for the derivative of P_{0t} we obtain

$$-\partial_t P_{0t} = \frac{\mathbb{E}[\eta_t]}{\int_0^\infty \mathbb{E}[\eta_s] ds}. \tag{9.36}$$

If we define

$$M_{ts} = \frac{\mathbb{E}_t[\eta_s]}{(-\partial_t P_{0t})}. \tag{9.37}$$

we obtain

$$V_t = \int_t^\infty (-\partial_s P_{0s}) M_{ts} ds \tag{9.38}$$

where M_{ts} is a unit-initialised positive martingale family, and we are back to the Flesaker-Hughston representation.

Another useful way to represent potentials is by the introduction of a square-integrable random variable X_∞ satisfying

$$\mathbb{E}\left[X_\infty^2\right] < \infty. \tag{9.39}$$

Then we define the martingale

$$X_t = \mathbb{E}_t\left[X_\infty\right] \tag{9.40}$$

and write

$$V_t = \mathbb{E}_t\left[(X_\infty - X_t)^2\right] \tag{9.41}$$

In other words, V_t is defined to be the *conditional variance* of X_∞, given information up to t.

We recall that for any random variable X, the conditional variance of X with respect to $\mathcal{F}_t$ is defined by

$$\mathrm{Var}_t[X] = \mathbb{E}_t[(X - \mathbb{E}_t[X])^2]. \tag{9.42}$$

One can check that V_t is a supermartingale, and that $\mathbb{E}[V_t] \to 0$ as $t \to \infty$.

In this approach the entire interest rate framework is captured in the specification of a single random variable X_∞. We shall have more to say about such conditional variance framework shortly.

9.5 Rational Models

Now suppose we let the positive martingale family M_{ab} be of the form

$$M_{ab} = \alpha_b + \beta_b M_a \tag{9.43}$$

where α_b and β_b are positive deterministic functions satisfying $\alpha_b + \beta_b = 1$, and M_a is any positive martingale, normalised so that initially we have $M_0 = 1$.

Then a short calculation shows that

$$P_{ab} = \frac{F_b + G_b M_a}{F_a + G_a M_a}, \tag{9.44}$$

where F_b and G_b are positive decreasing functions, satisfying

$$F_b + G_b = P_{0b} \tag{9.45}$$

where P_{0b} is the initial discount function.

Inspection shows that $P_{bb} = 1$, $0 < P_{ab} \leq 1$, and $\partial_b P_{ab} < 0$, the positive interest conditions.

This is the so-called *rational* model (Flesaker & Hughston 1996).

If M_a is chosen, for example, to be a geometric Brownian motion, then we obtain the *rational log-normal model.*

In the *extended* rational log-normal model we have

$$M_a = \exp\left(\int_0^a \sigma(s)\, dW_s - \tfrac{1}{2}\int_0^a \sigma^2(s)\, ds\right) \tag{9.46}$$

where $\sigma(s)$ is deterministic.

This model is one of the simplest of all interest rate models.

It admits completely analytic formulae for the valuation of caps, floors and swaptions of all maturities.

A short calculation shows that the short rate, in the case of a general rational model, is given by

$$r_t = -\frac{F'(t) + G'(t)M_t}{F(t) + G(t)M_t}. \tag{9.47}$$

It is not difficult to show then, in the case of the *extended rational log-normal model*, that r_t is a diffusion.

In other words, in the RLN model r_t satisfies a stochastic equation of the form

$$dr_t = \delta(t, r_t)\, dt + \gamma(t, r_t)\, dW_t \tag{9.48}$$

where $\delta(t, r)$ and $\gamma(t, r)$ are each deterministic functions of two variables.

It is an interesting exercise to show in this case that $\gamma(t, r)$ is a *quadratic polynomial* in the short rate.

The two positive roots to this equation correspond to (time dependent) upper and lower bounds on the interest rate process.

The RLN model is an important example of a completely tractable system of interest rate dynamics exhibiting many desirable qualitative features.

A relatively complete analysis of the valuation of caps and swaptions in the rational log-normal model has been given by Musiela & Rutkowski (1997).

Chapter 10

Multi-currency interest rate dynamics. Compatible exchange rate systems. Geometric analysis of foreign exchange volatility and correlation. Quanto effects. International models for interest rates and foreign exchange.

10.1 Interest rate and foreign exchange dynamics

Let us consider the problem of constructing an extension of the basic HJM framework suitable for the valuation of interest rate and foreign exchange derivatives.

We consider an international economy consisting of a set of n currencies, and for each currency a family of discount bonds denominated in that currency.

For such an economy it is possible to deduce a set of formulae for the price processes of these discount bonds and the associated exchange rates, subject to the conditions of no arbitrage.

In the multi-currency situation we do not wish to single out any preferred currency.

So we work with the natural measure P, and transform to the risk neutral measure associated with a choice of currency only for special applications.

In the multi-currency situation there is a numeraire process associated with each currency, and these are all related to one another via the exchange rate process.

If S_t^{ij} denotes the price of one unit of currency i in units of currency j (e.g., the price of one pound sterling in dollars), then the relation is given more specifically by

$$\xi_t^i S_t^{ij} = \xi_t^j, \tag{10.1}$$

where ξ_t^i denotes the price process for the numeraire asset, expressed in units of currency i.

The effect of the no arbitrage condition on the international interest rate and foreign exchange markets is to ensure the existence of a "global" numeraire asset, the value of which can be expressed in any currency.

If the market is complete, then the global numeraire is completely determined by the given asset processes. More generally, we simply assume the existence of a global pricing kernel.

The global numeraire has the property that the ratio of the value (in a given currency) of any nondividend-paying tradable asset to the value (in the same currency) of the global numeraire is a martingale with respect to natural measure.

An arbitrage-free complete system of interest rates and foreign exchange is called an "Amin-Jarrow" economy.

Let us write P^i_{ab} for the value (in units of currency i) at time a (time 0 is the present) of a default-free discount bond that matures at time b to deliver one unit of currency i.

We shall write B^i_a for the value at time a (in units of currency i) of a money market account for currency i, initialised to one unit of currency at time 0.

The money market account for currency i can be expressed in terms of the short rate r^i_s for the currency by the formula

$$B^i_a = \exp\left(\int_0^a r^i_s \, ds\right). \tag{10.2}$$

We shall write λ^i_a for the risk premium vector for currency i.

Thus λ^i_a determines the excess rate of return (above the short rate in currency i), per unit of volatility, for assets denominated in that particular currency.

For the discount bonds we have the following dynamics:

$$\frac{dP^i_{ab}}{P^i_{ab}} = r^i_a \, da + \Omega^i_{ab} \, (dW_a + \lambda^i_a \, da). \tag{10.3}$$

Here Ω^i_{ab} is the discount bond volatility vector for currency i.

For any given value of i, the dynamics (10.3) look much like the bond dynamics we have already considered.

However, in the present context, the multi-dimensional Brownian motion W^α_t drives the *whole economy*, and $\lambda_t^{i\alpha}$ is the risk premium vector for any asset denominated in that currency.

Now let us consider the process S_a^{ij} for the exchange rate. This must of course be of the form

$$\frac{dS_a^{ij}}{S_a^{ij}} = \mu_a^{ij}\,da + \nu_a^{ij}\,dW_a, \tag{10.4}$$

where μ_a^{ij} is the drift and ν_a^{ij} is the volatility vector.

In a complete market with no arbitrage, the drift and volatility take the following remarkable form:

$$\mu_a^{ij} = r_a^i - r_a^j + (\lambda_a^j - \lambda_a^i) \cdot \lambda_a^j \tag{10.5}$$

and

$$\nu_a^{ij} = \lambda_a^j - \lambda_a^i. \tag{10.6}$$

10.2 Compatible exchange rate systems*

Suppose we have an n-by-n matrix S_a^{ij} of positive Itô processes based on a multi-dimensional Brownian motion W_a^α.

We assume that S_a^{ij} satisfies the compatibility conditions

$$S_a^{ij} S_a^{jk} = S_a^{ik}, \tag{10.7}$$

and that $S_a^{ii} = 1$.

It follows that $S_a^{ij} = 1/S_a^{ji}$, and that $S_a^{ii} = 1$.

We call such a set of processes a "compatible exchange rate system".

What constraints does the form (10.7) place on the resulting exchange rate dynamics?

For any compatible exchange rate system there exists a set of positive processes ξ_a^i such that

$$S_a^{ij} = \xi_a^j/\xi_a^i. \tag{10.8}$$

The proof of this follows if we write (10.7) in the form in the form $S_a^{ij} = S_a^{kj}/S_a^{ki}$. Fixing a value k, we define $\xi_a^i = S_a^{ki}$ for all i.

This is well-defined procedure since S_a^{ij} is always positive, and shows that S_a^{ij} splits into a quotient.

The split (10.8) is not unique, since it is invariant under the transformation $\xi_a^i \to \pi_a \xi_a^i$ for any positive process π_a.

We shall investigate the consequences of this freedom later.

Suppose we therefore write

$$\frac{d\xi_a^i}{\xi_a^i} = R_a^i \, da + \lambda_a^i \, dW_a \tag{10.9}$$

for the stochastic equation satisfied by ξ_a^i, for some given choice of ξ_a^i.

Without loss of generality we define the process r_a^i by setting $R_a^i = r_a^i + \lambda_x^{i\,2}$. Then we have

$$\frac{d\xi_a^i}{\xi_a^i} = r_a^i \, da + \lambda_a^i (dW + \lambda_a^i \, da). \tag{10.10}$$

A short calculation making use of the Itô quotient relation shows that

$$\frac{dS_a^{ij}}{S_a^{ij}} = (r_a^j - r_a^i) \, da + \nu_a^{ij} \, (dW_a + \lambda_a^j \, da). \tag{10.11}$$

The exchange rate volatility ν_a^{ij} is thus given as indicated earlier by

$$\nu_a^{ij} = \lambda_a^j - \lambda_a^i. \tag{10.12}$$

Clearly we have $\nu_a^{ij} = -\nu_a^{ji}$. Thus we see that the splitting of ν_a^{ij} to a difference of two vector processes arises from the compatibility condition.

Recall that S_a^{ij} is the price of one unit of currency i in units of currency j.

Thus the volatility vector for the price of sterling in dollars is minus the volatility vector for the price of one dollar in sterling.

Now a currency is not a non-dividend paying asset.

The "dividend" earned by currency i is the interest it continuously accumulates in a money market account.

Thus in (10.11) the overall drift in the value of currency i is of the form

$$\mu_a^{ij} = r_a^j - r_a^i + \nu_a^{ij} \cdot \lambda_a^j. \tag{10.13}$$

This is given by the risk-free rate on the valuation currency j, less the “dividend yield” r_a^i, plus the excess rate of return.

The excess rate of return is given by the inner product of the volatility vector for currency i (when priced in units of currency j) times the relative risk vector for the valuation currency.

10.3 Geometric analysis of FX volatility

The volatility vectors for a compatible foreign exchange rate system have to fit together to form a *polytope* in the multidimensional Euclidean space in which the Wiener process takes its values.

This is on account of the relation $\nu_{ij} + \nu_{jk} + \nu_{ki} = 0$.

Thus for three currencies we have a triangle, four give a tetrahedron, and so on.

In that picture the vertices of the figure correspond to currencies, and the length of the edge joining two given vertices is the magnitude of the instantaneous volatility of the associated exchange rate.

The cosine of the angle between two edges (whether they intersect or not a common vertex) measures the instantaneous correlation between the movements in the given exchange rates.

The relation $\nu_t^{ij} = \lambda_t^j - \lambda_t^i$ allows one to take this set of ideas a step further, incorporating the relative risk into the picture.

In particular, if we fix an origin, then the relation $\nu_t^{ij} = \lambda_t^j - \lambda_t^i$ shows us that the system of risk premium vectors for the various currencies, viewed as emanating from the origin, determines the location and structure of the volatility ‘polytope’.

10.4 Scale transformations*

Now suppose that we are just given the process S_a^{ij}. To what extent does this process determine r_a^i and λ_a^i in a complete market free of arbitrage?

