Chapter 4

Bond Price Volatility: Duration and Convexity

Contents

Can anyone measure the ocean by handfuls
or measure the sky with his hands?
—Isaiah 40:12

Understanding how bond price and interest rate move with respect to each other is key to risk management of interest rate-sensitive securities. This chapter focuses on bond price volatility, which measures the extent of price movements when interest rates move. Two classic notions, **duration** and **convexity**, will be introduced for this purpose. A few applications of duration in risk management will also be presented.

4.1 Price Volatility

The sensitivity of the percentage bond price change to changes in interest rates, $(\partial P/P)/\partial y$, is what people have in mind for the price volatility. The *degree* of volatility is signified by the absolute value of $(\partial P/P)/\partial y$. For example, a bond with $(\partial P/P)/\partial y = -200$ is more volatile than one with $(\partial P/P)/\partial y = 100$. The sign of $(\partial P/P)/\partial y$ says about the *direction* of price changes with respect to interest rate changes. It is also not hard to see that $(\partial P/P)/\partial y < 0$ for bonds without embedded options.

For coupon bonds,

$$\frac{\partial P/P}{\partial y} = \frac{(C/y)\, n - \left(C/y^2\right) \left((1+y)^{n+1} - (1+y)\right) - nF}{(C/y) \left((1+y)^{n+1} - (1+y)\right) + F\,(1+y)}, \tag{4.1}$$

where n is the number of periods before maturity, y is the period required yield, F is the par value, and C is the coupon payment. In the above, C, F, and n must be independent of y; in other words, the cash flow should not be linked to interest rates or yields [280].

Price volatility increases as the coupon rate decreases, other things being equal. Figure 4.1 demonstrates this point clearly. As a consequence, zero-coupon bonds have the greatest volatility for a given maturity, and bonds selling at a deep discount will have greater volatility than those selling near or above par. Figure 4.1 also shows another important characteristic: Price volatility increases as the required yield decreases, other things being equal. So, bonds traded with higher yields have less volatility.

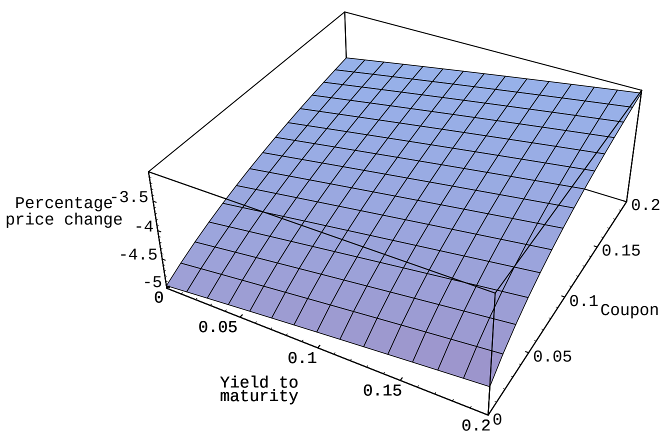

Figure 4.1: VOLATILITY WITH RESPECT TO COUPON RATE AND REQUIRED YIELD. Plotted is the percentage price change per percentage change in the required yield, or $(\partial P/P)/\partial y$. Bonds are assumed to pay semiannual coupon payments with a maturity date of September 15, 2000. The settlement date is September 15, 1995. Note that the degree of volatility is related to the magnitude of the value on the z-axis in the above graph.

Price volatility typically increases as the **term to maturity** becomes higher. That is, bonds with longer maturity have more *price* volatility, other things being equal. This is consistent with the preference for liquidity and with the empirical fact that long-term bond prices are more volatile than short-term ones. The *yields* of long-term bonds, however, are less volatile than those of short-term bonds [185]. Price volatility typically increases (but at a decreasing rate) with term to maturity as shown in Fig. 4.2. The above statement can be violated in extreme cases. Figure 4.3, for example, shows an example which goes against this typical trend of increasing volatility for longer maturity.

4.2 Duration

The **Macaulay duration**, first proposed in 1938 by Macaulay [541], is defined as the weighted average of the terms to maturity of a security's cash flows with the weights being each cash flow's present value as a percentage of the security's full price. Formally, it is

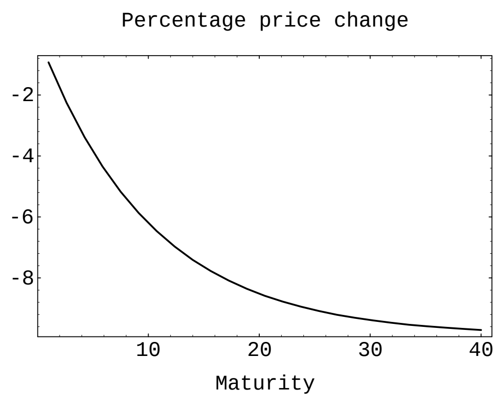

Figure 4.2: VOLATILITY WITH RESPECT TO TERMS TO MATURITY. Plotted is the percentage bond price change per percentage change in the required yield at various remaining terms to maturity. The annual coupon rate is 10%, with coupons paid semiannually. The yield to maturity is identical to the coupon rate.

