Chapter 5

The Term Structure of Interest Rates

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He pays least [...] who pays latest.
—Charles de Montesquieu (1689–1755),
The Spirit of Laws [586]

The term structure of interest rates is concerned with how the interest rates change with maturity and how they may evolve in time. It is fundamental to the valuation of fixed-income securities. This subject is important also because the term structure is the starting point of any stochastic theory of interest rate movements.

5.1 Introduction

The set of yields to maturity for bonds of equal quality, differing solely in their terms to maturity, forms the **term structure**. Often, this term refers exclusively to yields of zero-coupon bonds [565]. Term to maturity is the time period during which the issuer has promised to meet the conditions of the obligation. **Maturity** and **term** are usually used in place of "term to maturity." A **yield curve** plots various yields to maturity against maturity. It thus represents the prevailing interest rates for various terms. Typically, by yield curve is understood as that of the U.S. Treasuries. See Fig. 5.1 for a sample Treasury yield curve. A **par yield curve** is constructed from bonds trading near their par value.

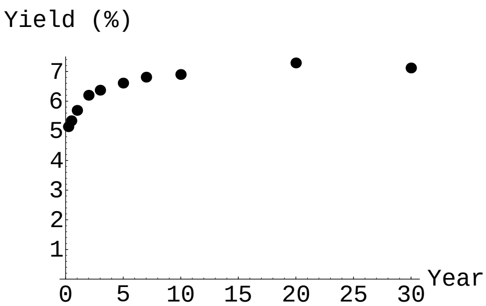

Figure 5.1: TREASURY YIELD CURVE. The Treasury yield curve as of May 3, 1996 published by the U.S. Treasury. It is based on bid quotations on the most actively traded Treasury securities as of 3:30 P.M. with information from the Federal Reserve Bank of New York.

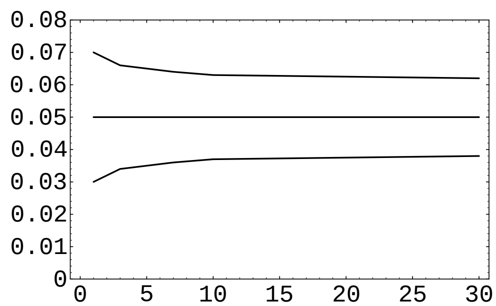

Figure 5.2: THREE TYPES OF YIELD CURVES.

Four yield curve shapes have been identified. A **normal** yield curve is upward sloping, an **inverted** yield curve is downward sloping, and a **flat** yield curve is flat (see Fig. 5.2). Finally, in a **humped** yield curve, the yield is upward sloping at first but then turns downward sloping. Several theories have been advanced to explain the shapes of the yield curve. We will survey them later in the chapter.

The U.S. Treasury yield curve is the most widely followed yield curve for the following reasons. Firstly, it spans a full range of maturities, from three months to thirty years. Secondly, the prices are representative since the Treasuries are extremely liquid and their market deep. Finally, as the Treasuries are backed by the full faith and credit of the U.S. government, they are perceived as having no credit risk [88].

The most recent Treasury issues for each maturity are known as the **on-the-run** or **current coupon** issues in the secondary market (see Fig. 5.3). Issues auctioned prior to the current coupon issues are referred to as **off-the-run** issues. On-the-run and off-the-run yield curves are based on their respective issues [283, 442].

The yield of any non-Treasury security must exceed the base interest rate offered by an on-the-run Treasury security of comparable maturity with a positive spread called the **yield spread** [284]. This spread reflects the risk premium of holding securities not issued by the government. The base interest rate is known as the **benchmark interest rate**.

All rates in this chapter will be period-based instead of being annualized. This simplifies presentation by eliminating reference to the compounding frequency.

5.2 Spot Rates

The t-period **spot rate** $S(t)$ is the period yield to maturity of a t-period zero-coupon bond. Hence, $(1+S(t))^{-t}$ is the present value of one dollar t periods from now. In particular, the one-period spot rate, called the **short rate**, plays an important role in modeling interest rate dynamics. A **spot rate curve** is a plot of spots rate against maturity. Its other names include **spot yield curve** and **zero-coupon yield curve**.

A major shortcoming of the yield to maturity concept can be seen clearly with the spot rate idea. Recall the bond price formula,

$$\text{PV} = \sum_{i=1}^{n} \frac{C}{(1+r)^i} + \frac{F}{(1+r)^n} = C\frac{1-(1+r)^{-n}}{r} + \frac{F}{(1+r)^n}.$$

A single interest rate is used to discount all future cash flows. Specifically, every cash flow is discounted at the same rate r. To see the inconsistency of this methodology, consider two riskless bonds of equal quality but *different* yields to maturity because of their different cash flow patterns. The above methodology would discount their *contemporaneous* cash flows with different rates, while common sense dictates that cash flows occurring at the same time should be discounted using the same rate, which is precisely the spot rate. We remark that the yield to maturity is roughly a weighted sum of spot rates with each weight being proportional to the dollar duration of the cash flow (see Exercise 5.2.1).

Any coupon bond can be viewed as a package of zero-coupon bonds. In other words, a bond with cash flow C_i at time i for $1 \leq i \leq n$ is equivalent to a package of zero-coupon bonds where the ith bond pays C_i dollars at time i. A bond can therefore be priced by

$$\text{PV} = \sum_{i=1}^{n} \frac{C}{(1+S(i))^i} + \frac{F}{(1+S(n))^n}. \tag{5.1}$$

	Curr	Securities	Prev Close		9:28	
3	–	11/13/97	5.10	5.24	5.11	5.25
6	–	2/12/98	5.13	5.34	5.12	5.33
1	–	8/20/98	5.20	5.49	5.19	5.48
2	5.875	7/31/99	100-03+	5.81	100-04+	5.80
3	6.000	8/15/00	100-03+	5.96	100-04+	5.95
5	6.000	7/31/02	99-23+	6.06	99-24	6.06
10	6.125	8/15/07	99-07	6.23	99-09	6.22
30	6.375	8/15/27	97-25+	6.54	97-27+	6.54

Figure 5.3: ON-THE-RUN U.S. TREASURY YIELD CURVE (AUGUST 18, 1997, 9:28 A.M. EDT). Source: Bloomberg.