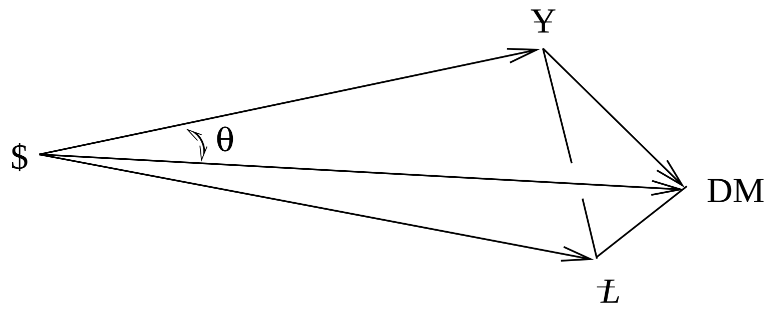

Figure 10.1: *Four currency tetrahedron. The six edge-length correspond to the volatilities of the six exchange rates for the four given currencies. The angles between edges determine the corresponding correlations.*

The exchange rate volatility ν_a^{ij} given by (10.12) is invariant under the transformation

$$\xi_a^i \to \pi_a \xi_a^i, \tag{10.14}$$

since the exchange rate S_a^{ij} itself is left unchanged.

Under the scale transformation (10.14) we find, after a short calculation, that the risk premium and short rate transform as follows:

$$\lambda_a^i \to \lambda_a^i + \Psi_a \tag{10.15}$$

$$r_a^i \to r_a^i + \Phi_a - \Psi_a^2 - \lambda_a^i \Psi_a. \tag{10.16}$$

Here the vector process Ψ_a and the scalar process Φ_a are defined by

$$\frac{d\pi_a}{\pi_a} = \Phi_a \, da + \Psi_a \, dW_a. \tag{10.17}$$

The process λ_a^i is thus determined by the exchange rate system up to a transformation of the form (10.15) for an arbitrary vector process Ψ_a.

Geometrically, this can be pictured as a translation of the entire volatility polytope in the direction given by Ψ_a.

One can think of such transformations as representing "global" change in the international economy. For example, one might have an overall drop in interest rates coupled with a general change in risk aversion as regards some particular source of risk.

Once λ_a^i is fixed, then the interest rates are determined up to an overall change of level Φ_a.

10.5 "Quanto" effects*

It is worth noting the effect that a transformation to the risk neutral measure associated with a given "domestic" currency j has on the bond price process for a "foreign" currency.

The process for the domestic bond price, when we transform to the risk neutral measure, becomes

$$\frac{dP^j_{ab}}{P^j_{ab}} = r^j_a \, da + \Omega^j_{ab} \, dW^j_a, \tag{10.18}$$

where $dW^j_a = dW_a + \lambda^j_a \, da$.

The bond process for the "foreign" currency i, which in the original measure is given by

$$\frac{dP^i_{ab}}{P^i_{ab}} = r^i_a \, da + \Omega^i_{ab} \, (dW_a + \lambda^i_a \, da), \tag{10.19}$$

transforms to

$$\frac{dP^i_{ab}}{P^i_{ab}} = r^i_a \, da + \Omega^i_{ab} \, (dW^j_a - \nu^{ij}_a \, da), \tag{10.20}$$

when expressed in terms of W^j_a, which is a Brownian motion in the risk neutral measure associated with currency j.

Note the appearance of $\nu^{ij}_a = \lambda^j_a - \lambda^i_a$, the foreign exchange volatility vector, in this formula.

The "quanto" correction term appearing here involves the inner product of the foreign discount bond volatility vector and the exchange rate volatility vector.

This can be re-expressed in more familiar terms as a product of the bond volatility level, the foreign exchange volatility level, and a correlation factor.

10.6 Martingale representation for FX and interest rate systems*

An Amin-Jarrow economy is completely characterised by a set of n one-parameter families of unit-initialised martingales denoted M^i_{as}, along with a set of initial term structure data P^i_{0s} for each currency, and a set of initial exchange rates S^{ij}_0.

We require that the initial exchange rates are compatible in the sense that $S_0^{ij} S_0^{jk} = S_0^{ik}$ (e.g. the price of sterling in dollars times the price of one dollar in yen gives the price of sterling in yen).

For positive interest rates we require, in addition to the above, that the martingales M_{as}^i are strictly positive, and that the initial discount functions P_{0s}^i exhibit positive interest in the sense that

$$0 < P_{0b}^i \leq 1 \quad \text{and} \quad \partial_b P_{0b}^i < 0 \tag{10.21}$$

for all maturities. Here ∂_b denotes differentiation with respect to b.

For the discount bonds in currency i we have an *integral representation* of the form

$$P_{ab}^i = \frac{\int_b^\infty (-\partial_s P_{0s}^i) M_{as}^i \, ds}{\int_a^\infty (-\partial_s P_{0s}^i) M_{as}^i \, ds}. \tag{10.22}$$

Here again P_{ab}^i denotes the value (in units of currency i) at time a (time 0 is the present) of a discount bond that matures at time b to deliver one unit of currency i.

Note that each discount bond is valued in its "own currency".

The system of exchange rates is then given by

$$S_a^{ij} = S_0^{ij} \frac{\int_a^\infty (-\partial_s P_{0s}^i) M_{as}^i \, ds}{\int_a^\infty (-\partial_s P_{0s}^j) M_{as}^j \, ds}. \tag{10.23}$$

Taking into account the given initial conditions, it follows that the compatibility conditions

$$S_a^{ij} S_a^{jk} = S_a^{ik} \tag{10.24}$$

are satisfied.

The numeraire process, which in currency i has the value ξ_a^i, is given by

$$\xi_a^i = \frac{\xi_0^i}{\int_a^\infty (-\partial_s P_{0s}^i) M_{as}^i \, ds}, \tag{10.25}$$

where initial values ξ_0^i are such that

$$S_0^{ij} = \xi_0^j / \xi_0^i. \tag{10.26}$$

The existence of such a system of initial values is ensured by the initial compatibility conditions on the exchange rates.

The basic martingales M^i_{as} are defined for all $s \geq a \geq 0$ (up to some time horizon), and satisfy

$$\mathbb{E}_a M^i_{bs} = M^i_{as}. \tag{10.27}$$

There is an explicit formula for the risk premium vector for each currency, given by

$$\lambda^i_a = -\frac{\int_a^\infty (-\partial_s P^i_{0s}) M^i_{as} \sigma^i_{as}\, ds}{\int_a^\infty (-\partial_s P^i_{0s}) M^i_{as}\, ds}. \tag{10.28}$$

Here the vector process σ^i_{as} is defined by

$$dM^i_{as} = M^i_{as} \sigma^i_{as} dW_a, \tag{10.29}$$

The discount bond volatilities are given in terms of the basic martingales according to the scheme

$$\Omega^i_{ab} = V^i_{ab} - V^i_{aa}, \tag{10.30}$$

where the "risk adjusted" volatility V^i_{ab} is given by

$$V^i_{ab} = \frac{\int_b^\infty \partial_s P_{0s} M^i_{as} \sigma^i_{as}\, ds}{\int_b^\infty \partial_s P_{0s} M^i_{as}\, ds}. \tag{10.31}$$

Thus $\lambda^i_a = -V^i_{aa}$, consistent with equation (10.28).

The short rate is given by

$$r^i_a = \frac{\partial_a P^i_{0a} M^i_{aa}}{\int_a^\infty (-\partial_s P^i_{0s}) M^i_{as}\, ds}. \tag{10.32}$$

With this information at hand we can verify again that the cross-currency process S^{ij}_t satisfies

$$\frac{dS^{ij}_a}{S^{ij}_a} = (r^j_a - r^i_a)\, da + (\lambda^j_a - \lambda^i_a)\, (dW_a + \lambda^j_a\, da). \tag{10.33}$$

We shall return to the matter of international interest rate and foreign exchange systems in greater depth in due course.

Chapter 11

Axiomatic framework for continuous asset price dynamics. Perpetual floating rate notes. Price processes for discount bonds. Dynamics of the state price density.

11.1 Axiomatic framework for continuous asset price dynamics

The idea now is to develop an axiomatic scheme that will ensure the existence of an arbitrage-free system of discount bonds over all time horizons, but that is general enough also to allow a place for other systems of assets.

The methodology that we propose, which in effect unifies a number of important features of the theory of interest rate modelling and the theory of volatility modelling, is based on a *conditional variance representation* for the state price density, and makes use of the *Wiener chaos expansion* technique in a novel way.

We model the unfolding of random market events in the usual way with the specification of a fixed probability space $(\Omega, \mathcal{F}, P)$ which we denote as Π.

We assume that the economy Π is equipped with the standard augmented filtration $\Phi = (\mathcal{F}_t)_{0\leq t\leq T^*}$ generated by a system of one or more independent Wiener processes $(W_t^\alpha)_{0\leq t\leq T^*}$ $(\alpha = 1, \cdots, k)$.

Here T^* represents a fixed time horizon, which for the moment we leave unspecified but eventually will be assumed to be infinite.

The probability measure P is to be interpreted as the "natural" measure, and filtration-dependent concepts (such as adaptedness or the martingale property) are defined relative to Φ.

We assume in this investigation that the random processes on Π followed by asset prices are continuous semimartingales adapted to Φ.

The absence of arbitrage in the economy will be characterised according to the following scheme.

We assume the existence of a continuous semimartingale ξ_t, adapted to Φ, which we call the "natural numeraire" process, satisfying $\xi_t > 0$ for all $t \in [0, T^*]$, such that the following three axioms hold:

(A1) There exists a strictly increasing (and hence "risk-free") asset with price process B_t (the money-market account).

(A2) If S_t is the price-process of any asset, and D_t is the adapted dividend rate for that asset, so that $D_t dt$ represents the small random dividend paid at time t, then the process M_t defined by

$$M_t = \frac{S_t}{\xi_t} + \int_0^t \frac{D_s}{\xi_s} ds$$

ia a martingale.

(A3) There exists an asset (a floating rate note) that offers a dividend rate sufficient to ensure that the value of the asset remains constant.

Now let us examine some of the *consequences* of these axioms.

Existence of risk adjustment density

Since the process B_t introduced in (A1) is by assumption continuous and strictly increasing, there exists an adapted process $r_t > 0$ such that

$$B_t = B_0 \exp\left(\int_0^t r_s ds\right). \tag{11.1}$$

Because the money market account is a non-dividend paying asset, it follows as a consequence of (A1) and (A2) that there exists a positive martingale Λ_t such that

$$\frac{B_t}{\xi_t} = \Lambda_t. \tag{11.2}$$

Since Λ_t is positive, there exists an adapted vector-valued process λ_t such that

$$d\Lambda_t = -\Lambda_t \lambda_t dW_t, \tag{11.3}$$

where here, and similarly elsewhere, we use the shorthand

$$\lambda_t dW_t = \sum_{\alpha=1}^{k} \lambda_t^\alpha dW_t^\alpha. \tag{11.4}$$

As a consequence of (11.3), we then have

$$\Lambda_t = \rho_0 \exp\left(-\int_0^t \lambda_s dW_s - \tfrac{1}{2}\int_0^t \lambda_s^2 ds\right). \tag{11.5}$$

Uniqueness of the money market account

At most one process B_t can exist satisfying axioms (A1) and (A2). For if B_t^* were another such increasing price process, then we would have

$$\frac{\Lambda_t}{B_t} = \frac{\rho_t^*}{B_t^*} \tag{11.6}$$

for some positive martingale ρ_t^*. But this relation implies that

$$\frac{d\Lambda_t}{\Lambda_t} = (r_t - r_t^*)dt + \frac{d\rho_t^*}{\rho_t^*} \tag{11.7}$$

which shows that for Λ_t and ρ_t^* both to be martingales we have $r_t = r_t^*$.

Dynamic equations for risky-assets

Axiom (A2) implies, in the case of a non-dividend-paying asset, that S_t can be written in the form

$$S_t = \frac{B_t M_t}{\Lambda_t} \tag{11.8}$$

where M_t is a martingale.

Thus, if we write $dM_t = \theta_t dW_t$ it is a straightforward exercise to verify that

$$dS_t = (r_t S_t + \lambda_t \psi_t)dt + \psi_t dW_t, \tag{11.9}$$

where the vector-valued process ψ_t is defined by

$$\psi_t = \frac{B_t \theta_t}{\Lambda_t} + \lambda_t S_t. \tag{11.10}$$

In particular, if the asset price S_t is positive, then M_t is positive, and we can write $\theta_t = (\sigma_t - \lambda_t)M_t$ for some vector-valued process σ_t, from which it follows that $\psi_t = \sigma_t S_t$.