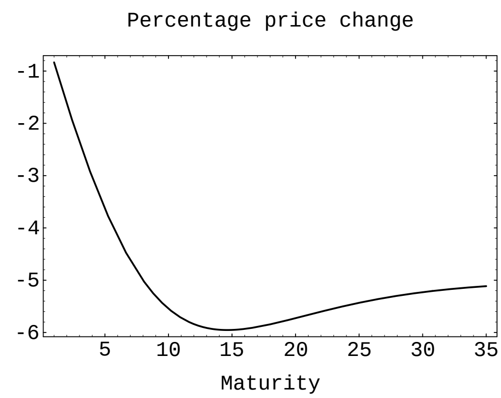

Figure 4.3: VOLATILITY WITH RESPECT TO TERMS TO MATURITY: AN ANOMALY. The annual coupon rate is 10%, with coupons paid semiannually, and the yield to maturity is 40% (a deep discount bond). The remaining terms to maturity are measured in half-years. The rest follows Fig. 4.2.

defined as

$$\mathrm{MD} \equiv \frac{1}{P} \sum_{i=1}^{n} \frac{iC_i}{(1+y)^i},$$

where n is the number of periods before maturity, y is the period interest rate or the required yield, C_i is the cash flow at time i, and P is the price of the security. Clearly, the Macaulay duration (in periods) is equal to

$$\mathrm{MD} = -(1+y) \frac{\partial P/P}{\partial y}. \tag{4.2}$$

This simple relationship was discovered by Hicks in 1939 [200, 451]. In particular, the Macaulay duration for option-free bonds is

$$\mathrm{MD} = \frac{1}{P} \left(\sum_{i=1}^{n} i \frac{C}{(1+y)^i} + n \frac{F}{(1+y)^n} \right). \tag{4.3}$$

Comment 4.2.1 We emphasize that the above equations hold only if the coupon C, the par value F, and the maturity n are independent of the yield y; in other words, the cash flow is not affected by changes in the yield. □

Based on (4.3), it is not hard to show that

$$\mathrm{MD} = \frac{c(1+y)((1+y)^n - 1) + ny(y-c)}{cy((1+y)^n - 1) + y^2}, \tag{4.4}$$

where c is the period coupon rate [303]. The above equation reduces to

$$\mathrm{MD} = \frac{(1+y)((1+y)^n - 1)}{y(1+y)^n}$$

when $c = y$. As another example, the Macaulay duration of a zero-coupon bond is n (corresponding to $c = 0$), exactly its term to maturity. In general, the Macaulay duration of a coupon bond is less than its maturity. In fact, (4.4) says the Macaulay duration of a coupon bond approaches $(1+y)/y$ as the maturity increases, independent of the coupon rate.

To convert the Macaulay duration to be year-based, modify (4.3) thus,

$$\text{MD} = \frac{1}{P}\left(\sum_{i=1}^{n} \frac{i}{k} \frac{C}{(1+(y/k))^i} + \frac{n}{k} \frac{F}{(1+(y/k))^n}\right),$$

where y is the *annual* yield, and k is the compounding frequency per annum. Now (4.2) becomes

$$\text{MD} = -\left(1+\frac{y}{k}\right)\frac{\partial P/P}{\partial y}.$$

Note that

$$\text{MD (in years)} = \frac{\text{MD (in periods)}}{k}.$$

Although the Macaulay duration has its origin in measuring the length of time a bond investment is outstanding, it should be seen mainly as measuring the sensitivity of price to change in market yield, that is, price volatility [303]. As a matter of fact, many, if not most, duration-related terminology cannot be comprehended otherwise. A related measure is **modified duration** defined as

$$\textsf{modified duration} \equiv -\frac{\partial P/P}{\partial y} = \frac{\text{MD}}{(1+y)}. \tag{4.5}$$

The rightmost equality above is valid if the cash flow is independent of changes in interest rates (see Comment 4.2.1). Modified duration is clearly positive for option-free bonds. It can be easily checked that the modified duration of a portfolio equals

$$\sum_i w_i D_i, \tag{4.6}$$

where D_i is the modified duration of the ith asset, and w_i is the market value of that asset expressed as a percentage of the market value of the portfolio. To measure the modified duration by the year, (4.5) should be changed to

$$\textsf{modified duration} = \frac{\text{MD (in years)}}{1+(y/k)},$$

where y is the annual yield, and k is the payment frequency per annum.