This pricing method incorporates information from the term structure by discounting each cash flow with the corresponding spot rate. The **discount factors**

$$d(i) \equiv \frac{1}{(1+S(i))^i}, \quad i = 1, \ldots, n$$

form the **market discount function** or simply **discount function** [699]. Note that $d(i)$ denotes the present value of one dollar i periods from now. In other words, the discount function denotes zero-coupon bond prices. It is the market discount function not the spot rate curve that is directly observable.

As a general principle, any riskless security having predetermined cash flows $C_1, \ldots, C_n$ should have a market price of

$$P = \sum_{i=1}^{n} C_i d(i). \tag{5.2}$$

If the market price is less than P, it is said to be **undervalued** or **cheap**, while if it is more than P, it is said to be **overvalued** or **rich**.

5.3 Extracting Spot Rates from the Yield Curve

Spot rates are typically extracted from the yields of coupon bonds because zero-coupon bonds may not be available. In fact, even if zero-coupon bonds exist, using them to derive spot rates can be problematic (see §22.1).

Start with the known $S(1)$ (short-term Treasuries are pure discount securities). $S(2)$ can be computed from a two-period coupon bond because of the following relationship,

$$P_2 = \frac{C}{1+S(1)} + \frac{C+100}{(1+S(2))^2},$$

where P_2 is the market price of the bond. In general, $S(n)$ can be recursively derived from (5.1) given the market price of the n-period coupon bond and $S(1), \ldots, S(n-1)$. The algorithm appears in Fig. 5.4.

Algorithm for extracting spot rates from coupon bonds:

```
input: n, C[1..n], P[1..n];
real   S[1..n], p, x;
S[1] := (100/P[1]) − 1;
p := P[1]/100;
for i = 2 to n {
        1. Solve P[i] = C[i] × p + (C[i] + 100)/(1 + x)^i
              for x;
        2. S[i] := x;
        3. p := p + (1 + x)^(−i);
}
return S[1..n];
```

Figure 5.4: ALGORITHM FOR EXTRACTING SPOT RATES FROM YIELD CURVE. $P[i]$ is the price (as a percentage of par) of the coupon bond maturing i periods from now, $C[i]$ is the coupon of the i-period bond expressed as a percentage of par, and n is the term of the longest maturity bond. We assume the first bond is a pure discount bond. The i-period spot rate, period-based, will be computed and stored in $S[i]$.

The correctness of the above algorithm is easy to see. First, the initialization step and Step 3 ensure that

$$p = \sum_{j=1}^{i-1} \frac{1}{(1+S(j))^j}$$

at the beginning of each loop. Hence, Step 1 solves for x such that

$$P_i = \sum_{j=1}^{i-1} \frac{C_i}{(1+S(j))^j} + \frac{C_i + 100}{(1+x)^i},$$

where C_i is the level coupon payment of bond i.

Each execution of Step 1 requires $O(1)$ arithmetic operations since

$$x = \left(\frac{C_i + 100}{P_i - C_i p} \right)^{1/i} - 1$$

and expressions like y^z can be computed as $\exp[\, z \times \ln y \,]$. (We use $\exp[\, y \,]$ and e^y synonymously.) Recall that we assume exponentiation and logarithm, usually implemented in hardware, take $O(1)$ time. Similarly, Step 3 runs in $O(1)$ time. The total running time is therefore linear in n.

Example 5.3.1 Suppose the one-year Treasury bill has a yield of 8%. Since this security is a zero-coupon bond, the one-year spot rate is 8%. Suppose two-year 10% Treasury notes are trading at 90. The two-year spot rate hence satisfies

$$90 = \frac{10}{1.08} + \frac{110}{(1+S(2))^2}.$$

It can be verified that $S(2) = 0.1672$, or 16.72%. □

In reality, computing the spot rates is not as clean-cut as the above **bootstrapping** procedure might suggest. Treasuries of the same maturity might be selling at different yields (the **multiple cash flow problem**), some maturities might be missing from the data points (the **incompleteness problem**), and Treasuries might not be of the same quality, and so on. Interpolation and fitting techniques are needed in practice to create a smooth spot rate curve (see Chapter 22). Such schemes, however, may lack economic justifications. Finally, with the advent of the STRIPS in 1985, prices and yields of default-free zero-coupon bonds have become available and, since 1989, are published daily in *The Wall Street Journal*.

5.4 Static Spread

The following strategy is sometimes employed to compare risky bonds. Consider a risky bond with cash flows $C_1, \ldots, C_n$ and selling for P. What would the same cash flows fetch were they from a riskless Treasury security? Of course, it is

$$P^* = \sum_{t=1}^{n} C_t d(t) = \sum_{t=1}^{n} C_t \,(1+S(t))^{-t}.$$

Typically, $P^* > P$. So we look for the s that, when added to the spot rate curve, gives the bond's market price, that is,

$$P = \sum_{t=1}^{n} C_t \left(1 + s + S(t)\right)^{-t}.$$

This s is called the **static spread**. Static spread is hence the spread that a risky bond would realize over the entire Treasury spot rate curve *if the bond is held to maturity*; it signifies the amount that the whole spot rate curve should shift *in parallel* in order to price the bond correctly [288].

Suppose the spot rate curve is flat at r. If a bond is selling for P with cash flows $C, \dots, C, C + F$, then the static spread is the s such that

$$P = \sum_{t=1}^{n} C \left(1 + r + s\right)^{-t} + F \left(1 + r + s\right)^{-n}.$$

The table below tabulates various price/static spread combinations for a 5%, 15-year bond paying semiannual interest under a flat, 7.8% spot rate curve.