In that case the dynamical equation satisfied by S_t can be written in the form

$$\frac{dS_t}{S_t} = (r_t + \lambda_t \sigma_t)dt + \sigma_t dW_t, \tag{11.11}$$

where σ_t is the adapted vector-valued volatility process for the given asset, and λ_t has the interpretation of the market risk premium.

We recognise (11.11) as the dynamics of a risky asset with limited liability in a market with no arbitrage.

However the dynamical equation

$$dS_t = (r_t S_t + \lambda_t \psi_t)dt + \psi_t dW_t, \tag{11.12}$$

has the advantage of holding in the more general situation for assets such as portfolio positions including borrowing, short sales, or derivatives, where the value of the position may swing into the red as well as the black.

Risky assets with dividend

In the case of a dividend paying asset these formulae need to be modified slightly, and in place of (11.12) we obtain

$$dS_t = (r_t S_t - D_t + \lambda_t \psi_t)dt + \psi_t dW_t \tag{11.13}$$

as a consequence of (A2), with ψ_t defined as before according to $\psi_t = B_t\theta_t/\Lambda_t + \lambda_t S_t$.

Then if S_t is positive we can introduce a proportional dividend rate δ_t by the relation $D_t = \delta_t S_t$, and we obtain the simplified expression

$$\frac{dS_t}{S_t} = (r_t - \delta_t + \lambda_t \sigma_t)dt + \sigma_t dW_t, \tag{11.14}$$

where σ_t is defined as before by $\psi_t = \sigma_t S_t$.

Clearly, (11.14) conforms to the familiar dynamics of a dividend or interest paying asset with limited liability.

For example, if S_t is the price of a foreign currency, then δ_t corresponds to the overnight rate for that currency. We consider the case of a foreign currency in greater detail later.

Assets of constant value

Now let us examine axiom (A3) more closely. Such a "cash" asset that maintains a constant value has the interpretation of being a *floating rate note*.

Equation (11.14) shows that if we set $S_t = 1$ for all $t \in [0, T^*]$ then the "dividend" rate offered for such an instrument must be r_t. It follows that

$$\frac{1}{\xi_t} + \int_0^t \frac{r_s}{\xi_s} ds \qquad \text{is a martingale.} \tag{11.15}$$

In particular since r_t and ξ_t are positive we deduce that

$$\mathbb{E}\left[\frac{1}{\xi_t}\right] < \infty \tag{11.16}$$

and

$$\mathbb{E}\left[\int_0^t \frac{r_s}{\xi_s} ds\right] < \infty \tag{11.17}$$

for all $t \in [0, T^*]$.

11.2 Price processes for discount bonds

To proceed further we introduce a system of discount bonds on the economy Π.

More precisely, this will be the discount bond system associated with the base currency in terms of which the other assets on Π are priced and with respect to which the money market process B_t is defined.

The discount bond price processes will be denoted P_{tT}, where $0 \le t \le T \le T^* \le \infty$.

We shall as usual regard the zero-coupon bond for a given value of T as a default-free contract that pays one unit of the base currency at time T.

Then P_{tT} denotes the price of the bond at time t, and by the definition of the contract we require that $P_{TT} = 1$ for all $T \in [0, T^*]$.

For the moment we make no other assumptions concerning the discount bond processes other than those properties applicable to all assets implicit in axioms (A1), (A2), and (A3), though later we add a further important assumption concerning the asymptotic behaviour of the bond prices in the case of an infinite time horizon.

Since P_{tT} represents the price process of a non-dividend-paying asset for each value of $T \in [0, T^*]$, it follows from axiom (A2) that P_{tT}/ξ_t is a martingale, and hence that there exists a family of positive martingales M_{tT} such that

$$P_{tT} = \frac{B_t M_{tT}}{\Lambda_t}. \tag{11.18}$$

Because M_{tT} is a positive martingale for each bond maturity date $T \in [0, T^*]$, there exists a vector-valued process Ω_{tT} such that

$$\frac{dM_{tT}}{M_{tT}} = (\Omega_{tT} - \lambda_t) dW_t. \tag{11.19}$$

We thus that the dynamics of the discount bond system are given by

$$\frac{dP_{tT}}{P_{tT}} = (r_t + \lambda_t \Omega_{tT}) dt + \Omega_{tT} dW_t. \tag{11.20}$$

We recognise Ω_{tT} as being the T-maturity discount bond vector relative volatility process.

It then follows, by integrating (11.20), if we make use of the relation $P_{tt} = 1$, that the discount bond price processes can be represented in the form

$$P_{tT} = P_{0tT} \frac{\exp\left(\int_0^t \lambda_s \Omega_{sT} ds + \int_0^t \Omega_{sT} dW_s - \frac{1}{2}\int_0^t \Omega_{sT}^2 ds\right)}{\exp\left(\int_0^t \lambda_s \Omega_{st} ds + \int_0^t \Omega_{st} dW_s - \frac{1}{2}\int_0^t \Omega_{st}^2 ds\right)}, \tag{11.21}$$

and that the money market account process is given by a corresponding expression of the form

$$B_t = \frac{B_0}{P_{0t} \exp\left(\int_0^t \lambda_s \Omega_{st} ds + \int_0^t \Omega_{st} dW_s - \frac{1}{2}\int_0^t \Omega_{st}^2 ds\right)}. \tag{11.22}$$

Here we have used the notation $P_{0tT} = P_{0T}/P_{0t}$ for the t-forward price made at time 0 for a T-maturity discount bond.

The volatility structure approach

An interesting feature of the expressions (11.21) and (11.22) is that the discount bond system and the money market account can be represented directly in terms of the market risk premium process λ_t and the bond volatility process Ω_{tT}, together with the initial discount function P_{0t}, without direct reference to the short rate r_t.

It is therefore legitimate to regard λ_t and Ω_{tT} as being subject to an exogenous specification.

Indeed, historically this observation is of considerable significance since it forms the basis of the approach to interest rate derivatives pricing frequently used in practice according to which one "models the volatility structure".

In such an approach one typically assumes market completeness, then transforms to the risk neutral measure to eliminate the market risk premium, and then models the bond volatility process exogenously, calibrating it to a suitable given set of market interest rate option data.

It has been a problematic feature of the volatility approach, however, that if λ_t and Ω_{tT} are specified exogenously, then there is no guarantee that axiom (A1) is satisfied—that is to say, the resulting interest rates need not be positive.

Additionally, there is no reason to suppose, a priori, that the bond volatilities will take on a given form in the risk neutral measure.

Let us therefore put to one side the "volatility structure" approach, and return to the consideration of the assumptions (A1), (A2), and (A3) in the context of a term structure model.

Martingale relations

Because the discount bonds are non-dividend-paying assets, it follows as a consequence of (A2) that the martingale relations

$$E\left[\frac{P_{tT}}{\xi_t}\right] < \infty \tag{11.23}$$

and

$$\frac{P_{tT}}{\xi_t} = \mathbb{E}_t\left[\frac{P_{uT}}{\xi_u}\right] \tag{11.24}$$

hold for all $0 \leq t \leq u \leq T \leq T^*$.

Here $\mathbb{E}_t[-]$ denotes as usual the conditional expectation with respect to the σ-algebra $\mathcal{F}_t$.

It follows from (11.23) by setting $t = T$ that the existence of the discount bond system implies that the inequality

$$\mathbb{E}\left[\frac{1}{\xi_t}\right] < \infty \tag{11.25}$$

holds for all $t \in [0, T^*]$.

It is interesting to note, as was shown by Baxter (1997), that the inequality (11.17) is the additional assumption required to ensure the differentiability of the bond price system with respect to the maturity date.

Instantaneous forward rates

In other words, as a consequence of (11.17) there exists a family of continuous semimartingales f_{tu}, adapted to Φ, for all $0 \leq t \leq u \leq T^*$, such that

$$P_{tT} = \exp\left(-\int_t^T f_{tu}du\right). \tag{11.26}$$

It then follows that

$$-\partial_T \ln P_{tT} = f_{tT}, \tag{11.27}$$

where ∂_T denotes differentiation with respect to T, and also that

$$\lim_{t\to T} f_{tT} = r_T \tag{11.28}$$

and

$$\lim_{t\to T} \Omega_{tT} = 0. \tag{11.29}$$

The importance of the existence of the instantaneous forward rates is that the class of interest rate models under consideration here is equivalent to the family of all positive interest HJM models (Heath, Jarrow and Morton 1992) defined over the relevant time horizon.

We take the view here nevertheless that the instantaneous forward rates are in some sense secondary, and that primary significance should be attached to modelling the natural numeraire process ξ_t.

Risk neutral valuation formula

In particular, setting $u = T$ in (11.24) we obtain the pricing formula

$$P_{tT} = \xi_t \mathbb{E}_t\left[\frac{1}{\xi_T}\right]. \tag{11.30}$$

Thus, once axioms (A1), (A2), and (A3) have been specified, the associated discount bond system is also determined.

We note that P_{tT} is unchanged if we multiply ξ_t by a positive constant.

11.3 Dynamics of the state price density

It is be useful now to introduce the related process $V_t = 1/\xi_t$ which has the interpretation of being the state price density.

It follows from equation (11.1) that $V_t = \Lambda_t/B_t$, and from (11.16) we have $\mathbb{E}[V_t] < \infty$ for all $t \in [0, T^*]$.

In particular, since B_t is $\mathcal{F}_t$-measurable and increasing we deduce that

$$\mathbb{E}_t\left[V_T\right] = \mathbb{E}_t\left[\frac{\Lambda_T}{B_T}\right] < \mathbb{E}_t\left[\frac{\Lambda_T}{B_t}\right] = \frac{\mathbb{E}_t\left[\Lambda_T\right]}{B_t} = \frac{\Lambda_t}{B_t} = V_t, \tag{11.31}$$

for $t < T$.

In other words, we have $\mathbb{E}_t[V_T] < V_t$, and thus we see that V_t is a supermartingale.

Now writing the risk neutral valuation formula in the form

$$P_{tT} = \mathbb{E}_t\left[\frac{V_T}{V_t}\right], \tag{11.32}$$

we see that $P_{tT} < 1$ for all $t < T$.

Pricing kernel

The quotient $K_{tT} = V_T/V_t$ can be regarded as a "pricing kernel" for derivatives (Constantinides 1992).

In particular, suppose that H_t is for $t \in [0, T]$ the price process of a derivative asset on Π with a European-style payoff H_T at time T.

Then by (A2) we have

$$H_t = \mathbb{E}_t\left[K_{tT} H_T\right], \tag{11.33}$$

a relation that remains valid independently of any hedgeability considerations.

Note that no assumption of market completeness is made in our axiomatic scheme.

Properties of the state price density

It follows from the dynamical equations for B_t and Λ_t that the dynamics of V_t are given by

$$dV_t = -r_t V_t dt - \lambda_t V_t dW_t. \tag{11.34}$$

Therefore, given V_t we can recover the short rate r_t and the market risk premium process λ_t.

Integrating (11.34) from t to T we get

$$V_T = V_t - \int_t^T r_s V_s ds - \int_t^T \lambda_s V_s dW_s. \tag{11.35}$$

Taking the conditional expectation of each side of (11.35) we obtain

$$\mathbb{E}_t [V_T] = V_t - \mathbb{E}_t \left[\int_t^T r_s V_s ds \right]. \tag{11.36}$$

Dividing by V_t we then arrive at the formula

$$P_{tT} = 1 - \mathbb{E}_t \left[\int_t^T K_{ts} r_s ds \right], \tag{11.37}$$

which has a natural economic interpretation from which a number of interesting consequences can be deduced.

It follows for example as a corollary of (11.37) that for any two maturity dates T_1 and T_2 we have

$$P_{tT_1} - P_{tT_2} = \mathbb{E}_t \left[\int_{T_1}^{T_2} K_{ts} r_s ds \right]. \tag{11.38}$$

Therefore if $T_2 > T_1$, we deduce that $P_{tT_2} < P_{tT_1}$, and hence that the random forward price

$$P_{tT_1T_2} = \frac{P_{tT_2}}{P_{tT_1}}, \tag{11.39}$$

made at time t for purchase at time T_1 of a T_2-maturity discount bond satisfies

$$0 < P_{tT_1T_2} \leq 1 \tag{11.40}$$

for all $0 \leq t \leq T_1 \leq T_2 < \infty$.