Taylor's expansion implies the following approximation formula

$$\textsf{percentage price change} \approx -\textsf{modified duration} \times \textsf{yield change}. \tag{4.7}$$

Mathematically speaking, the modified duration is simply the negation of the first derivative of the bond price with respect to yield divided by the price. As an example, the modified duration of an option-free bond, whose cash flow is hence fixed, is equal to minus (4.1). The value is also equal to (4.4) divided by $(1+y)$ and with the substitution $c = C/F$. See Fig. 4.4 for illustration.

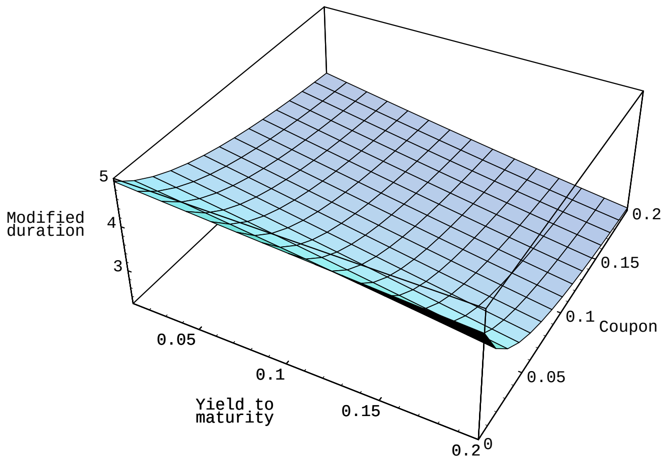

Figure 4.4: MODIFIED DURATION WITH RESPECT TO COUPON RATE AND REQUIRED YIELD. The bond is identical to the one in Fig. 4.1.

Example 4.2.2 Consider a bond whose modified duration is 11.54 with a yield of 10%. This means that, if yields increase instantaneously from 10% to 10.1%, the approximate percentage price change would be $-11.54 \times 0.001 = -0.01154$, or -1.154%. □

Cash flows of securities with an embedded option depend on interest rate movements. The duration measures introduced up to now are hence inappropriate for them. To see this point, suppose the required yield decreases. The Macaulay duration will be lengthened. However, for securities whose cash flows actually increase as a result of yield decline, the Macaulay duration may decrease. For this reason, the Macaulay duration should be used only for securities whose cash flows do not change with yields.

A general formula to measure volatility is

$$\frac{P_- - P_+}{P_0\,(y_+ - y_-)}, \tag{4.8}$$

where P_- is the price if yield is decreased by Δy, P_+ is the price if yield is increased by Δy, P_0 is the initial price, y is the initial yield, $y_+ \equiv y + \Delta y$, $y_- \equiv y - \Delta y$, and Δy is sufficiently small. This is called **effective duration**. A less accurate, albeit computationally economical formula for effective duration is to use **forward difference**

$$\frac{P_+ - P_0}{P_0\,\Delta y}$$

instead of the **central difference** in (4.8). Effective duration is most useful in cases where yield changes alter the cash flows and where the cash flows are so complex that simple formulae such as (4.1) are unavailable. This duration measure strengthens our contention that duration should be looked upon as a measure of volatility and not average term to maturity. In fact, it is possible for the duration of a security to be longer than its maturity or even negative [279].

In principle, one can compute the effective duration of almost any financial instrument. The prices P_+ and P_- are usually expected values themselves. One particular methodology that takes into account the option features of securities, called the **option-adjusted spread** (**OAS**), gives rise to the **option-adjusted spread duration** [47, 288]. We will return to this topic later in the book.

4.2.1 Continuous compounding

Under continuous compounding, the formula for duration is slightly changed. To start with, the price of a bond is now $P = \sum_i C_i e^{-yt_i}$, where t_i denotes the time when payment C_i is made. Since duration measures the average time before a bond holder receives cash payments, we have

$$\textsf{duration} \text{ (continuous compounding)} \equiv \frac{\sum_i t_i C_i e^{-yt_i}}{P} = -\frac{\partial P/P}{\partial y}. \tag{4.9}$$

Unlike the Macaulay duration in (4.2), the extra $1 + y$ term disappears.

Example 4.2.3 The duration of an n-period zero-coupon bond is n. □

For the rest of the book, we shall mean by duration the mathematical expression $-(\partial P/P)/\partial y$ or its approximation, effective duration. As a consequence,

$$\textsf{percentage price change} \approx -\textsf{duration} \times \textsf{yield change}. \tag{4.10}$$

The principal applications of duration are in hedging and asset/liability matching [47].