Price (% of par)	98	98.5	99	99.5	100	100.5	101
Static spread (%)	0.435	0.375	0.316	0.258	0.200	0.142	0.085

Traditionally, spread refers to the difference between the yield to maturity of the bond under consideration and that of a Treasury security of comparable maturity. In the case of an upward-sloping (normal) spot rate curve, for example, the magnitude of the difference between the static spread and the traditional spread tends to be greater when the maturity of the bond is shorter or the yield curve is steeper [281].

Programming assignment 5.4.1 Write a program to compute the static spread. The inputs are the payment frequency per annum, the annual coupon rate as a percentage of par, the market price as a percentage of par, the number of remaining coupon payments, and the market discount factors read from a file. ◊

5.5 Spot Rates and Yield Curve

Many relationships hold between spot rates and yields to maturity. First, the spot rate dominates the corresponding yield to maturity at the same maturity if the yield curve is normal. Formally, let y_k denote the yield to maturity for the k-period riskless security. The normality of the yield curve means $y_k > y_{k-1}$. Our claim was $S(k) \geq y_k$ (see Exercise 5.5.1(1)). Similarly, the spot rate is dominated by the corresponding yield to maturity if the yield curve is inverted. Of course, if the yield curve is flat, the spot rate curve coincides with the yield curve. Secondly, the spot rate dominates the corresponding yield to maturity if the spot rate curve is normal (i.e., $S(1) < S(2) < \cdots$). It becomes smaller when the spot rate curve is inverted (see Exercise 5.5.1(2)).

The above analysis illustrates the so-called **coupon effect** of coupon bonds on their yields to maturity [752]. For instance, under a normal spot rate curve, a coupon bond has a lower yield than the spot rate of the same maturity. Picking a zero-coupon bond over a coupon bond based purely on its higher yield to maturity, which coincides with the spot rate, is therefore flawed.

It is often the case that the spot rate curve has the same **shape** as the yield curve. That is, if the spot rate curve is inverted, then the yield curve is, too, and a similar result holds under the normal case. A numerical example can be contrived, however, to demonstrate that this is only a trend, not a mathematical truth. Consider a three-period bond that pays a coupon of \$1 per period and repays the principal of \$100 at the end of the third period. Take the following three spot rates, $S(1) = 0.1$, $S(2) = 0.9$, and $S(3) = 0.901$. From (32.3) with $C = 1$ and $F = 100$, we derive the following yields to maturity, $y_1 = 0.1$, $y_2 = 0.8873$, and $y_3 = 0.8851$. This set of yields to maturity is clearly not strictly increasing.

When the final principal payment is relatively insignificant, the spot rate curve and the yield curve share the same general shape. Such is the case with high coupon rates and long maturities (see Exercise 5.5.3). Unless stated otherwise, this typical agreement in shape will be assumed for the rest of the chapter.

5.6 Forward Rates

The yield curve not only contains the prevailing interest rate structure but also information regarding future interest rates currently expected by the market known as the **forward rates**. Since $S(i)$ denote the i-period spot rate, \$1 will grow to be $(1+S(i))^i$ at time i. Suppose a person invests \$1 in riskless securities for j periods and, at time j, invests the proceeds in riskless securities for another $i-j$ periods ($i > j$). What does this person expect the future value of this \$1 investment to be at time i? Let $S(j,i)$ denote the $(i-j)$-period spot rate j periods from now. Note $S(j) = S(0,j)$. The investor in question will receive $(1+S(j))^j(1+S(j,i))^{i-j}$ at time i. If the above value equals $(1+S(i))^i$, that is, $S(j,i)$ is equal to

$$f(j,i) \equiv \left[\frac{(1+S(i))^i}{(1+S(j))^j} \right]^{1/(i-j)} - 1, \tag{5.3}$$

then the investor would be indifferent between the two investment approaches. Figure 5.5 illustrates the time line for spot rates and forward rates.

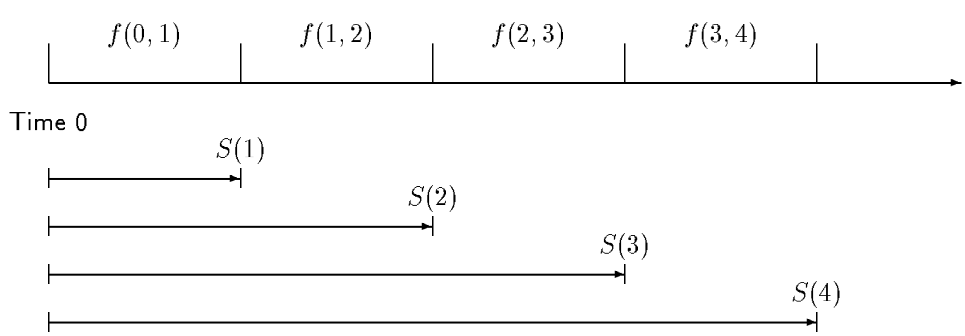

Figure 5.5: THE TIME LINE FOR SPOT AND FORWARD RATES.

The forward rates computed by (5.3) are **implied forward rates**, more precisely, the **$(i-j)$-period forward rate j periods from now**. In the above argument, we were not assuming any a priori relationship between the implied forward rate $f(j,i)$ and the actual future spot rate $S(j,i)$; this is the subject of the term structure theories to which we shall turn shortly. Rather, we were merely looking for the future spot rate that, *if realized*, will equate the two investment strategies. Forward rates with a duration of a single period are called **instantaneous forward rates** or **one-period forward rates**.

We pause here to mention briefly that the term "forward rate" is not always used consistently in the literature. Some authors use it to mean future spot rate, whereas others may use it to denote implied forward rate exclusively. This book adopts the latter convention, regarding forward rate and implied forward rate as synonymous.

If the spot rate curve is normal, the forward rate dominates the corresponding spot rate,

$$f(j,i) > S(i) > \cdots > S(j). \tag{5.4}$$

This claim can be easily extracted from (5.3). If, on the other hand, the spot rate curve is inverted, then the forward rate is dominated by the corresponding spot rate,

$$f(j,i) < S(i) < \cdots < S(j). \tag{5.5}$$

See Fig. 5.6 for illustration.