This in turn implies the positivity of all forward rates.

Interpretation of the instantaneous forward rates

Another interesting corollary of (11.37) follows if we differentiate each side of this equation with respect to T, from which we deduce that

$$f_{tT} P_{tT} = \mathbb{E}_t [K_{tT} r_T]. \tag{11.41}$$

This relation shows that the instantaneous forward rates can be interpreted as the value, at time t, future-valued to time T, of the contingent claim that pays the short rate r_T at time T on a unit principal.

It follows that the term structure density $\rho_t(x)$ for tenor $x = T - t$ is the value at time t of an instrument that pays the rate r_T at time T on a unit principal.

Equation (11.37) says that ownership of a T-maturity discount bond is equivalent to ownership of one unit of the cash asset, but without the right to the dividend flow of the cash asset from time t to time T.

To put the matter in another way, a money-lender will be willing at time t to part with one unit of cash in exchange for a discount bond maturing at time T together with a continuous flow of interest from time t to time T.

Equivalently, to hold a T-maturity floating-rate note is the same as holding a T-maturity discount bond together with the right to a continuous stream of interest from time t to T.

Chapter 12

The conditional variance representation for the state price density. Interest rate models as elements of $L^2(\Omega, \mathcal{F}, P)$. Elements of Wiener chaos. First chaos models.

12.1 The conditional variance representation

Now suppose we consider the case of an interest rate system with an infinite time horizon $T^* = \infty$. It follows from (11.37) that

$$P_{0T} = 1 - \mathbb{E}\left[\int_0^T K_{0s} r_s ds\right]. \tag{12.1}$$

This relation can be interpreted as saying that the value of a T-maturity discount bond at time 0 is one unit of cash less the present value of the interest stream from time 0 to time T.

The idea is that by holding the discount bond one forgoes the dividends associated with the cash until the maturity date of the bond—at which point one acquires the cash.

On the role of potential

The ownership of a discount bond that never matures (i.e. matures at $T = \infty$) is equivalent to ownership of a unit of floating rate note stripped of its interest stream for all time—in other words, the ownership of nothing.

As a consequence we conclude that

$$\lim_{T\to\infty} P_{0T} = 0, \tag{12.2}$$

or equivalently

$$V_0 = \mathbb{E}\left[\int_0^\infty r_s V_s ds\right]. \tag{12.3}$$

Indeed, we shall now take it as part of the definition of a discount bond system that $T^* = \infty$ and that the natural numeraire ξ_t and the interest rate r_t are such that (12.2) holds, or equivalently

$$\xi_0 \mathbb{E}\left[\int_0^\infty \frac{r_s}{\xi_s} ds\right] = 1. \tag{12.4}$$

Alternatively, it follows from (11.32) that the asymptotic condition (12.2) holds if and only if

$$\lim_{T\to\infty} \mathbb{E}\left[V_T\right] = 0. \tag{12.5}$$

This is the condition that the process V_t is a "potential", i.e. a positive supermartingale with the property that its expectation vanishes in the limit.

Thus, as was pointed out by Rogers (1997), it should be regarded as an essential element of interest rate theory that the state price density should have this property.

Recursive relation of the state price density

We see therefore that once an appropriate asymptotic condition has been placed on the discount bond system we have the key relation

$$V_t = \mathbb{E}_t\left[\int_t^\infty r_s V_s ds\right]. \tag{12.6}$$

This formula has the economic interpretation that a floating rate note that promises to pay the rate r_t on a unit principal in perpetuity necessarily has the value unity.

An alternative expression for V_t can be deduced from (12.6) if we define the increasing process

$$A_t = \int_0^t r_s V_s ds. \tag{12.7}$$

Then we obtain the relation $V_t = \mathbb{E}_t\left[A_\infty\right] - A_t$ as discussed earlier.

This forms the basis of the Flesaker-Hughston framework and its extensions (see, e.g., Flesaker and Hughston 1996, 1997, 1998, Rutkowski 1997, Musiela and Rutkowski 1997, Rogers 1997, James and Webber 2000, Hunt and Kennedy 2000, Jin and Glasserman 2001).

In the present investigation, we take an alternative point of view and emphasize a rather different feature of the state price density that emerges in this context: namely, that V_t can be interpreted as a *conditional variance*.

This makes use of an idea appearing in Meyer (1966). More precisely, let σ_t be a vector process satisfying

$$\sigma_t^2 = r_t V_t. \tag{12.8}$$

Then we can define a random variable X_∞ by the formula

$$X_\infty = \int_0^\infty \sigma_s dW_s. \tag{12.9}$$

The existence of X_∞ is guaranteed by virtue of axiom (A3) which implies that

$$\mathbb{E}\left[\int_0^\infty r_s V_s ds\right] < \infty. \tag{12.10}$$

It follows immediately then by virtue of the Ito isometry that

$$\begin{aligned} V_t &= \mathbb{E}_t\left[\int_t^\infty \sigma_s^2 ds\right] \\ &= \mathbb{E}_t\left[\left(\int_t^\infty \sigma_s dW_s\right)^2\right] \\ &= \mathbb{E}_t\left[\left(\int_0^\infty \sigma_s dW_s - \int_0^t \sigma_s dW_s\right)^2\right]. \end{aligned} \tag{12.11}$$

However, because

$$\mathbb{E}_t\left[X_\infty\right] = \int_0^t \sigma_s dW_s, \tag{12.12}$$

we deduce that

$$V_t = \mathbb{E}_t\left[\left(X_\infty - \mathbb{E}_t\left[X_\infty\right]\right)^2\right], \tag{12.13}$$

which we recognise as the conditional variance of X_∞ with respect to the σ-algebra $\mathcal{F}_t$.

In particular we note that $X_\infty \in L^2(\Omega, \mathcal{F}, P)$.

We shall take the view that the random variable X_∞ should in some sense be regarded as the "primitive" in the construction of the associated interest rate system.

12.2 Interest rate models as elements of $L^2(\Omega, \mathcal{F}, P)$

Let us now recapitulate what we have learned so far.

The market is characterised by a probability space $\Pi = (\Omega, \mathcal{F}, P)$ which we can assume to be the classical Wiener space associated with a system of n independent Brownian motions.

If we assume the existence of an arbitrage-free system of discount bonds on Π then it follows from the considerations of the previous sections that there exists a random variable $X_\infty \in L^2(\Pi)$ with zero mean such that the state price density V_t is given by the conditional variance

$$V_t = \mathbb{E}_t\left[(X_\infty - \mathbb{E}_t[X_\infty])^2\right] \tag{12.14}$$

and the discount bond system is given by

$$P_{tT} = \frac{\mathbb{E}_t[V_T]}{V_t}. \tag{12.15}$$

The state-price density is fully determined by the random variable X_∞.

Conversely, given the state-price density process, we can determine the short rate process and then use the relation $\sigma_t^2 = r_t V_t$ to construct the integrand in the expression for the corresponding asymptotic random variable X_∞.

We therefore have a correspondence between arbitrage-free positive interest rate models and square-integrable zero-mean random variables on the Wiener space Π.

Interestingly, this space has a very rich natural structure that can be exploited in the analysis of the associated interest rate systems.

The key point is that we can represent X_∞, and therefore characterise the corresponding interest rate system, by use of a Wiener chaos expansion.

In particular, the integrand σ_s in the defining equation (12.9) can be expanded in a unique way in a series of the form

$$\sigma_s = \phi_s + \int_0^s \phi_{ss_1} dW_{s_1} + \int_0^s \int_0^{s_1} \phi_{ss_1s_2} dW_{s_2} dW_{s_1} + \cdots . \tag{12.16}$$

Inserting this expression into (12.9) we then obtain the following representation for the random variable X_∞:

$$X_\infty = \int_0^\infty \phi_s dW_s + \int_0^\infty \int_0^s \phi_{ss_1} dW_{s_1} dW_s + \cdots . \tag{12.17}$$

The integrands $\phi_s = \phi^{\alpha}(s), \phi_{ss_1} = \phi^{\alpha\alpha_1}(s, s_1), \phi_{ss_1s_2} = \phi^{\alpha\alpha_1\alpha_2}(s, s_1, s_2)$, and so on, appearing here are deterministic tensor-valued functions, where $s \geq s_1 \geq s_2 \geq \cdots$.

Then for the expectation of the square of the random variable X_∞ we have

$$\mathbb{E}\left[X_\infty^2\right] = \int_0^\infty \phi_s^2 ds + \int_0^\infty \int_{s_1=0}^{s} \phi_{ss_1}^2 ds_1 ds + \cdots \quad . \tag{12.18}$$

It should be evident by consideration of formula (12.13) that for each choice of X_∞ we obtain a specific interest rate model.

Nesting of interest rate models

In addition, the different models thus arising are nested in a natural way.

To be precise, by an interest rate model we mean the filtered probability space Π together with the pair (V_t, P_{tT}).

We shall call an interest rate model that only contains terms up to order n in the expansion of X_∞ an n^{th}-order chaos model.

If X_∞ contains only the n^{th} order term we shall call the resulting interest rate model a "pure" chaos model of order n. It should be evident that the n^{th}-order chaos models are contained as a subset of the m^{th}-order chaos models, for all $n < m$.

Despite the relatively high level of abstraction in the overall framework, the inputs of such models are simply the deterministic functions ϕ_s, ϕ_{s,s_1}, $\phi_{s.s_1,s_2}$ and so on.

It follows that interest rate models can be classified according to their chaos structure, and indeed all positive interest HJM models based on a Brownian filtration can be systematically built up in this way.

12.3 Elements of Wiener chaos

Before we embark upon the analysis of specific interest rate models it will be helpful first if we review briefly in a little more detail the basics of the Wiener chaos technique.

This will also give us the opportunity to develop the notation further. The material discussed in this section is for the most part well established, and we refer the reader for example to Nualart (1995), Øksendal (1997) or Teichmann (2002) for further details. The foundations of the chaos technique can be found in Wiener (1938) and Ito (1951).

The applications of Wiener chaos to problems in finance were pioneered by Lacoste (1996).

Let H be a real Hilbert space with scalar product $\langle \cdot, \cdot \rangle$.

Given an element $h \in H$, its norm will be denoted $\|h\|$. We introduce a field of random variables $W = \{W_h, h \in H\}$.

We say that W is a Gaussian field if W is a Gaussian family of random variables with zero mean such that $\mathbb{E}[W_g W_h] = \langle g, h \rangle$ for all $g, h \in H$.

Under this definition the map $h \to W_h$ is a linear isometry of the space H onto a closed subspace of $L^2(\Omega, \mathcal{F}, P)$, which we denote by $\mathcal{H}_1$.

It follows immediately that $W_{(ag+bh)} = aW_g + bW_h$ for any $a, b \in R$ and $g, h \in H$.

The elements of $\mathcal{H}_1$ are zero-mean Gaussian random variables.

Next we introduce the Hermite polynomials $H_n(x)$, defined by the formula

$$H_n(x) = \frac{1}{n!}(-1)^n \mathrm{e}^{\frac{1}{2}x^2} \frac{d^n}{dx^n}(\mathrm{e}^{-\frac{1}{2}x^2}), \qquad n \geq 1, \tag{12.19}$$

and $H_0(x) = 1$.

These polynomials play a fundamental role in the Wiener chaos expansion.

The Hermite polynomials of degree one, two, three and four are $H_1(x) = x$, $H_2(x) = \frac{1}{2}(x^2 - 1)$, $H_3(x) = \frac{1}{6}(x^3 - 3x)$, and $H_4(x) = \frac{1}{24}(x^4 - 6x^2 + 3)$ respectively.

Let X and Y be random variables with a jointly Gaussian distribution such that $\mathbb{E}[X] = \mathbb{E}[Y] = 0$, and $\mathbb{E}[X^2] = \mathbb{E}[Y^2] = 1$.

Then for all $n, m \geq 0$ we have

$$\mathbb{E}[H_n(X) H_m(Y)] = \frac{1}{n!} \delta_{nm} \left(\mathbb{E}\left[XY \right] \right)^n . \tag{12.20}$$

For each $n \geq 1$ we denote by $\mathcal{H}_n$ the linear subspace of $L^2(\Omega, \mathcal{F}, P)$ generated by the random variables $\{H_n(W_h), h \in H, \|h\| = 1\}$, with the convention that $\mathcal{H}_0$ denotes the constants.