4.2.2 Applications to immunization

Suppose a fund manager has a liability in the future. Buying coupon bonds to meet that liability incurs certain risks which are missing from zero-coupon bonds. If the interest rate rises subsequent to the purchase, then the interest on interest from the reinvestment of the coupon payments will increase. But, assuming the investment horizon is shorter than the maturity of the bond, a capital loss will occur for the sale of the bond. The reverse is true if the interest rate falls. The result is uncertainty in meeting future liabilities. Such a situation naturally arises when, for example, a bank issues certificates of deposit and has to invest the proceeds in such a way that the future value can cover the liabilities whichever way the interest rate moves. Can one find a bond or a bond portfolio that is immunized against interest rate changes?

A portfolio is said to be **immunized** for a holding period if its value at the end of the period, for *any* rate movement during the holding period, is at least as large as it would have been had the interest rate remained constant during the period. The idea of immunization is due to Redington in 1952 [638].

Amazingly, the answer to the above question is as elegant as it is simple: Construct a bond or a bond portfolio whose Macaulay duration is equal to the length of the investment horizon and whose present value is equal to the present value of the single future liability [305]. This means a liability of \$100,000 twelve years from now should be matched by a portfolio with a Macaulay duration of twelve years and with a future value of \$100,000.

When these two conditions are satisfied, losses from the interest on interest will be compensated by gains in the sale price when the interest rate falls, and losses from the sale price will be compensated by the gains in the interest on interest when the interest rate rises. Figure 4.5 illustrates the connection between Macaulay duration and immunization.

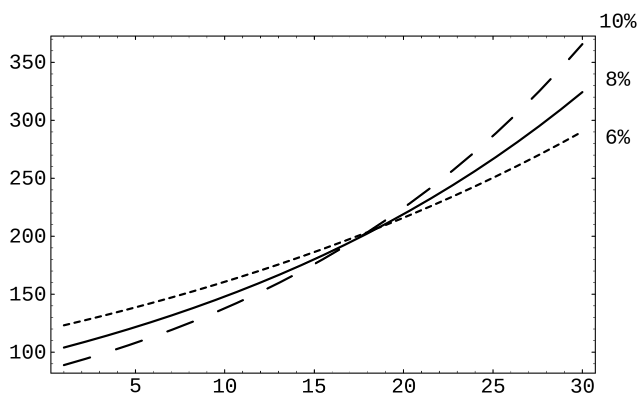

Figure 4.5: VALUES OF AN 8%, 15-YEAR BOND UNDER THREE INTEREST RATE SCENARIOS. Plotted is the value of an 8%, 15-year bond at every period until maturity when the interest rate is unchanged at 8% (solid line), increased by 25% instantaneously to 10% (dashed line), and decreased by 25% instantaneously to 6% (dotted line). At the time which corresponds to the Macaulay duration $m = 17.9837$ (half years), the curves converge. At any time before m, a rate decline adds to the return and a rate increase subtracts from the return; the opposite holds at any time after m. The Macaulay duration corresponds to that point in time at which small interest rate changes now will leave the future value relatively unchanged [89].

The above claim can be verified as follows. Suppose the liability is a certain L at time m and the current interest rate is y. We are looking for a coupon bond such that

(1) its future value, FV, is L at time m;

(2) $\partial \text{FV}/\partial y = 0$;

(3) FV is convex for all $y > 0$.

Condition (1) says the bond has to meet the obligation. Conditions (2) and (3) say the obligation has to be met whichever way the interest rate moves. These conditions together mean the liability L is the bond's minimum future value: The bond's future value will be at least L at the investment horizon.

Let

$$\text{FV} = L = (1+y)^m P$$

and P be the present value of L discounted at the current interest rate y. Now,

$$\frac{\partial \text{FV}}{\partial y} = m\,(1+y)^{m-1} P + (1+y)^m \frac{\partial P}{\partial y}. \tag{4.11}$$

The condition $\partial \text{FV}/\partial y = 0$ leads to

$$m = -(1+y)\,\frac{\partial P/P}{\partial y}. \tag{4.12}$$

This simple identity says the Macaulay duration is equal to the length of the investment horizon m. The y that satisfies (4.12) is called a critical point. Finally, since

$$\text{FV} = \sum_{i=1}^{n} \frac{C}{(1+y)^{i-m}} + \frac{F}{(1+y)^{n-m}},$$

we have

$$\frac{\partial^2 \text{FV}}{\partial y^2} = \sum_{i=1}^{n} \frac{(m-i)(m-i-1)C}{(1+y)^{i-m+2}} + \frac{(m-n)(m-n-1)F}{(1+y)^{n-m+2}}, \tag{4.13}$$

which is positive for $y > -1$ because $(m-i)(m-i+1)$ is either zero or positive. Note that FV is convex for every $y > -1$, not just around the critical point where the Macaulay duration is equal to the investment horizon. As a result, the critical point is actually the **global** minimum of FV for $y > -1$ (see Fig. 4.6). Hence (3) is satisfied automatically.