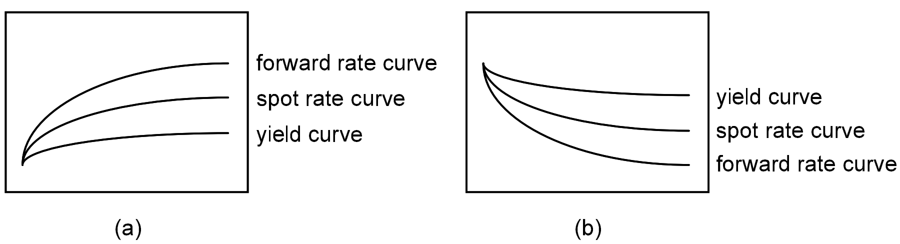

Figure 5.6: Yield curve, spot rate curve, and forward rate curve. When the yield curve is normal, it is dominated by the spot rate curve, which in turn is dominated by the forward rate curve if the spot rate curve is normal. When the yield curve is inverted, on the other hand, it dominates the spot rate curve, which in turn dominates the forward rate curve if the spot rate curve is inverted. (The forward rate curve is a plot of one-period forward rates).

Example 5.6.1 Suppose the following spot rates are extracted from the yield curve.

Period	1	2	3	4	5	6	7	8	9	10
Rate (%)	4.00	4.20	4.30	4.50	4.70	4.85	5.00	5.25	5.40	5.50

The following are the nine one-period forward rates, starting one period from now.

Period	1	2	3	4	5	6	7	8	9
Rate (%)	4.40	4.50	5.10	5.50	5.60	5.91	7.02	6.61	6.40

If \$1 is invested in a ten-period zero-coupon bond, it will grow to be $(1+0.055)^{10} = 1.708$. An alternative strategy is to invest \$1 in one-period zero-coupon bonds at 4% and reinvest at the one-period forward rates. The final result,

$$1.04 \times 1.044 \times 1.045 \times 1.051 \times 1.055 \times 1.056 \times 1.0591 \times 1.0702 \times 1.0661 \times 1.064 = 1.708,$$

is exactly the same, as expected. □

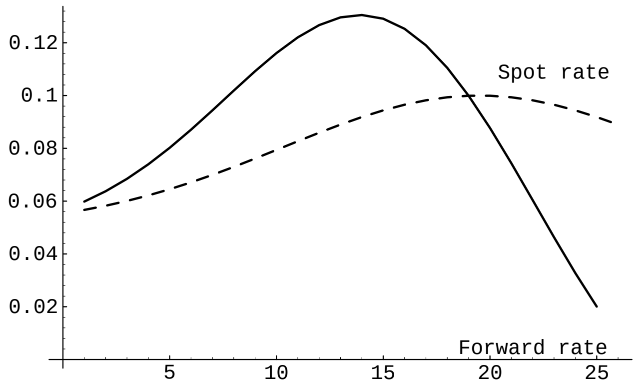

Figure 5.7: SPOT RATE CURVE AND FORWARD RATE CURVE. The forward rate curve is built upon one-period forward rates.

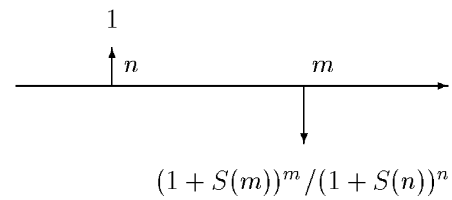

Figure 5.8: LOCKING IN THE FORWARD RATE. By trading zero-coupon bonds of maturities n and m in the right proportion, the forward rate $f(n, m)$ can be locked in today.

5.6.1 Locking in the forward rate

Forward rates are implied by the current yield curve and may or may not be realized in the future. However, one can lock in the forward rate $f(n, m)$ today by buying one unit of the n-period zero-coupon bond at price $1/(1+S(n))^n$ and shorting $(1+S(m))^m/(1+S(n))^n$ units of the m-year zero-coupon bond. There is no net initial investment as the cash inflow and the cash outflow, both at $1/(1+S(n))^n$, cancel out. Furthermore, there will be a \$1 cash inflow at time n and $(1+S(m))^m/(1+S(n))^n$ dollars of cash outflow at time m. See Fig. 5.8 for illustration.

Now that forward rates can be locked in today, they should not be negative. However, forward rates derived by (5.3) may be negative when the spot rate curve is steeply downward sloping. Therefore, not all spot rate curves are theoretically sound.

The future value of \$1 at time t can be derived in two ways. One can buy t-period zero-coupon bonds and receive $(1+S(t))^t$. An alternative is to buy one-period zero-coupon bonds today and a series of such bonds at the forward rates as they mature. The future value of this approach is $(1+S(1))(1+f(1,2))\cdots(1+f(t-1,t))$. Since they are identical,

$$S(t) = ((1+S(1))(1+f(1,2))\cdots(1+f(t-1,t)))^{1/t} - 1. \tag{5.6}$$

Hence, forward rates can also be used to determine the spot rate curve. Mathematically, forward rates, spot rates, and the yield curve can be derived from each other. (The coupon rates of the coupon bonds making up the yield curve need to be specified. Why?) The above equation shows that the one-period forward rates $f(s, s+1)$ can serve as the building block for term structure theories. We emphasize that (5.7) holds whether the future interest rates are random or not. In fact, in a certain economy, that equation reduces to

$$S(t) = ((1+S(1))(1+S(1,2))\cdots(1+S(t-1,t)))^{1/t} - 1$$

(see Exercise 5.6.5).

5.6.2 Term structure of credit spreads

People discount the cash flows from a corporate bond by adding a constant **credit spread** to the Treasury spot rate curve to reflect the risk premium. But this assumption of same credit spread at all maturities runs counter to the common sense that credit spread should rise with maturity because it is more likely for a corporation to be out of business within ten years than one year. A theory of term structure of credit spreads is needed.