For $n = 1$, we recover the space $\mathcal{H}_1$ of zero mean Gaussian random variables.

It should be evident from (12.20) that $\mathcal{H}_n$ and $\mathcal{H}_m$ are orthogonal for $n \neq m$.

The subspace $\mathcal{H}_n$ is called the *Wiener chaos of order* n.

If we denote by $\mathcal{G}$ the σ-field generated by the random variables $\{W_h, h \in H\}$, then the space $L^2(\Omega, \mathcal{G}, P)$ can be decomposed into the following infinite orthogonal sum of the subspaces $\mathcal{H}_n$:

$$L^2(\Omega, \mathcal{G}, P) = \oplus_{n=0}^{\infty} \mathcal{H}_n. \tag{12.21}$$

This fundamental decomposition of $L^2(\Omega, \mathcal{G}, P)$ leads to the representation of any element of this space by series of terms resulting from the orthogonal projection of the given element on to the various chaos subspaces.

Now let us reduce the generality of the underlying Hilbert space and consider the case $H = L^2(R_+, \mathcal{B}, \mu)$, where $\mathcal{B}$ denotes the Borel σ-algebra on R_+ and μ is the Lebesgue measure.

In this case any element of the n^{th}-order Wiener chaos can be represented as an Ito integral of a square integrable function.

More precisely, let us consider the subspace Δ^n of R_+^n defined by

$$\Delta^n = \{(s, s_1, \cdots, s_{n-1}) \in R_+^n; 0 \leq s_{n-1} \leq \cdots \leq s_1 \leq s \leq \infty\}. \tag{12.22}$$

Also, let the function $\phi_n : R_+^n \to R$, be square integrable in the sense that

$$\int_0^{\infty} \int_0^{s} \cdots \int_0^{s_{n-1}} \phi_n^2(s, s_1, \cdots, s_{n-1}) ds_{n-1} \cdots ds_1 ds < \infty. \tag{12.23}$$

Then if we let W_t denote a one-dimensional Brownian motion, we can verify that the random variable $I_n(\phi_n)$ defined by the multiple Ito integral

$$I_n(\phi_n) = \int_0^{\infty} \int_0^{s} \cdots \int_0^{s_{n-1}} \phi_n(s, s_1, \cdots, s_{n-1}) dW_{s_{n-1}} \cdots dW_{s_1} dW_s \tag{12.24}$$

is an element of the n^{th} Wiener chaos subspace $\mathcal{H}_n$.

Indeed, the integral on the right hand side of the equation above is an Ito integral on Δ^n since the integrand is adapted and square integrable.

Now let us write $\mathcal{F}_{\infty}^W$ for the σ-field generated by W_t over the totality of the infinite time horizon.

By combining expression (12.24) with the decomposition (12.21), one is led to the result that any square integrable random variable $X \in L^2(\Omega, \mathcal{F}^W_\infty, P)$ can be expressed as a chaos expansion according to the scheme

$$X = \sum_{n=0}^{\infty} I_n(\phi_n), \tag{12.25}$$

where the deterministic functions $\phi_n \in L^2(R^n_+)$ are uniquely determined by the random variable X (see, e.g., Revuz and Yor 2001).

Inner product formulae for $L^2(\Pi)$

It is a straightforward exercise to verify explicitly by use of the Ito isometry and the stochastic Fubini theorem (interchange of integration and expectation) that elements of distinct chaos spaces are orthogonal.

For example, if $X \in \mathcal{H}_1$, and $Y \in \mathcal{H}_2$ we have

$$X = \int_0^\infty \phi(s) dW_s, \qquad \text{and} \qquad Y = \int_0^\infty \int_0^s \phi(s, s_1) dW_{s_1} dW_s, \tag{12.26}$$

for some choice of $\phi(s) \in L^2(R^1_+)$ and $\phi(s, s_1) \in L^2(R^2_+)$, and thus

$$\begin{aligned} \mathbb{E}[XY] &= \mathbb{E}\left[\int_0^\infty \phi(s) dW_s \int_0^\infty \int_0^s \phi(s, s_1) dW_{s_1} dW_s\right] \\ &= \mathbb{E}\left[\int_0^\infty \int_0^s \phi(s)\phi(s, s_1) dW_{s_1} ds\right] \\ &= \int_0^\infty \mathbb{E}\left[\int_0^s \phi(s)\phi(s, s_1) dW_{s_1}\right] ds \\ &= 0. \end{aligned} \tag{12.27}$$

On the other hand, if $A, B \in \mathcal{H}_2$ are two elements of the same chaos, e.g.,

$$A = \int_0^\infty \int_0^s \alpha(s, s_1) dW_{s_1} dW_s, \qquad B = \int_0^\infty \int_0^s \beta(s, s_1) dW_{s_1} dW_s, \tag{12.28}$$

then their inner product is given by

$$\begin{aligned}\mathbb{E}[AB] &= \mathbb{E}\left[\int_0^\infty \int_0^s \alpha(s,s_1)dW_{s_1}dW_s \int_0^\infty \int_0^s \beta(s,s_1)dW_{s_1}dW_s\right] \\ &= \mathbb{E}\left[\int_0^\infty \left(\int_0^s \alpha(s,s_1)dW_{s_1} \int_0^s \beta(s,s_1)dW_{s_1}\right) ds\right] \\ &= \int_0^\infty \mathbb{E}\left[\int_0^s \alpha(s,s_1)dW_{s_1} \int_0^s \beta(s,s_1)dW_{s_1}\right] ds \\ &= \int_0^\infty \int_0^s \alpha(s,s_1)\beta(s,s_1)ds_1 ds. \qquad (12.29)\end{aligned}$$

Thus the random variables A and B are orthogonal in $\mathcal{H}_2$ if and only if the corresponding elements of $L^2(R^2_+)$ are orthogonal.

Factorisable chaos elements

Another useful result arises in the case for which $\phi_n(t_1, t_2, \cdots, t_n)$ is "factorisable" in the special form

$$\phi_n(s, s_1, \cdots, s_{n-1}) = h(s)h(s_1)\cdots h(s_{n-1}), \qquad (12.30)$$

for some element $h(t) \in L^2(R^1_+)$ with unit norm.

Then for this choice of ϕ_n we have the relation $I_n(\phi_n) = H_n(W_h)$, where $H_n(W_h)$ is the n^{th} Hermite polynomial formed from the unit-norm Gaussian random variable W_h defined by

$$W_h = \int_0^\infty h(s)dW_s, \qquad \int_0^\infty h^2(s)ds = 1. \qquad (12.31)$$

We note, in particular, that

$$\exp\left[\alpha W_h - \frac{1}{2}\alpha^2\right] = \sum_{n=0}^{\infty} \alpha^n H_n(W_h). \qquad (12.32)$$

The formulae presented in this section apply in the case of the Wiener chaos based on a standard one-dimensional Brownian motion.

The extension to the general case of a multidimensional Brownian motion is straightforward, and consists of replacing the deterministic coefficients ϕ_s, ϕ_{ss_1}, $\phi_{ss_1s_2}$, etc., with tensorial expressions of the form $\phi^{\alpha}(s)$, $\phi^{\alpha\alpha_1}(s, s_1)$, $\phi^{\alpha\alpha_1\alpha_2}(s, s_1, s_2)$, and so on.

12.4 First chaos models

Now we proceed to consider in more detail the structure and classification of interest rate models according to the scheme outlined in the previous sections.

The first Wiener chaos offers the simplest application of the method and gives rise to a deterministic interest rate model.

One should remember that the majority of the applications of interest rate theory start from the deterministic case, so this case should not be regarded as trivial.

Indeed, the chaos framework offers new insights into the relation between deterministic models and their stochastic generalisations.

It is interesting to note in this connection that even in the case of a deterministic interest rate model there is still a random variable underpinning the dynamics.

For simplicity we shall assume that the dimension of the Brownian motion is one.

In the case of a first chaos model we then write

$$X_\infty = \int_0^\infty \phi_s dW_s, \tag{12.33}$$

where ϕ_s is a deterministic function of one variable.

A straightforward calculation by use of the Ito isometry confirms that the corresponding expression for the potential is given by

$$V_t = \int_t^\infty \phi_s^2 ds. \tag{12.34}$$

This is clearly a positive supermartingale that tends to zero in expectation, and it is evident that the interest rate model that arises is deterministic.

The corresponding expression for the discount bonds is

$$P_{tT} = \frac{\int_T^\infty \phi_s^2 ds}{\int_t^\infty \phi_s^2 ds}. \tag{12.35}$$

Thus, the first chaos is sufficient to characterise a deterministic interest rate structure.

In other words, we can identify the space of positive interest yield curves with the first chaos.

As a simple example, suppose we take

$$\phi_s = \sqrt{R}\mathrm{e}^{-\frac{1}{2}Rs} \tag{12.36}$$

for the first chaos expansion.

Then the associated discount bond becomes

$$P_{tT} = \mathrm{e}^{-R(T-t)}. \tag{12.37}$$

We remark that there is a direct link between the chaos structure presented here and the applications of information geometry considered earlier in our discussion of the space of admissible yield curves.

Chapter 13

Second chaos models. Factorisable second chaos models. Foreign exchange systems.

13.1 Second chaos models

The second chaos models are the simplest models that introduce stochasticity.

In a single-factor second chaos model the random variable X_∞ can be represented in the form

$$X_\infty = \int_0^\infty \sigma_s dW_s, \tag{13.1}$$

with the adapted process σ_s given by

$$\sigma_s = \phi_s + \int_0^s \phi_{ss_1} dW_{s_1}. \tag{13.2}$$

Here $\phi_s = \phi(s)$ is a deterministic function of one variable, and $\phi_{ss_1} = \phi(s, s_1)$ is a deterministic function of two variables.

The second chaos representation for X_∞ is then given by

$$X_\infty = \int_0^\infty \phi_s dW_s + \int_0^\infty \int_0^s \phi_{ss_1} dW_{s_1} dW_s. \tag{13.3}$$

In the case of a second chaos model we can think of the deterministic coefficients $\phi(s)$ and $\phi(s, s_1)$ as supplying just enough freedom to allow for calibration to the initial yield curve and a complete set of caplet prices for all tenors and maturities.

It is a straightforward exercise to show as a consequence of equation

$$V_t = \mathbb{E}_t \left[(X_\infty - \mathbb{E}_t [X_\infty])^2 \right], \tag{13.4}$$

that we are then led to the following expression for the state price density:

$$V_t = \int_t^\infty \left(\phi_s + \int_0^t \phi_{ss_1} dW_{s_1} \right)^2 ds + \int_t^\infty \int_t^s \phi_{ss_1}^2 ds_1 ds. \tag{13.5}$$

The derivation of formula (13.5) can be established most directly if we write

$$V_t = \int_t^\infty M_{ts} ds, \tag{13.6}$$

where the positive martingale family M_{ts} is defined for $0 \le t \le s \le \infty$ by the relation

$$M_{ts} = \mathbb{E}_t \left[\sigma_s^2 \right]. \tag{13.7}$$

The fact that V_t can be represented in this way follows as a consequence of

$$V_t = \mathbb{E}_t \left[\int_t^\infty \sigma_s^2 ds \right]. \tag{13.8}$$

Then a short calculation making use of the relation (13.2) and the conditional Ito isometry gives

$$M_{ts} = \left(\phi_s + \int_0^t \phi_{ss_1} dW_{s_1} \right)^2 + \int_t^s \phi_{ss_1}^2 ds_1. \tag{13.9}$$

To check that the expression appearing on the right hand side of (13.9) is indeed a martingale we note that

$$M_{ts} = R_{ts}^2 - Q_{ts} + Q_{ss}, \tag{13.10}$$

where, for each value of s, R_{ts} is the martingale

$$R_{ts} = \phi_s + \int_0^t \phi_{ss_1} dW_{s_1} \tag{13.11}$$

and Q_{ts} is the associated quadratic variation:

$$Q_{ts} = \int_0^t \phi_{ss_1}^2 ds_1. \tag{13.12}$$

If R_{ts} is a martingale and Q_{ts} is its quadratic variation, then $R_{ts}^2 - Q_{ts}$ is also a martingale, and hence so is M_{ts} since Q_{ss} is deterministic and independent of t.