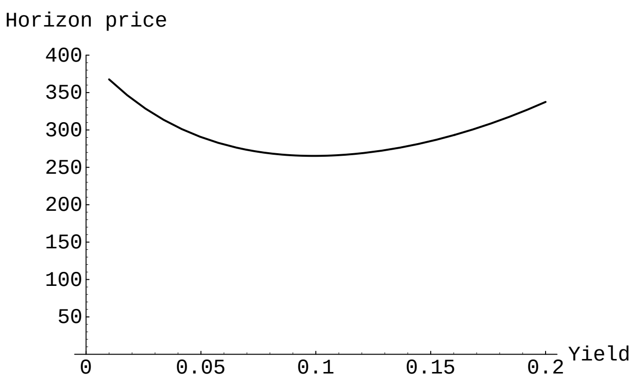

Figure 4.6: IMMUNIZATION AND INVESTMENT HORIZON. Plotted is the future value of a bond at the investment horizon. The yield at which the graph is minimized equates the bond's Macaulay duration with the horizon. In this example, the bond pays semiannual coupons at an annual rate of 10% for 30 years. The investment horizon is ten years. The future value is minimized at $y = 9.90878\%$. One can verify that the Macaulay duration at the annual yield of 9.90878% is exactly 10 years.

We conclude that, for a portfolio whose Macaulay duration is equal to the investment horizon, its future value at the horizon can only increase if the interest rate moves. Note that this is true regardless of the direction and the size of the change in the interest rate, as long as the rate remains positive. The implication for immunization is thus established.

If there is no single bond whose Macaulay duration matches our liability, a portfolio of bonds will do. Suppose we look for a portfolio of two bonds A and B which gives the Macaulay duration sought for. Mathematically, we solve the following set of linear equations

$$\begin{aligned} 1 &= w_A + w_B \\ D &= w_A D_A + w_B D_B \end{aligned} \tag{4.14}$$

for w_A and w_B, where D_i is the Macaulay duration of bond i and w_i is the (to be determined) market value of bond i expressed as a percentage of the market value of the portfolio. That this works follows from (4.6).

We have been dealing with immunizing a *single* liability. The extension to multiple liabilities can be carried along the same line. But we make a few comments first. Conditions (1)–(3) still constitute the sufficient conditions for immunization. However, the convexity condition (3) no longer holds automatically. In fact, even if it happens to hold near the current interest rate y, it may not hold globally for all $y > 0$ as before, rendering the immunization valid merely for *small* interest rate movements. We illustrate this point by an example. Consider a liability stream with two payments. Employ the above analysis to immunize this liability stream using a zero-coupon bond with a matching Macaulay duration. Suppose the first liability is due imminently and the interest rate rises instantaneously. Since a zero-coupon bond does not have interest income, the higher rate lowers the asset value without helping it to meet the second liability. As a result, depending on the magnitude of the interest rate movement, the second liability may not be met when viewed at that point in time.

Let there be liabilities of size L_i at time i and cash inflows A_i at time i. The net present value of these cash flows is

$$P = \sum_i (A_i - L_i)(1+y)^{-i}.$$

Conditions (1)–(3) require that $P = 0$, $dP/dy = 0$, and $d^2P/dy^2 > 0$ for $y > 0$, respectively. These three requirements guarantee that the cash inflows suffice to cover the liabilities no matter how the rate moves instantaneously, as long as it remains non-negative. If condition (3) is relaxed to "$d^2P/dy^2 > 0$ for the critical point y," a more likely event, then the immunization is valid only for a small rate movement around the current rate y. In this more general setting, the distribution of durations of individual assets must have a wider range than that of the liabilities to achieve immunization (see Exercise 4.2.9).

Of course, a stream of liabilities can always be immunized with a matching stream of zero-coupon bonds. This is called **cash matching**, and the resulting portfolio of bonds is called **dedicated portfolio** [699]. There are two problems with this approach: Zero-coupon bonds may be missing for certain maturities, and they typically carry lower yields.