Suppose we postulate that the price of a corporate bond equals that of the Treasury times the probability of solvency. Since the probability of solvency is simply one minus the probability of default, it must hold that

$$\text{probability of default} = 1 - \frac{\text{price of corporate zero}}{\text{price of Treasury zero}}.$$

The above equation can be used to compute the probability of default for corporate bonds with, say, one year to maturity. Then, we can calculate the **forward probability of default**, the conditional probability of default in the second year, given that the corporation does not default in the first year. The formula is clearly

$$\begin{aligned}&(1 - \text{probability of default } (1 \text{ year})) \times (1 - \text{forward probability of default})\\ &= \frac{\text{price of corporate zero } (2 \text{ year})}{\text{price of Treasury zero } (2 \text{ year})}.\end{aligned}$$

This procedure can apparently be carried out to longer maturities.

5.7 Spot/Forward Rates under Continuous Compounding

The mathematics can be simplified tremendously with continuous compounding. To start with, modify (5.1) to be

$$\text{PV} = \sum_{i=1}^{n} Ce^{-iS(i)} + Fe^{-nS(n)}.$$

In particular, the market discount function becomes

$$d(n) = e^{-nS(n)}. \tag{5.7}$$

A bootstrapping procedure similar to the one in Fig. 5.4 can be used to calculate the spot rates under continuous compounding. Alternatively, we can retain the same algorithm by converting the resulting spot rates into their equivalent continuous compounded rates.

The formula for the forward rate is also much simplified as

$$f(j,i) = \frac{iS(i) - jS(j)}{i-j}. \tag{5.8}$$

In particular, the one-period forward rate is

$$f(j,j+1) = (j+1)\,S(j+1) - jS(j) = -\ln\left(\frac{d(j+1)}{d(j)}\right) \tag{5.9}$$

The formula corresponding to (5.7) is

$$S(t) = \frac{f(0,1) + f(1,2) + \cdots + f(t-1,t)}{t}. \tag{5.10}$$

Recall that all the rates above are period rates and $f(0,1) = S(1)$. To annualize them, simply multiply the numbers by the number of periods per annum. For example, the following table calculates one-period forward rates from spot rates using (5.10).

Period	1	2	3	4	5
Spot rate (% per period)	6.00	6.50	7.00	7.50	8.00
One-period forward rate (% per period)		7.00	8.00	9.00	10.00

Rewrite the forward rate in (5.9) as

$$f(j,i) = S(i) + (S(i) - S(j))\,\frac{j}{i-j}.$$

The above equation becomes

$$f(T, T+\Delta T) = S(T+\Delta T) + (S(T+\Delta T) - S(T))\,\frac{T}{\Delta T}$$

under *continuous* time instead of discrete time. Letting $\Delta T \to 0$, we arrive at

$$f(T,T) \equiv \lim_{\Delta T \to 0} f(T, T+\Delta T) \equiv f(T) = S(T) + T\,\frac{\partial S}{\partial T}. \tag{5.11}$$

$f(T)$ is the instantaneous forward rate at time T. Note that $f(T) > S(T)$ if and only if $\partial S/\partial T > 0$.

5.8 Spot/Forward Rates under Simple Compounding

We shall be brief here since the basic principles are similar. Equation (5.1) becomes

$$\text{PV} = \sum_{i=1}^{n} \frac{C}{1+iS(i)} + \frac{F}{1+nS(n)}.$$

In particular, the market discount function becomes

$$d(n) = (1+nS(n))^{-1}. \tag{5.12}$$

The $(i-j)$-period forward rate j periods from now is

$$f(j,i) = \frac{(1+iS(i))(1+jS(j))^{-1} - 1}{i-j}. \tag{5.13}$$

Recall that all the rates above are period rates. To annualize them, simply multiply the numbers by the number of periods per annum.

5.9 Term Structure Theories

Term structure theories attempt to explain the relationships among interest rates of various maturities. As the spot rate curve is most critical to the valuation of securities, the term structure theories discussed below will be limited to the spot rate curve.

5.9.1 Expectations theory

Unbiased expectations theory

According to the **unbiased expectations theory** attributed to Irving Fisher, the forward rate represents the expected future spot rate for the period in question; mathematically,

$$f(a, b) = E[\, S(a, b)\,], \tag{5.14}$$

where $E[\cdot]$ denotes mathematical expectation [565, 699]. Note that this theory does not say the forward rate is an accurate predictor for the future spot rate. It merely asserts that it does not deviate from the future spot rate systematically. Although this theory is the most widely accepted explanation for the shape of the term structure, it is rejected by most empirical studies dating back at least to Macaulay [541, 546, 666] with the possible exception of the period prior to the founding of the Federal Reserve System in 1915 [551, 654].

Let us use the numbers from Example 5.6.1 to illustrate some important points. Suppose an investor has a horizon of two periods. She could invest \$1 for the full two periods at the two-period spot rate of 4.2%, receiving \$1.0858 at the end with certainty. Call it the **maturity strategy**. Alternatively, she can invest \$1 now for only one period at the one-period spot rate of 4% and reinvest \$1.04 for another period at an unknown future spot rate with an expected value of $E[\, S(1, 2)\,]$, receiving an average of $1.04 \times (1 + E[\, S(1, 2)\,])$ at the end of the horizon. Call it the **rollover strategy**. If $E[\, S(1, 2)\,]$ is greater than the forward rate $f(1, 2) = 4.4\%$ for the period under consideration, then the investor will get more than $1.04 \times (1 + 0.044) = 1.0858$ on the average. Suppose that were the consensus of the market. Then, people would follow the rollover strategy instead of the maturity strategy, and the two-period spot rate would rise because of lack of demand. The one-period spot rate would also fall because of strong demand. Hence, we conclude that, in an equilibrium, the expected future spot rate cannot be greater than the forward rate. Similarly, in an equilibrium, the expected future spot rate cannot be lower than the forward rate either. Only when the two rates are equal do investors become indifferent between the two strategies.