On the other hand Q_{ss} is just the extra term required to ensure M_{ts} is positive for all $0 \le t \le s \le \infty$, as is clear from expression (13.9).

The discount bond system can then be put into the Flesaker-Hughston form

$$P_{tT} = \frac{\int_T^\infty M_{ts} ds}{\int_t^\infty M_{ts} ds}, \tag{13.13}$$

and the initial term structure that corresponds to this system is given by

$$P_{0T} = \frac{\int_T^\infty M_{0s} ds}{\int_0^\infty M_{0s} ds}. \tag{13.14}$$

More explicitly, we have $M_{0s} = \phi_s^2 + \int_0^s \phi_{ss_1}^2 ds_1$ and hence:

$$P_{0T} = \frac{\int_T^\infty \left(\phi_s^2 + \int_0^s \phi_{ss_1}^2 ds_1\right) ds}{\int_0^\infty \left(\phi_s^2 + \int_0^s \phi_{ss_1}^2 ds_1\right) ds}. \tag{13.15}$$

Clearly, by an overall adjustment of the scale of X_∞ we can set the denominator in (13.15) to unity.

With this choice of normalisation the corresponding term structure density is given by $\rho(T) = M_{0T}$.

Expressions for the discount bond volatility and the market price of risk arising in the case of a general second chaos model

Making use of the Ito quotient identity

$$\frac{d\,(A_t/B_t)}{(A_t/B_t)} = \frac{dA_t}{A_t} - \frac{dB_t}{B_t} + \frac{(dB_t)^2}{B_t^2} - \frac{dA_t dB_t}{A_t B_t}, \tag{13.16}$$

we deduce that the discount bond volatility is given by

$$\Omega_{tT} = \frac{\int_T^\infty U_{ts} ds}{\int_T^\infty M_{ts} ds} - \frac{\int_t^\infty U_{ts} ds}{\int_t^\infty M_{ts} ds}, \tag{13.17}$$

and that the market risk premium vector is given by

$$\lambda_t = -\frac{\int_t^\infty U_{ts} ds}{\int_t^\infty M_{ts} ds}. \tag{13.18}$$

Here for convenience we have introduced the vector-valued process U_{ts} defined by $U_{ts} = 2R_{ts}\phi_{st}$. We note that the constraint $\Omega_{TT} = 0$ is automatically satisfied.

The instantaneous forward rate process f_{tT} can be calculated by use of the formula $f_{tT} = -\partial_T \ln P_{tT}$ and we find

$$f_{tT} = \frac{M_{tT}}{\int_T^\infty M_{ts} ds}. \tag{13.19}$$

The short rate process is given analogously by the formula

$$r_t = \frac{M_{tt}}{\int_t^\infty M_{ts} ds}, \tag{13.20}$$

which is equivalent to the relation $\sigma_t^2 = r_t V_t$.

At first glance, the expressions related to the second chaos might look complicated.

However the only exogenously specified ingredients are the deterministic functions ϕ_s and ϕ_{ss_1}.

In fact, all the formulae above can be expressed in terms of the underlying Gaussian random variables R_{ts}.

Option pricing in a second chaos model

We observe that for fixed values of t and s the random variable M_{ts} defined by (13.9) is given by the square of a Gaussian random variable, plus a constant.

Therefore, for fixed t and T the random variable

$$Z_{tT} = \int_T^\infty M_{ts} ds, \tag{13.21}$$

can be understood as the integral of a parametric family of squared Gaussian random variables, plus a constant.

The next step is to define the joint distribution function of the random variables Z_{tT_1}, and Z_{tT_2} by

$$F_{tT_1T_2}(x, y) = \text{Prob}\left[Z_{tT_1} \leq x \text{ and } Z_{tT_2} \leq y\right]. \tag{13.22}$$

We denote the corresponding joint density function by $f_{tT_1T_2}(x, y)$.

Now the payoff for a call option that expires at time t and is written on a T-maturity discount bond is

$$H_t = (P_{tT} - K)^+, \tag{13.23}$$

for some strike K.

Therefore, according to (11.33) the price of this instrument is

$$H_0 = E\left[V_t\left(P_{tT} - K\right)^+\right]. \tag{13.24}$$

By virtue of (13.13) this is evidently equivalent to

$$H_0 = E\left[\left(Z_{tT} - KZ_{tt}\right)^+\right], \tag{13.25}$$

which can be written in terms of the density function $f(x, y)$ in the form

$$H_0 = \int_0^\infty \int_0^\infty f(x, y)\,(x - Ky)^+\,dxdy. \tag{13.26}$$

Analogous formulae can be derived for other types of options.

13.2 Factorisable second chaos models

A considerable simplification can be achieved when the second chaos coefficient ϕ_{ss_1} separates, that is to say, when ϕ_{ss_1} can be written as a finite sum of products of functions of one variable.

In this situation we obtain a model characterised by *a finite set of state variables*.

We shall examine in some detail the case where there is a single such term, and set

$$\phi_s = \alpha_s \tag{13.27}$$

and

$$\phi_{ss_1} = \beta_s \gamma_{s_1}, \tag{13.28}$$

where α_s, β_s and γ_{s_1} are deterministic functions of one variable.

The resulting "factorisable" second chaos model then depends on a single state variable. This model is completely tractable in the sense that it leads to closed-form expressions both for bond prices and various types of options on bond prices, which we discuss at greater length below.

First we observe that in the factorisable case we have

$$\phi_s + \int_0^t \phi_{ss_1} dW_{s_1} = \alpha_s + \beta_s R_t, \tag{13.29}$$

where the Gaussian martingale R_t is defined by

$$R_t = \int_0^t \gamma_{s_1} dW_{s_1}. \tag{13.30}$$

At any given time t, the random variable R_t is the sole state variable that characterises the interest rate system in this model.

If we define the corresponding quadratic variation process Q_t by

$$Q_t = \int_0^t \gamma_s^2 ds, \tag{13.31}$$

then it follows that the process $R_t^2 - Q_t$ is also a martingale, and the positive martingale family M_{ts} defined by (13.9) reduces to expression

$$M_{ts} = \alpha_s^2 + \beta_s^2 Q_s + 2\alpha_s\beta_s R_t + \beta_s^2 \left(R_t^2 - Q_t\right). \tag{13.32}$$

Clearly, $Q_s \geq Q_t$ for all $s \geq t$, so $M_{ts} > 0$ for all values of R_t.

For the integral of M_{ts} we can write

$$\int_T^\infty M_{ts} ds = A_T + B_T R_t + C_T \left(R_t^2 - Q_t\right), \tag{13.33}$$

where for convenience in what follows we define the following processes:

$$\begin{aligned} A_t &= \int_t^\infty \left(\alpha_s^2 + \beta_s^2 Q_s\right) ds, \\ B_t &= 2\int_t^\infty \alpha_s\beta_s ds, \\ C_t &= \int_t^\infty \beta_s^2 ds. \end{aligned} \tag{13.34}$$

Setting $T = t$ in (13.33) we see that the state price density is given by

$$V_t = A_t + B_t R_t + C_t \left(R_t^2 - Q_t\right), \tag{13.35}$$

and thus that the discount bond price can be written as the ratio of a pair of quadratic polynomials in the state variable R_t :

$$P_{tT} = \frac{A_T + B_T R_t + C_T \left(R_t^2 - Q_t\right)}{A_t + B_t R_t + C_t \left(R_t^2 - Q_t\right)}. \tag{13.36}$$

Given these expressions, it is then a straightforward exercise to work out formulae for the bond volatility, the market price of risk, the short rate, and the instantaneous forward rates, all of which depend upon R_t.

Because R_t is a Gaussian martingale, it is in principle straightforward to simulate the dynamical trajectories of these quantities.

Valuation of options in second chaos models

The present value H_0 of a European-style call option with strike K exercisable at time t on a discount bond with maturity T is given by

$$H_0 = \mathbb{E}\left[V_t\left(P_{tT} - K\right)^+\right]. \tag{13.37}$$

Now clearly, according to

$$V_t = A_t + B_t R_t + C_t\left(R_t^2 - Q_t\right), \tag{13.38}$$

and

$$P_{tT} = \frac{A_T + B_T R_t + C_T\left(R_t^2 - Q_t\right)}{A_t + B_t R_t + C_t\left(R_t^2 - Q_t\right)}, \tag{13.39}$$

we have

$$\begin{aligned} V_t\left(P_{tT} - K\right) &= \left(A_T - KA_t\right) - \left(C_T - KC_t\right)Q_t \\ &+ \left(B_T - KB_t\right)R_t + \left(C_T - KC_t\right)R_t^2. \end{aligned} \tag{13.40}$$

To proceed let us therefore now fix t, T and K, and introduce the standard normally distributed random variable $Z = R_t/\sqrt{Q_t}$.

Then (13.40) above can be written in the form

$$V_t(P_{tT} - K) = A + BZ + CZ^2. \tag{13.41}$$

Here the quantities A, B and C are defined by:

$$\begin{aligned} A &= (A_T - KA_t) - (C_T - KC_t)Q_t, \\ B &= (B_T - KB_t)Q_t^{1/2}, \\ C &= (C_T - KC_t)Q_t. \end{aligned} \tag{13.42}$$

Therefore if we construct the polynomial $\mathcal{P}(z) = A + Bz + Cz^2$, we see that the value of the call option is given by

$$H_0 = \frac{1}{\sqrt{2\pi}}\int_{\mathcal{P}(z)\geq 0}\mathcal{P}(z)e^{-\frac{1}{2}z^2}dz, \tag{13.43}$$

which by an analysis of the roots of $\mathcal{P}(z)$ can be reduced to a simple explicit expression involving the normal distribution function and its density.

Analogous formulae can then be deduced for various other types of options, as we shall indicate shortly.

Explicit formulae for options on discount bonds

Let us proceed then case by case to examine the behaviour of the polynomial $\mathcal{P}(z)$ more closely.

First we distinguish the cases $C = 0$ and $C \neq 0$. If $C = 0$ then $\mathcal{P}(z)$ is linear, and for the value of the call option we obtain

$$H_0 = AN(-z_0) + B\rho(z_0) \tag{13.44}$$

when $B > 0$, and

$$H_0 = AN(z_0) - B\rho(z_0) \tag{13.45}$$

when $B < 0$. Here $z_0 = -A/B$ is the single root of $\mathcal{P}(z)$, $N(z)$ is the standard normal distribution function, and $\rho(z)$ is the standard normal density function.

If $C \neq 0$, then we need to consider the sign of the discriminant $\Delta = B^2 - 4AC$.

If $\Delta \leq 0$ then for $C > 0$ the option is guaranteed to expire in the money, and we have $H_0 = P_{0T} - KP_{0t}$.

If $C < 0$ then the option will expire out of the money and $H_0 = 0$.

If $\Delta > 0$ then, again, we have to consider the cases $C > 0$ and $C < 0$.

Let us write

$$z_1 = \frac{-B - \sqrt{\Delta}}{2C}, \qquad z_2 = \frac{-B + \sqrt{\Delta}}{2C} \tag{13.46}$$

for the roots of $\mathcal{P}(z)$. Then if $C > 0$ we obtain

$$\begin{aligned} H_0 &= (P_{0T} - KP_{0t})\left(N(z_1) + N(-z_2)\right) \\ &- \frac{1}{2}\left(B - \sqrt{\Delta}\right)\rho(z_1) + \frac{1}{2}\left(B + \sqrt{\Delta}\right)\rho(z_2), \end{aligned} \tag{13.47}$$

and if $C < 0$ we obtain

$$\begin{aligned} H_0 &= (P_{0T} - KP_{0t})\left(N(z_1) - N(z_2)\right) \\ &- \frac{1}{2}\left(B - \sqrt{\Delta}\right)\rho(z_1) + \frac{1}{2}\left(B + \sqrt{\Delta}\right)\rho(z_2). \end{aligned} \tag{13.48}$$

Thus we see that in the factorisable second-chaos framework the pricing of options on discount bonds is completely tractable.

More generally, the value of an option on any predesignated set of deterministic cash-flows is also tractable, for example an option on a coupon bond.

To obtain the above formulae, we have set $A_0 = 1$. This can be achieved without loss of generality by changing the scale of X_∞.