Immunization is a dynamic process. Once established, it has to be **rebalanced** continuously to ensure that the Macaulay duration is equal to the remainder of the horizon. There are three reasons. First, the Macaulay duration decreases as time passes, and, except for zero-coupon bonds, the decrement is not equal to the decrement in terms to maturity [185]. This phenomenon is called **duration drift** [213]. This point can be easily seen by considering a coupon bond whose Macaulay duration matches the investment horizon. This bond's maturity date hence lies beyond the investment horizon. At horizon, the remaining term to horizon reaches zero, but the bond's duration is still positive. Hence, immunization needs to be re-established even if interest rate never changes. Another reason is that interest rate will fluctuate during the holding period, while in our derivation of the conditions for immunization, it was assumed that interest rate changes *instantaneously* after immunization is established and then stays there. One further reason for the need of rebalancing arises from the possibility that the duration of assets and liabilities may not change at the same rate [596].

Comment 4.2.4 In Comment 4.2.1, it was asserted that Macaulay duration (4.2) is applicable only when the cash flow does not depend on interest rates. The steps leading to (4.12), however, can be used to generalize the concept of Macaulay duration as the point in time at which the future value of the security is immune to changes in interest rate today. Figure 4.5 demonstrates this point clearly. □

In the absence of interest rate changes, it suffices to match the present values of the future liability and the investment. To see this point, assume again the liability is L at time m. The present value is therefore $L/(1+y)^m$. A coupon bond with an equal present value will grow to be exactly L at time m. In fact, it is not hard to show that, at any point in time, the present value of the bond plus the cash incurred by reinvesting the coupon payments exactly matches the present value of the liability. It was assumed above that the yield per period y does not change over time, and all the coupon payments were reinvested at the same yield.

When the duration of the liabilities and that of the assets are mismatched, adverse interest rate movements can quickly wipe out the equity. A bank that finances long-term mortgage investments with short-term credit from the savings accounts runs such a risk. Other institutions that require duration matching are pension funds and life insurance companies [666].

4.2.3 Macaulay duration of floating-rate instruments

A **floating-rate instrument** makes interest rate payment based on some publicized index, such as the prime rate, LIBOR (London Interbank Offered Rate), the U.S. Treasury bill rate, the constant maturity Treasury rate, or the Cost of Funds Index [303]. Instead of being locked into a set number, the coupon rate is **reset** periodically to reflect the prevailing market interest rate. Herein lies their attractiveness, especially in periods of abnormally high interest rates.

Start with the simple case where the coupon rate c equals the market yield y. For simplicity, we work in period rates instead of annual rates and, without loss of generality, assume the principal is \$1. The cash flow is therefore

$$(\overbrace{y, y, \ldots, y, y+1}^{n}).$$

This cash flow implies the instrument is priced at par.

Suppose the first reset date is j periods from now where $0 \leq j \leq n-1$, and reset will be performed thereafter. This means the coupon payment at time $j+1$ starts to reflect the market yield. For example, if $j=0$, every coupon payment reflects the prevailing market interest. On the other hand, if $j=1$, a more typical case, interest rate movements during the first period would not affect the first coupon payment. The cash flow can be seen as

$$(\overbrace{c, c, \ldots, c}^{j}, \overbrace{y, \ldots, y, y+1}^{n-j}),$$

where c is a constant and $y=c$.

The Macaulay duration of such a floating-rate instrument is therefore

$$\begin{aligned} -(1+y)\left.\frac{\partial P/P}{\partial y}\right|_{c=y} &= \sum_{i=1}^{j}\left(i\,\frac{y}{(1+y)^i}\right) + \sum_{i=j+1}^{n}\left(i\,\frac{y}{(1+y)^i} - \frac{1}{(1+y)^{i-1}}\right) + n\,\frac{1}{(1+y)^n} \\ &= \text{MD} - \sum_{i=j+1}^{n}\frac{1}{(1+y)^{i-1}} = \frac{(1+y)\left(1-(1+y)^{-j}\right)}{y}, \end{aligned} \qquad (4.15)$$

where MD is the Macaulay duration of an otherwise identical fixed-rate instrument. Formulae for the general case are more complex but do not involve new ideas [303].

The first thing to note is that the duration is independent of n, the maturity of the bond. Floating-rate instruments are typically less sensitive to interest rate changes than fixed-rate instruments. Furthermore, the more distant the first reset date, the more volatile the instrument, which is intuitively obvious. In the limiting case of $j = 0$, the duration becomes zero. If every coupon is adjusted to reflect the market yield, then there is no interest rate risk. In the typical $j = 1$ case, the duration is only one (period). In comparison, a bond that pays 5% per period for 30 periods has a duration of 16.14 periods, or roughly eight years if this 15-year bond pays semiannual interest. The above observations partially explain the attractiveness of floating-rate instruments.