A normal spot rate curve, according to the theory, is therefore due to the fact that the market expects the future spot rate to rise. Formally, since $f(j, j+1) > S(j+1)$ if and only if $S(j+1) > S(j)$ from (5.3), the theory says

$$E[\, S(j, j+1)\,] > S(j+1) \text{ if and only if } S(j+1) > S(j).$$

Conversely, the theory says the spot rate is *expected* to fall if and only if the spot rate curve is inverted [653].

Since the term structure has been upward sloping about 80% of the time, the unbiased expectations theory would have to imply that investors have expected interest rates to rise 80% of the time. This does not seem plausible. The theory also implies that all bonds, regardless of their different maturities, are expected to earn the same riskless return (see Exercise 5.9.1) [442, 498]. This is not credible either, because it means investors are indifferent to risk.

Other versions of the expectations theory

At least four other flavors of expectations theory have been separated in the literature. They are inconsistent with each other for subtle reasons [201, 447]. Expectations theory also plays a critical role to all the other theories, which differ by how risks are treated [447].

Consider a theory that says the expected returns on all possible riskless bond strategies are equal for all holding periods [201]. Then, in particular, $(1+S(2))^2 = (1+S(1))E[\,1+S(1,2)\,]$ because of the equivalency of buying a two-period bond and rolling over two one-period bonds. After rearrangement, we have

$$E[\,1+S(1,2)\,] = \frac{(1+S(2))^2}{1+S(1)}.$$

Now consider the following two one-period strategies. The first one buys a two-period bond and sells it after one period. The expected return is clearly $E\left[\,(1+S(1,2))^{-1}\,\right]\,(1+S(2))^2$. The second strategy buys a one-period bond with a return of $1+S(1)$. The said theory imposes the condition that $E\left[\,(1+S(1,2))^{-1}\,\right]\,(1+S(2))^2 = 1+S(1)$, i.e.,

$$\frac{(1+S(2))^2}{1+S(1)} = \frac{1}{E[\,(1+S(1,2))^{-1}\,]}.$$

Equating the above equations to obtain

$$E\left[\frac{1}{1+S(1,2)}\right] = \frac{1}{E[\,1+S(1,2)\,]}.$$

But this is impossible save for a certain economy. The reason is **Jensen's inequality**, which states that $E[\,g(X)\,] > g(E[\,X\,])$ for any random variable X with positive variance and convex function g (i.e., $g''(x) > 0$). Now, take $g(x) = (1+x)^{-1}$ to prove our point. So this version of expectations theory is untenable.

Another version of expectations theory is the **local expectations theory** [201, 338, 447]. Unlike the previous theory, it only postulates that the expected rate of return of any bond over a single period should equal the then prevailing one-period spot rate, that is,

$$\frac{E\left[\,(1+S(1,n))^{-(n-1)}\,\right]}{(1+S(n))^{-n}} = 1+S(1) \quad \text{for } any \;\; n > 1. \tag{5.15}$$

This theory will form the basis of many stochastic interest rate models later. The expected difference between the one-period holding period return and the prevailing spot rate,

$$\frac{E\left[\,(1+S(1,n))^{-(n-1)}\,\right]}{(1+S(n))^{-n}} - (1+S(1)),$$

is called the **holding premium**.

Each version of the expectations theory postulates certain expected difference, called **liquidity premium** or **term premium**, to be zero. For the unbiased expectations theory, the liquidity premium is $f(a,b) - E[\,S(a,b)\,]$, whereas for the local expectations theory, the liquidity premium is defined to be the holding premium [456, 599]. The inconsistency between versions of the expectations theory alluded to earlier then follows. However, these incompatibility results go away if the theories instead postulate *non-zero* premiums [128]. One version[1] of the unbiased expectations theory says

$$f(a,b) - E[\,S(a,b)\,] = p(a,b),$$

where p denote the liquidity premium [33]. Thus, today's forward rate overestimates future spot rate if $p(a,b) > 0$. There is evidence that $p_t(a,b)$ is neither constant nor time-independent [36, 292].

5.9.2 Liquidity preference theory

The **liquidity preference theory** holds that investors demand a risk premium for holding long-term bonds [447]. An alternative phrase for liquidity preference is "propensity to hoard" [706]. The liquidity preference theory is attributed to Hicks [565].

Consider an investor with a holding period of two. If the investor chooses the maturity strategy and is forced to sell the two-period bonds because of an unexpected need for cash, he would face the **interest rate risk** and the **price risk** because the prices of those bonds depend on the prevailing interest rate at the time of the sale. This risk is absent from the rollover strategy. In fact, prices of bonds of longer maturity are more sensitive to interest rate changes (see §4.1). As a consequence, the investor would demand a higher return for longer-term bonds. Mathematically, this means $f(a,b) > E[\,S(a,b)\,]$ [398]. From (5.3) and the above inequality, we have

$$(1+S(i))^i(1+E[\,S(i,2i)])^i < (1+S(i))^i(1+f(i,2i))^i = (1+S(2i))^{2i}.$$

If the spot rate curve is inverted, hence $S(2i) < S(i)$, then

$$E[\,S(i,2i)\,] \ll S(2i) < S(i).$$

The market therefore has to expect the interest rate to decline *substantially* in order for an inverted spot rate curve to be observed.

A normal spot rate curve can be consistent with declining expected interest rates. (The unbiased expectations theory is not consistent, however, with such a possibility.) Only when the interest rate is expected to fall below a threshold does the spot rate curve starts to become inverted. Since non-zero liquidity premium is reasonably supported by evidence, the unbiased expectations theory cannot be valid [565].

The liquidity preference theory seems to be consistent with the typically upward-sloping yield curve. Even if people expect the rate to decline and rise equally frequently, the theory asserts that the curve is upward sloping more often. This is because a rising expected interest rate is associated only with a normal spot rate curve, and a declining expected interest rate can sometimes be associated with a normal spot rate curve [185, 699].