Valuation of swaptions

Now we shall demonstrate that in the factorisable second-chaos framework we can also derive explicit results for a swaption that pays $(S_{tn} - K)^+$ at a series of future dates T_i, for some strike K, where $i = 1, \cdots, n$, and S_{tn} is the swap rate

$$S_{tn} = \frac{1 - P_{tT_n}}{\sum_{i=1}^{n} P_{tT_i}}. \tag{13.49}$$

The effective payoff at expiry t is therefore equal to

$$H_t = \left(1 - P_{tT_n} - K \sum_{i=1}^{n} P_{tT_i}\right)^+, \tag{13.50}$$

and the price for this instrument at present is

$$H_0 = \mathbb{E}\left[V_t \left(1 - P_{tT_n} - K \sum_{i=1}^{n} P_{tT_i}\right)^+\right]. \tag{13.51}$$

The analysis turns out to be quite similar to the bond option case.

In the case of a swaption we define the quantities

$$\begin{aligned} A^* &= \left(A_t - A_{T_n} - K \sum_{i=1}^{n} A_{T_i}\right) - \left(C_t - C_T - K \sum_{i=1}^{n} C_{T_i}\right) Q_t \\ B^* &= \left(B_t - B_{T_n} - K \sum_{i=1}^{n} B_{T_i}\right) Q_t^{1/2} \\ C^* &= \left(C_t - C_{T_n} - K \sum_{i=1}^{n} C_{T_i}\right) Q_t. \end{aligned} \tag{13.52}$$

for fixed t and T_i.

The value of the swaption is then given by

$$H_0 = \frac{1}{\sqrt{2\pi}} \int_{\mathcal{P}(z) \geq 0} \mathcal{P}(z) e^{-\frac{1}{2}z^2} dz, \tag{13.53}$$

where in the present case the polynomial $\mathcal{P}(z)$ is given by $\mathcal{P}(z) = A^* + B^*z + C^*z^2$.

When $C^* = 0$ we have

$$H_0^* = A^*N(-z_0^*) + B^*\rho(z_0^*) \tag{13.54}$$

for $B^* > 0$, and

$$H_0^* = A^*N(z_0^*) - B^*\rho(z_0^*) \tag{13.55}$$

for $B^* < 0$. Here $z_0^* = -A^*/B^*$.

When $C^* \neq 0$ then we have to consider the discriminant $\Delta^* = B^{*2} - 4A^*C^*$.

For $\Delta^* \leq 0$ we have that, for $C^* > 0$ the contract is guaranteed to pay off and the value at present is $H_0^* = P_{0t} - P_{0T_n} - K\sum_{i=1}^n P_{0T_i}$.

On the other hand in the case that $C^* < 0$ the contract will expire worthless and $H_0^* = 0$.

Finally, when $\Delta^* > 0$ we define the two roots of $\mathcal{P}(z)$ by

$$z_1^* = \frac{-B^* - \sqrt{\Delta^*}}{2C^*}, \qquad z_2^* = \frac{-B^* + \sqrt{\Delta}}{2C^*}. \tag{13.56}$$

The value of the swaption contract is then given by

$$\begin{aligned} H_0^* &= \left(P_{0t} - P_{0T_n} - K\sum_{i=1}^n P_{0T_i}\right)\left(N(z_1^*) + N(-z_2^*)\right) \\ &- \frac{1}{2}\left(B^* - \sqrt{\Delta^*}\right)\rho(z_1^*) + \frac{1}{2}\left(B^* + \sqrt{\Delta^*}\right)\rho(z_2^*), \end{aligned} \tag{13.57}$$

when $C^* > 0$; whereas if $C^* < 0$ we get

$$\begin{aligned} H_0^* &= \left(P_{0t} - P_{0T_n} - K\sum_{i=1}^n P_{0T_i}\right)\left(N(z_1^*) - N(z_2^*)\right) \\ &- \frac{1}{2}\left(B^* - \sqrt{\Delta^*}\right)\rho(z_1^*) + \frac{1}{2}\left(B^* + \sqrt{\Delta^*}\right)\rho(z_2^*). \end{aligned} \tag{13.58}$$

It is a remarkable feature of the factorisable second chaos models that they admit tractable closed-form expressions for both options and swaptions.

13.3 Foreign exchange systems

In conclusion we consider how the framework presented here generalises to the situation where there is a foreign exchange system, with a family of discount bonds associated to each currency.

It will be demonstrated that a chaotic representation exists for the entirety of such an international system of interest rates and foreign exchange.

As a byproduct of this result, we are also led to a simple class of stochastic volatility models for general asset pricing dynamics.

For convenience we shall write S_t^{ij} for the price of one unit of currency i in units of currency j.

Here $i, j = 0, 1, \cdots, N$, and we may think of the case $i = 0$ as referring to the particular base currency with respect to which the axioms (A1), (A2), and (A3) are framed.

In fact, there is ultimately no special significance to the choice of base currency: the entire system is symmetrical in the ensemble of currencies.

We shall assume in the present investigation, as before, that the foreign exchange market is "frictionless" in the sense that

$$S_t^{ij} S_t^{jk} = S_t^{ik} \tag{13.59}$$

for all i, j, k.

Let us write B_t^i for the value in units of currency i of a money-market account in that currency, initialised to one unit of currency i.

We assume that for each currency there exists a strictly increasing money-market asset, with a corresponding strictly positive short rate process r_t^i such that

$$B_t^i = B_0^i \exp\left(\int_0^t r_s^i ds\right). \tag{13.60}$$

Constant value assets

We also assume the existence of a floating rate note in each currency.

That is to say, for each i we assume the existence of an asset of constant value in units of currency i, paying a dividend at the rate r_t^i.

Derivative of the exchange rate process as a ratio system

Writing S_t^{i0} for the value of one unit of currency i in units of the base currency, we see that the product $S_t^{i0}B_t^i$ represents the base-currency price of a non-dividend paying asset.

Therefore by axiom (A2) we deduce for each value of i that

$$M_t^i = \frac{S_t^{i0}B_t^i}{\xi_t} \qquad (13.61)$$

is a martingale, from which it follows that the process V_t^i defined by

$$V_t^i = \frac{S_t^{i0}}{\xi_t}, \qquad (13.62)$$

is a supermartingale.

Since $S_t^{ij}S_t^{j0} = S_t^{i0}$ for all i, j, we thus deduce that

$$S_t^{ij} = \frac{V_t^i}{V_t^j}. \qquad (13.63)$$

This gives us a general expression for the exchange-rate process as a *ratio of supermartingales*.

As a consequence we deduce that the dynamics of S_t^{ij} are given by

$$\frac{dS_t^{ij}}{S_t^{ij}} = \left[r_t^j - r_t^i + \lambda_t^j\left(\lambda_t^j - \lambda_t^i\right)\right]dt + \left(\lambda_t^j - \lambda_t^i\right)dW_t, \qquad (13.64)$$

where λ_t^i is the market price of risk process associated with assets that are denominated in currency i.

The derivation of (13.64) follows directly from the relation

$$dV_t^i = -r_t^iV_t^i dt - \lambda_t^iV_t^i dW_t \qquad (13.65)$$

together with the Ito quotient rule.

It is interesting to note that in the general arbitrage-free exchange rate dynamics the FX volatility is completely determined by the associated market price of risk processes.

Foreign discount bonds

Let us consider the discount bond system for foreign currency number i. We denote by P_{tT}^i the value at time t of a bond that pays one unit of currency i at time T.

In this case $S_t^{i0} P_{tT}^i$ is the base-currency price of a non-dividend paying asset, and therefore $S_t^{i0} P_{tT}^i / \xi_t$ is a martingale by (A2).

It follows that $S_t^{i0} P_{tT}^i / \xi_t = \mathbb{E}_t \left[S_T^{i0} P_{TT}^i / \xi_T \right]$.

Thus, from $S_t^{i0}/\xi_t = V_t^i$ and $P_{TT}^i = 1$, we deduce from this line of argument that

$$P_{tT}^i = \frac{\mathbb{E}_t \left[V_T^i \right]}{V_t^i}. \tag{13.66}$$

Asymptotic behaviour

Now we make the additional assumption that $\lim_{T \to \infty} P_{0T}^i = 0$ for all i.

It follows that a conditional variance representation exists for the state-price density associated with each currency.

In other words, there exists a set of random variables $X_\infty^i \in L^2(\Omega, \mathcal{F}, P)$ for $i = 0, 1, \cdots, N$ such that

$$V_t^i = \mathbb{E}_t \left[\left(X_\infty^i - \mathbb{E}_t \left[X_\infty^i \right] \right)^2 \right]. \tag{13.67}$$

These random variables then each admit a chaos representation in terms of the vector Wiener process W_t^α $(\alpha = 1, \cdots, k)$.

We see that once the random variables X_∞^i have been specified for $i = 0, 1, \cdots, N$ then the international system of interest and foreign exchange is completely determined by the relations

$$S_t^{ij} = \frac{V_t^i}{V_t^j}, \tag{13.68}$$

$$P_{tT}^i = \frac{\mathbb{E}_t \left[V_T^i \right]}{V_t^i} \tag{13.69}$$

and

$$V_t^i = \mathbb{E}_t \left[\left(X_\infty^i - \mathbb{E}_t \left[X_\infty^i \right] \right)^2 \right]. \tag{13.70}$$

We can refer to the random variables X_∞^i as the generators of the corresponding interest rate and foreign exchange system.

It should be evident that although we have consistently used the language of foreign exchange in the discussion above, the matrix process S_t^{ij} can be used to characterise the price of any asset in terms of another, providing that these prices are always positive and that we interpret the associated short-rate systems as continuous dividend streams.

As a consequence we see that the generic model for such a “basic” asset price is a process of the form

$$S_t = \frac{\mathbb{E}_t\left[(Y_\infty - \mathbb{E}_t\left[Y_\infty\right])^2\right]}{\mathbb{E}_t\left[(X_\infty - \mathbb{E}_t\left[X_\infty\right])^2\right]}, \tag{13.71}$$

that is, a ratio of conditional variances, where X_∞ and Y_∞ are elements of $L^2(\Omega, \mathcal{F}, P)$.

For example, if we think of S_t as a dollar-valued share price (and we approximate the dividend flow as continuous–an equity index might work better for that!) then X_∞ carries the information of the dollar risk premium, and the dollar interest rate, whereas Y_∞ carries the information that is more specific to the particular stock.

The simplest models leading to a nontrivial asset price stochasticity are those for which at least one of X_∞ or Y_∞ is an element of the second chaos.

Chapter 14

Real and nominal interest rates. Models for inflation. Valuation of index-linked bonds and other inflation related products. General principles for the design of inflation-linked products.

14.1 Inflation linked bonds*

Now we consider a general model of inflation and inflation-linked derivatives.

The idea is to formulate an approach to the valuation of inflation derivatives that is as close as possible to the methodologies for valuing foreign exchange and interest rate derivatives.

The theory of inflation has aspects that relate to both interest rates and foreign exchange. In particular, a useful way of thinking about inflation is to treat the consumer price index (CPI) as if it were the price of a foreign currency.

We begin by considering an economy consisting of discount bonds and index-linked discount bonds.

The indexing of the index-linked discount bonds is with respect to the consumer price index which at time a has the value C_a.

We think of C_a as representing the value, in units of the domestic currency (henceforth, dollars) of a typical basket of goods and services at that time.

An increase in C_a over an interval of time then indicates that there has been inflation over that period.

We shall define an inflation linked discount bond to be a bond which pays out C_b at the maturity date b. In other words, the inflation linked bond pays out enough in dollars to buy a unit of goods and services at that time.

Our problem is to formulate a general theory for the price processes of the consumer price index and index-linked bonds, and tie this in with the HJM theory of interest rate derivatives.

Indexation is debt is not a new idea. An early example occurs in 1742 when Massachusetts issued bills linked to the price of silver on the London Exchange. The risks in indexation to a single commodity became apparent a few years later when the price of silver rose in excess over general prices.

As a consequence, a law was passed in Massachusetts requiring a wider base of commodities for indexation.

In 1780 notes were issued again, indexed this time with the intention of preserving the value of notes issued as wages to soldiers in the American Revolution.

In this case, both the principal and the interest of the notes were indexed to the combined market value of five bushels of corn, sixty-eight and four-sevenths pounds of beef, ten pounds of sheep wool, and sixteen pounds of sole leather.