4.2.4 Applications to hedging

Hedging aims at offsetting the price fluctuations of the position to be hedged by the hedging instruments in the opposite direction, leaving the overall wealth relatively unchanged [189]. Since a change in the market interest rate may not bring about the same yield changes in different instruments, we define **yield beta** to be

$$\text{yield beta} \equiv \frac{\text{change in yield for the hedged security}}{\text{change in yield for the hedging instrument}},$$

which measures their relative yield changes. Define **dollar duration** as

$$\text{dollar duration} \equiv \text{modified duration} \times \text{initial price} \ (\% \text{ of par}) = -\frac{\partial P}{\partial y},$$

where y is the yield and P is the price as a percentage of par. A tangent on the price-yield curve such as the one shown in Fig. 3.8 therefore denotes the dollar duration at a given yield (modulo the sign). The approximate dollar price change per \$100 of par value can then by computed by

$$\text{price change} \approx -\text{dollar duration} \times \text{yield change}.$$

The related **price value of a basis point** or simply **basis point value** (**BPV**), defined as the dollar duration divided by 10,000, measures the price change for a one basis-point change in the interest rate. One **basis point** equals 0.01%.

The **hedge ratio** is defined as

$$h \equiv \frac{\text{dollar duration of the hedged security}}{\text{dollar dration of the hedging security}} \times \text{yield beta}. \tag{4.16}$$

Hedging is accomplished when the value of the hedging security is h times that of the hedged security since, then,

$$\begin{aligned} &\text{dollar price change of the hedged security} \\ = \; &-\text{hedge ratio} \times \text{dollar price change of the hedging security}. \end{aligned}$$

Example 4.2.5 Suppose an investor wants to hedge bond A with a duration of seven using bond B with a duration of eight. For simplicity, assume the yield beta is one and both bonds are selling at par. The hedge ratio is hence 7/8. This means an investor who is long \$1 million of bond A should short \$7/8 million of bond B. □

4.3 Convexity

Recall

$$\textsf{percentage change in price } (\Delta P/P) \approx -\textsf{ duration} \times \textsf{yield change}.$$

The above formula works for very small yield changes. To provide better approximations when yield changes are larger, second-order terms in the Taylor expansion are needed,

$$\frac{\Delta P}{P} \approx \frac{\partial P}{\partial y}\frac{1}{P}\Delta y + \frac{1}{2}\frac{\partial^2 P}{\partial y^2}\frac{1}{P}(\Delta y)^2.$$

If we define **convexity** as

$$\textsf{convexity (in periods)} \equiv \frac{\partial^2 P}{\partial y^2}\frac{1}{P}, \tag{4.17}$$

then the improved approximation formula becomes

$$\frac{\Delta P}{P} \approx -\textsf{duration} \times \Delta y + \frac{1}{2} \times \textsf{convexity} \times (\Delta y)^2.$$

See Fig. 4.7 for illustration.

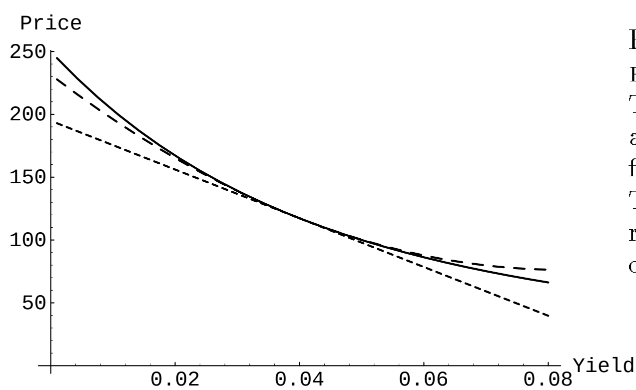

Figure 4.7: LINEAR AND QUADRATIC APPROXIMATIONS TO BOND PRICE CHANGES. The dotted line is the result of duration-based approximation, while the dashed line, which fits better, utilizes the convexity information. The bond in question has 30 periods to maturity with a period coupon rate of 5%, and the current yield is 4% per period.

Convexity should be considered in computing the price change when the magnitude of the interest rate change is non-negligible. Graphically, convexity measures the curvature of the price-yield relationship. Convexity measured in periods and convexity measured in years are related by

$$\textsf{convexity (in years)} = \frac{\textsf{convexity (in periods)}}{k^2},$$

where k is the payment frequency.