[1] Sometimes called the **biased expectations hypothesis** [565].

5.9.3 Market segmentation theory

The **market segmentation theory** holds that investors are restricted to bonds of certain maturities either by law, preferences, or customs. For instance, life insurance companies generally prefer long-term bonds, while commercial banks favor shorter-term ones. The spot rates are determined within each maturity sector separately [565, 699].

The market segmentation theory is closely related to the **preferred habitats theory** of Culbertson, Modigliani, and Sutch [283, 546]. This theory holds that the investor's investment horizon determines the riskiness of bonds. An investment horizon of five years will prefer a five-year zero-coupon bond, demanding higher returns from both two- and seven-year bonds, for example, because the former choice has reinvestment risk and the latter has price risk. Hence, in contrast to the liquidity preference theory, we may have $f(a,b) < E[\,S(a,b)\,]$ if the market is dominated by long-term investors [666].

5.10 Duration and Immunization Revisited

Recall that the kind of rate change considered before for duration was parallel shift under flat spot rate curves. It affected the bonds in the portfolio by the same *amount* for all maturities. For this reason, a bond portfolio thus immunized is said to be **immunized under parallel interest rate shifts**. In reality, long-term rates and short-term rates do not in general move by the same amount, or even in the same direction. Duration matching no longer works if rate change is not parallel. Transactions costs arising from periodical rebalancing will affect returns, too. We now study duration and immunization under the more general framework of non-flat spot rate curves.

5.10.1 Duration measures

Duration measures the relative price change as the spot rate curve shifts by the same amount. To put it another way, the percentage price change roughly equals the duration multiplied by the size of the parallel shift in the spot rate curve.

Term structure was assumed to be flat in Chapter 3. To extend the duration concept to the more general case where the spot rate curve is no longer flat, we proceed as follows. Let $S(1), S(2), \ldots$ define the current term structure and

$$P(y) \equiv \sum_i \frac{C_i}{(1+S(i)+y)^i}.$$

Then duration is simply

$$-\frac{\partial P(y)/P(0)}{\partial y}\Bigg|_{y=0} = -\lim_{\Delta y \to 0} \frac{\sum_i \frac{C_i}{(1+S(i))^i} - \sum_i \frac{C_i}{(1+S(i)+\Delta y)^i}}{\Delta y \sum_i \frac{C_i}{(1+S(i))^i}} = \frac{\sum_i \frac{iC_i}{(1+S(i))^{i+1}}}{\sum_i \frac{C_i}{(1+S(i))^i}}$$

by shifting the whole term structure to $S(1)+\Delta y, S(2)+\Delta y, \ldots$.

In the above derivation, the simple linear relationship between duration and Macaulay duration in (4.5) breaks down. One way to regain it is to resort to a different kind of term structure change, the **proportional shift**, defined as

$$\frac{\Delta(1+S(i))}{1+S(i)} = \frac{\Delta(1+S(1))}{1+S(1)}$$

for all i [277]. Here, $\Delta(x)$ denotes the change in x when the short-term rate is shifted by Δy. Duration now becomes

$$(1+S(1))^{-1}\left[\frac{\sum_i \frac{iC_i}{(1+S(i))^i}}{\sum_i \frac{C_i}{(1+S(i))^i}}\right]. \tag{5.16}$$

As a result, if we define **Macaulay's second duration** to be the number within the brackets above, then

$$\text{duration} = \frac{\text{Macaulay's second duration}}{(1+S(1))}$$

is regained. This measure is also called **Bierwag's duration** [65, 451].

Empirically speaking, long-term rates change less than short-term ones. To incorporate this fact into the duration measure, one may postulate such **nonproportional shifts** as

$$\frac{\Delta(1+S(i))}{1+S(i)} = K^{i-1}\frac{\Delta(1+S(1))}{1+S(1)}, \quad K < 1. \tag{5.17}$$

Parallel shift does not accurately reflect market reality. As the market segmentation theory claims, long-term rates may not correlate well with short-term rates; in fact, they often move in opposite directions. Short-term rates are also historically more volatile. Practitioners sometimes break the spot rate curve into segments and measure the duration for a parallel shift in each segment [422]. We mention in passing that there is evidence showing that, despite its theoretical limitations, duration provides as good an estimate for price volatility as more sophisticated measures [303].

Duration can also be applied to custom changes in the yield curve. Here, changes in the yield are specified over the entire term structure and parametrized by a single number Δy as before. For example, we may define the **short-end duration** as the effective duration using the following two term structure changes. Let the one-year market yield be changed by ± 50 basis points ($\pm 0.5\%$). The amounts of yield changes for maturity $1 \leq i \leq 10$ are set at $\pm 50 \times (11-i)/10$ basis points. Yields of maturities longer than ten remain intact. If the yield curve is normal, a $+50$ basis point change corresponds to **flattening** of the yield curve, while a -50 basis point change corresponds to **steepening** of the yield curve. **Long-end duration** can be specified similarly. We emphasize that these specific differentials over the term structure are just a few possibilities (see Exercise 5.7.3 for another possibility).

Although duration can have many variants, the one feature that all share is that the term structure can only shift in a preset pattern. This follows from the very nature of differential calculus. Besides parallel, proportional, nonproportional, and custom shifts, the influential notion of **key rate durations** by Ho is another alternative (see Chapter 27).

5.10.2 Immunization

The case of no rate changes

It was argued in §4.2.2 that, in the absence of interest rate changes and assuming a flat yield curve, it suffices to match the present values of the future liability and the investment to achieve immunization. The above observation can be easily generalized to the case where the term structure is not flat, the restriction being that the future spot rates be equal to the forward rates (the future term structure is hence completely specified today).