14.2 Payout structures for inflation-linked products*

We denote by P^N_{ab} the value of a nominal discount bond at time a with maturity at time b. At maturity the nominal discount bond pays one dollar.

Then a typical inflation-linked derivative has a payout or payouts given by functions of nominal discount bonds (at various times and of various maturities) and the consumer price index (at various times). Some examples are as follows.

(a) *Inflation cap*. This pays out if inflation (as measured by percentage appreciation in the CPI) exceeds a certain threshold K over a given period.

Thus if the period in question is the interval (a, b), then the payout H_b at time b is given by:

$$H_b = X \max\left[\left(\frac{C_b}{C_a} - 1\right) - K, 0\right], \tag{14.1}$$

where X is some dollar notional.

In practice the payout would have to be delayed to some still later date c (to allow for official publication of the relevant CPI figure), so the effective payout is

$$H_b = XP_{bc}^N \max\left[\left(\frac{C_b}{C_a} - 1\right) - K, 0\right]. \tag{14.2}$$

(b) *Inflation swap.* For a succession of intervals (a_i, b_i) $(i = 1, \ldots, n)$ we receive the inflation rate

$$I_{ab} = \frac{C_b}{C_a} - 1 \tag{14.3}$$

for that interval (with payment delayed to some slightly later time c_i), and pay a fixed rate, all on a fixed notional.

(c) *Zero strike floors on inflation.* Here the idea is to protect the receiver of the inflation leg in an inflation swap against a deflation scenario.

Thus instead of simply receiving I_{ab}, which can go negative (deflation), one receives $\max[I_{ab}, 0]$.

(d) *Inflation swaption.* This confers the right to enter into an inflation swap (e.g., as a payer of the fixed rate) at some specified future time, with a given "strike" fixed rate.

(e) *Inflation protected annuity.* This pays a fixed "real" annuity on the future dates a_i:

$$H_{a_i} = \frac{fNC_{a_i}}{C_0}. \tag{14.4}$$

Here f is the nominal annuity rate (e.g., 5%), N is the notional.

The effect of the CPI is to inflate the actual payment appropriately.

(f) *Knockout option.* A typical structure, for example, might pay if the total inflation exceeds a certain threshold K at time T.

Knockout would occur if the total inflation drops below a certain specified critical level K' between time t and T.

$$H_T = N \max\left[\left(\frac{C_T}{C_a} - 1\right) - K, 0\right] \text{ unless } \left(\frac{C_T}{C_a} - 1\right) - K' \leq 0 \text{ at some time } a$$
$$\text{in the interval } t \leq a \leq T, \text{ in which case } H_T = 0. \tag{14.5}$$

There are many variations on this kind of structure.

The basic idea is to make the option premium cheaper by having the contract specify a cancelling of the structure in the event of certain circumstances.

(g) *Cap on "real" interest rates.* This might, for example, pay off

$$H_b = X \max[L^R_{ab} - K, 0]. \qquad (14.6)$$

Real rates are not necessarily available as a basis for contract specification. Instead we can use a proxy.

(h) *Proxy cap on "real" interest rates.* This instead would pay

$$H_b = X \max[L^N_{ab} - I_{ab} - K, 0], \qquad (14.7)$$

where L_{ab} is the relevant per-period Libor rate.

Then if the Libor rate exceeds the inflation rate over the given interval by more than a specified amount, there is a payoff.

Here we have used the difference between the Libor rate and the inflation rate as a convenient proxy for the "real" interest rate over the given interval.

Clearly, more "exotic" structures can also easily be represented. Analogues both from the FX world (treating C_a as a foreign exchange rate), and the interest rate world (treating I_{ab} as a kind of "rate") can be formulated.

14.3 General theory of inflation*

There are three ingredients: the "nominal" discount bonds P^N_{ab}, the "real" discount bonds P^R_{ab}, and the consumer price index C_a.

The real discount bonds are defined as follows.

By P^R_{ab} we mean intuitively the price at time a, in units of goods and services, for one unit of goods and services to be delivered at time b.

Thus P^R_{ab} is the discount function that characterises "real" interest rates. If we lived in a pure barter economy, with no money, then P^R_{ab} would define the term structure of interest rates.

For example, if the price of bread happened to be a good proxy for goods and services in general, then one "unit" of goods and services could be represented by 100 loaves of bread.

The real term structure of interest rates would then supply information like how many loaves of bread you should in principle be willing to part with today in exchange for a sure delivery of 100 loaves one year from now.

The answer might be, say, 97 loaves, and that enables us to define the one-year real interest rate.

Associated with the system of real discount bonds we have a corresponding system of real interest rates. We denote a typical real rate with the notation L^R_{ab}.

The index-linked discount bonds are related to the real discount bonds by the consumer price index, which acts as a kind of exchange rate.

In other words, if we multiply the P^R_{ab} by C_a, that gives us the dollar value of the b-maturity real discount bond at time a.

In the foreign exchange analogy, we think of the nominal (dollar) discount bonds as the "domestic" bonds. We think of the real discount bonds as "foreign" discount bonds, and the CPI plays the role of the exchange rate.

Note that the actual inflation rate I_{ab} for the period (a, b) is not strictly analogous to an interest rate in the usual sense – it is only known at time b (or later!).

It is thus best thought of as an appreciation in an asset price.

But in that case what is the relation between "real" rates, "nominal" rates, and "inflation" rates?

Clearly care is required, and we must not confuse categories just because these are all loosely referred to as "rates".

Part of the goal is to gain some insight into the relation between these various "rates".

14.4 Price processes for nominal discount bonds*

As usual in an HJM type framework, we assume an economy where uncertainty in the future is modelled by a multi-dimensional Brownian motion defined with respect to the natural probability measure.

Assuming no arbitrage, and thus the existence of a risk premium vector, we can write the dynamics for the price processes of the nominal discount bonds in the form

$$\frac{dP^N_{ab}}{P^N_{ab}} = (r^N_a + \lambda^N_a \Omega^N_{ab})\, da + \Omega^N_{ab}\, dW_a. \tag{14.8}$$

Here r^N_a is the nominal short rate, λ^N_a is the nominal risk premium vector, Ω^N_{ab} is the nominal vector volatility, and W_a is the Brownian motion vector.

By analogy, for the real discount bonds we have

$$\frac{dP^R_{ab}}{P^R_{ab}} = (r^R_a + \lambda^R_a \Omega^R_{ab})\, da + \Omega^R_{ab}\, dW_a. \tag{14.9}$$

It then follows by virtue of the foreign exchange analogy that the price dynamics for the consumer price index are

$$\frac{dC_a}{C_a} = [r^N_a - r^R_a + \lambda^N_a (\lambda^N_a - \lambda^R_a)]\, da + (\lambda^N_a - \lambda^R_a)\, dW_a. \tag{14.10}$$

We note that the CPI volatility vector can be expressed as the difference between the nominal and real risk premium vectors.

Thus we can write

$$\frac{dC_a}{C_a} = (r^N_a - r^R_a + \lambda^N_a \nu_a)]\, da + \nu_a\, dW_a, \tag{14.11}$$

where $\nu_a = \lambda^N_a - \lambda^R_a$ is the CPI volatility.

In the absence of a risk premium, we see that the drift of the CPI is given by the difference between the nominal short rate and the real short rate.

In reality, the drift of the CPI contains another term, given by the product of the nominal risk premium vector and the CPI volatility vector.

Thus if by the instantaneous rate of inflation I_a we mean the drift process for the consumer price index, we have:

$$I_a = r^N_a - r^R_a + \lambda^N_a \nu_a. \tag{14.12}$$

This is an expression of the so-called “Fisher equation”, which relates the inflation rate to the nominal interest rate minus the real interest rate plus a risk premium term.

14.5 Transfer to the nominal risk neutral measure*

For the valuation of derivatives we want to introduce a change of measure such that the ratio of any of the nominal bonds P^N_{ab} to the nominal dollar money market account is a martingale.

Suppose we write B^N_a for the nominal money market account, which satisfies

$$dB^N_a = r^N_a B^N_a \, da. \tag{14.13}$$

Then we introduce a new probability measure P^N as usual according to the scheme

$$\mathbb{E}^N_a[X_b] = \frac{\mathbb{E}_a[\Lambda_b X_b]}{\mathbb{E}_a[\Lambda_b]}, \tag{14.14}$$

where $\mathbb{E}^N_a$ denotes conditional expectation with respect to the measure P^N given the filtration up to time a, and where X_b is any random variable adapted to time b.

We call P^N the nominal (or dollar) risk neutral measure.

Here the change of measure density process Λ_a is defined by

$$\Lambda_a = \exp\left(- \int_0^a \lambda^N_s \, dW_s - \frac{1}{2} \int_0^a \left(\lambda^N_s\right)^2 \, ds \right). \tag{14.15}$$

With respect to P^N the process W^N_a defined by

$$dW^N_a = dW_a + \lambda^N_a \, da \tag{14.16}$$

is a Brownian motion.

Then for the processes P^N_{ab} and P^R_{ab} we can write

$$\frac{dP^N_{ab}}{P^N_{ab}} = r^N_a \, da + \Omega^N_{ab} \, dW^N_a \tag{14.17}$$

and

$$\frac{dP^R_{ab}}{P^R_{ab}} = (r^N_a - \nu_a \Omega^R_{ab}) \, da + \Omega^R_{ab} \, dW^N_a. \tag{14.18}$$

We note that the process (14.18) for the real discount bonds picks up a "quanto" term in the drift in the nominal risk neutral measure.

This is appropriate since the real discount bonds are not denominated in dollars.

The process for the consumer price index in the risk neutral measure is:

$$\frac{dC_a}{C_a} = (r_a^N - r_a^R)\, da + \nu_a \, dW_a^N. \tag{14.19}$$

Thus in the risk neutral measure the nominal risk premium term disappears, and we see that the drift on the CPI is given by the difference between the nominal and real interest rates.

The process is like that of a foreign currency, and we can think of the real interest rate as playing the role of the "foreign" interest rate.

Normally we expect r_a^N and r_a^R both to be positive.

There are good economic arguments to support the idea that both nominal and real interest rates should be positive.

We note that by construction the ratio process P_{ab}^N/B_a is a martingale in the nominal risk neutral measure.

So is $C_a P_{ab}^R/B_a$, where $C_a P_{ab}^R$ is the (dollar) value of an index linked discount bond.

14.6 Valuation of inflation linked derivatives*

Now let H_T be a random variable corresponding to the payout of an inflation linked derivative.

We can think of H_T as depending in a general way of the values of nominal discount bonds, and the consumer price index at times between the present and the maturity date T.

There are many examples of inflation linked derivatives for which the payout depends in a direct way only on the nominal discount bonds and the consumer price index, but not on the real discount bonds.

These we shall call "index linked" derivatives, and it should be noted that these structures are in principle more straightforward to value and hedge than inflation linked derivatives, that also involve real interest rates.

The basic derivatives valuation formula is given in the risk neutral valuation scheme by

$$H_0 = \mathbb{E}^N \left[\frac{H_N}{B_T} \right]. \tag{14.20}$$

In particular, we can consider the case where H_T is the payout C_T of an index linked discount bond, normalised by the value of today's CPI. Then we have

$$P^R_{0T} = \mathbb{E}^N \left[\frac{C_T/C_0}{B_T} \right] \tag{14.21}$$

which shows that today's market for index linked bonds tells us the initial real discount function.

In reality, we have to work a bit harder, on account of the lagging effect, and the fact that we generally have to work with coupon bonds.

Finally, by use of the foreign exchange analogy let us consider a simple Black-Scholes type model for the valuation of index derivatives.

Let us assume deterministic interest rates (nominal and real), and a deterministic CPI volatility, with a prescribed local volatility function ν_t. Then for the CPI process we can write:

$$C_t = \frac{C_0 P^R_{0t}}{P^N_{0t}} \exp\left(\int_0^t \nu_s \, dW_s - \frac{1}{2} \int_0^t \nu_s^2 \, ds \right), \tag{14.22}$$

where the expression $C_0 P^R_{0t}/P^N_{0t}$ is the forward value for the CPI.

In this case the situation is entirely analogous to the corresponding problem for foreign exchange, and by use of the Black-Scholes formula we can get a crude valuation for some products in this way, though of course care is required in the case of longer dated structures.

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