Convexity is related to duration. Define **dollar convexity** as

$$\textsf{dollar convexity} \equiv \textsf{convexity} \times \textsf{initial price} \; (\% \text{ of par}) = \frac{\partial^2 P}{\partial y^2}.$$

Since dollar duration equals $-\partial P/\partial y$, dollar convexity measures the rate of change of dollar duration,

$$\textsf{dollar convexity} = -\frac{\partial(\textsf{dollar duration})}{\partial y}.$$

The convexity of an option-free bond equals

$$\frac{1}{P}\left[\sum_{i=1}^{n} i\,(i+1)\,\frac{C}{(1+y)^{i+2}} + n\,(n+1)\,\frac{F}{(1+y)^{n+2}}\right],$$

which can be simplified to

$$\frac{1}{P}\left[\frac{2C}{y^3}\left(1-\frac{1}{(1+y)^n}\right)-\frac{2Cn}{y^2(1+y)^{n+1}}+\frac{n\,(n+1)\,(F-(C/y))}{(1+y)^{n+2}}\right]. \tag{4.18}$$

This formula makes computing convexity easy.

The convexity of an option-free bond is clearly positive. As a consequence, the percentage price change is always positive whether the interest rate movement is up or down. Positive convexity is a plus for an investor who is long bonds because the price does not decline as much when the interest rate increases and it increases more than proportionately when the interest rate decreases. In other words, the price rises more for a given rate decline than it falls for a similar rate increase. Hence, among two bonds with the same duration, the one with a higher convexity is more valuable, other things being equal.

In analogy with (4.9), the convexity under continuous compounding is

$$\textsf{convexity} \text{ (continuous compounding)} \equiv \frac{\sum_i t_i^2\, C_i e^{-y t_i}}{P} = \frac{\partial^2 P/P}{\partial y^2},$$

where t_i denotes the time when the cash flow C_i occurs. This formulation assumes C_i is independent of yield y. It can be shown that the convexity of an option-free bond increases as its coupon rate decreases (see Exercise 4.3.4). Furthermore, for a given yield and duration, the convexity decreases as the coupon decreases [283].

A general formula for convexity is

$$\frac{P_+ + P_- - 2P_0}{P_0\,(0.5\,(y_+ - y_-))^2}, \tag{4.19}$$

where P_- is the price if yield is decreased by Δy, P_+ is the price if yield is increased by Δy, P_0 is the initial price, y is the initial yield, $y_+ \equiv y + \Delta y$, $y_- \equiv y - \Delta y$, and Δy is sufficiently small. (Note that $(y_+ - y_-)/2 = \Delta y$.) This is called **effective convexity**. Effective convexity becomes essential when, for example, a bond's cash flow is interest rate-sensitive.

4.3.1 Immunization and convexity

The two-bond immunization scheme in §4.2.2 clearly shows that many two-bond portfolios (equivalently, (D_A, D_B) pairs) satisfy the linear equations of (4.14). The question naturally arises as to which pair is to be preferred.

As convexity is a desirable feature, we turn the question into one of maximizing the portfolio convexity among all the duration-matched portfolios. Let there be n kinds of bonds available with bond i having duration D_i and convexity C_i. We shall be solving the following constrained optimization problem,

$$\begin{array}{ll} \text{Maximize} & w_1 C_1 + w_2 C_2 + \cdots + w_n C_n \\ \text{subject to} & 1 = w_1 + w_2 + \cdots + w_n \\ & D = w_1 D_1 + w_2 D_2 + \cdots + w_n D_n \\ & 0 \leq D_l \leq D_1 < D_2 < \cdots < D_n \leq D_u. \end{array}$$

The function to be optimized, $w_1C_1 + w_2C_2 + \cdots + w_nC_n$, is called the **objective function**, and the other equalities and inequalities make up the **constraints**. Note that, typically, $D_l = 0$ (cash) and $D_u = 30$ (30-year zeros). We shall further impose $0 \leq w_i \leq 1$. Incidentally, the above optimization problem is also a **linear programming problem** as all the functions are linear. The optimal solution usually implies a **barbell portfolio**, so called because the portfolio contains bonds at the two extreme ends of the duration spectrum (see Exercise 4.3.6). Many fundamental problems in finance and economics are best cast as optimization problems [214, 239, 243, 791].

4.4 Concluding Remarks and Additional Reading

Duration and convexity measure the risk of changes in interest rate levels. Other types of risks such as the frequency of large movements in interest rates are ignored [533]. They furthermore assume parallel shifts in the yield curve, whereas yield changes are not always parallel in reality (we will say more about this issue in Chapter 5).

Closed-form formulae for duration and convexity can be found in [83, 178]. See [451] for a penetrating review. Additional immunization techniques can be found in [175, 283, 479]. See [182, 243, 477] for more information on linear programming.