Let L be the liability at time m. We have

$$\text{PV} = \sum_{i=1}^{n} \frac{C}{(1+S(i))^i} + \frac{F}{(1+S(n))^n} = \frac{L}{(1+S(m))^m}.$$

From the above equation, the present value of the liability at time $k \leq m$ is

$$\frac{L}{(1+S(k,m))^{m-k}} = \text{PV} \times (1+S(k))^k$$

by (5.3) and the premise that $f(a,b) = S(a,b)$. On the other hand, the present value of the bond plus the reinvestments of the coupon payments at the same time is

$$\begin{aligned}
&\sum_{i=1}^{k} C\,(1+S(i,k))^{k-i} + \sum_{i=1}^{n-k} \frac{C}{(1+S(k,i+k))^i} + \frac{F}{(1+S(k,n))^{n-k}} \\
= \;& \sum_{i=1}^{k} \frac{C\,(1+S(k))^k}{(1+S(i))^i} + \sum_{i=1}^{n-k} \frac{C\,(1+S(k))^k}{(1+S(i+k))^{i+k}} + \frac{F\,(1+S(k))^k}{(1+S(n))^n} \\
= \;& \text{PV} \times (1+S(k))^k,
\end{aligned}$$

which matches the liability precisely. (We adopted $S(s,s) = 0$ for convenience.) In the absence of unpredictable term structure changes, therefore, duration-matching is not needed for immunizing a liability. Furthermore, once set up, the scheme does not need to be rebalanced.

The case of certain rate movements

A general result in §4.2.2 said a future liability can be immunized with a portfolio of positive cash inflows that has the same present value and Macaulay duration. This conclusion can be extended to any parallel shift in the spot rate curve which may not be flat.

We shall be working with continuous compounding. The liability L is T years from now. Without loss of generality, assume the portfolio consists only of two types of zero-coupon bonds maturing at t_1 and t_2 with $t_1 < T < t_2$. Let there be n_i bonds maturing at time t_i. Assume $L = 1$ for simplicity. The portfolio's present value is

$$V = n_1 e^{-S(t_1)\,t_1} + n_2 e^{-S(t_2)\,t_2} = e^{-S(T)\,T}$$

with duration

$$\frac{n_1 t_1 e^{-S(t_1)\,t_1} + n_2 t_2 e^{-S(t_2)\,t_2}}{V} = T.$$

These two equations imply

$$n_1 e^{-S(t_1)\,t_1} = \frac{V\,(t_2 - T)}{t_2 - t_1} \quad \text{and} \quad n_2 e^{-S(t_2)\,t_2} = \frac{V\,(t_1 - T)}{t_1 - t_2}. \tag{5.18}$$

Now, shift the spot rate curve by $\delta \neq 0$. The portfolio's present value becomes

$$\begin{aligned} n_1 e^{-(S(t_1)+\delta)\,t_1} + n_2 e^{-(S(t_2)+\delta)\,t_2} &= e^{-\delta t_1} \frac{V\,(t_2 - T)}{t_2 - t_1} + e^{-\delta t_2} \frac{V\,(t_1 - T)}{t_1 - t_2} \\ &= \frac{V}{t_2 - t_1} \left(e^{-\delta t_1} (t_2 - T) + e^{-\delta t_2} (T - t_1) \right) \end{aligned}$$

after plugging in (5.19). In contrast, the liability's present value after the parallel shift is

$$e^{-(S(T)+\delta)\,T} = e^{-\delta T} V.$$

It is not hard to prove that

$$\frac{V}{t_2 - t_1} \left(e^{-\delta t_1} (t_2 - T) + e^{-\delta t_2} (T - t_1) \right) > e^{-\delta T} V.$$

See Fig. 5.9 for illustration.

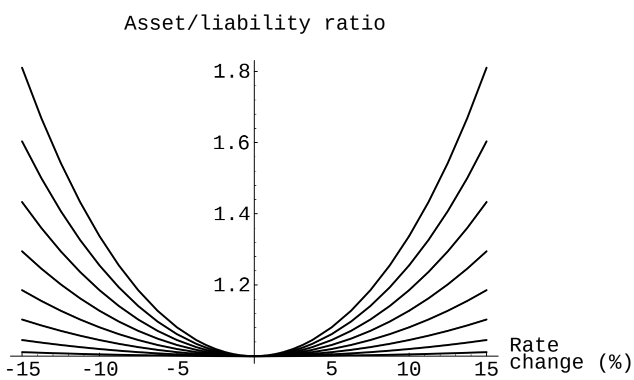

Figure 5.9: ASSET/LIABILITY RATIO UNDER PARALLEL SHIFTS IN THE SPOT RATE CURVE. Each curve is the result of a pair of zero-coupon bonds with maturities (t_1, t_2) to immunize a liability $T = 10$ years away. All curves have a minimum value of one when there are no shifts. Changes in interest rates move the immunized portfolio value ahead of the liability, and the advantage is more pronounced the more t_1 and t_2 are away from T.

A duration-matched portfolio under parallel shifts in the spot rate curve therefore implies free lunch in that any *instantaneous* interest rate change generates profits. Furthermore, no investor would hold the T-year bond because a portfolio of t_1- and t_2-year bonds has a higher return for any interest rate shock. This logic is flawed, however. Changes in the portfolio value *through time* should have been taken into account as well, and comparisons should have been based on holding period return [176, 752]. We will pick up this issue again in Chapter 14.

Immunization and convexity revisited

A barbell portfolio often arises from maximizing the portfolio convexity. This point was elaborated in §4.3.1. Higher convexity may be undesirable, however, when it comes to immunization. Recall that convexity is defined in terms of parallel shifts in the term structure. Once this condition is compromised, as is often the case in reality, the more dispersed the cash flows, the more exposed the portfolio is to the **shape risk** (or the **twist risk**) [175, 213].

Despite these reservations, there is evidence that immunization with the Macaulay duration, still widely used [86], is as effective as with alternative duration measures [376]. One possible reason is that roughly parallel shifts in the spot rate curve are responsible for more than 80% of the movements in interest rates [524].

Additional Reading

Consult [283, Chapter 9] for more information on the term structure of credit spread. Pointers to empirical studies of the expectations theory can be found in [129]. See [212, 228] for alternative approaches to immunization.