Chapter 16

Hedging

Contents

When we are still wearing our heavy clothes for protection against the cold of winter, not only are ready-made spring clothes already on the way to retail stores, but in factories light clothes are being woven which we will wear next summer, while yarns are being spun for the heavy clothing we will use the following winter.
—Carl Menger (1840–1921),
Principles of Economics [568, p. 79]

Hedging strategies made their appearances throughout this book. How can it be otherwise when one of the principal uses of derivatives is in the management of risks? In this chapter, we focus on hedging with non-interest rate derivatives. The issue of hedging with interest rate derivatives will be covered in Chapter 21.

16.1 Introduction

A **hedge** is a position that offsets the price risk of some other position. For hedging to be possible, the returns on the two positions must be correlated. In fact, the more correlated their returns are, the more effective the hedge will be [557]. This has implications for the design of financial instruments. In the case of futures contract, for instance, the contract should not be too broad (including too many deliverable grades) or too narrow [420].

One common thread throughout this book has been the management of risks. **Risk management** means selecting and maintaining portfolios with defined exposure to risks. Of course, part of the job is to decide which risks one is to be exposed to and which risks to be protected against. For instance, a hedge reduces or even eliminates risk exposure. The ideal instrument is one that provides cash flows equal but opposite to the existing exposure. We will illustrate the simple principle with examples from various derivative products. In an efficient market, speculation has zero expected return. The costs of hedging are therefore not large.

16.2 Hedging and Futures

The most straightforward way of hedging uses forward contracts (buy 2 million Japanese yen one month forward, say). Due to daily settlements, futures contracts are harder to analyze than forward contracts. Luckily, the forward price and futures price are generally very close to each other when the maturities of the two contracts are identical. Results obtained for forward prices can therefore be assumed to be true for futures prices as well.

16.2.1 Futures and spot prices

Two forces prevent the prevailing prices in the spot market and the futures market at any time from diverging too much. One is the delivery mechanism and the other is hedging. Hedging relates the futures price and the spot price through arbitrage between the two markets. If the futures price exceeds the spot price by more than the carrying charges, hedgers can buy in the spot market and sell futures without risk (see Exercise 12.3.7). Carrying charges, we recall, are the cost of holding physical inventories between now and the maturity of the futures contract. It follows that the futures price cannot exceed the prevailing spot price by more than the carrying charges. In practice, the futures price does not necessarily exceed the spot price by exactly the relevant carrying charges because inventories have what Kaldor termed the convenience yield derived from their availability when buyers need them [420].

16.2.2 Hedgers, speculators, and arbitrageurs

A company that is due to sell an asset at a particular time in the future can hedge by taking a short futures position. This is known as a **selling** or **short** hedge. The purpose is to lock in a selling price or, with fixed-income securities, a yield. If the price of the asset goes down, the company does not fare well on the sale of the asset, but makes a gain on the short futures position. If the price of the asset goes up, the reverse is true. Clearly, selling hedge is a temporary substitute for a later cash market sale of the underlying asset. A company due to buy an asset in the future can hedge by taking a long futures position. This is known as a **buying** or **long** hedge. Clearly, buying hedge is used when one plans to buy the cash asset at a later date. The purpose is to establish a fixed purchase price. Such strategies work because spot and futures prices tend to move in the same direction by roughly equal amounts, reacting to the same economic factors. They are correlated, in other words.

A person who gains or loses from the difference between the spot and futures prices is said to **speculate on the basis**. Simultaneous purchase and sale of futures contracts on two different, but related, assets is referred to as a **spread**. An investor who speculates by using spreads is called a **spreader** [88, 699].

Hedgers buy and sell futures contracts to offset risky positions in the spot market; speculators bet on price movements and hope to make a profit; **arbitrageurs** lock in riskless profits by simultaneously entering into transactions in two or more markets, an act called arbitrage.

A hedger is someone whose net position in the spot market is offset by positions in the futures market. A **short hedger** is long in the spot market and short in the futures market. A **long hedger** does the opposite. Those who are net long or net short overall are speculators. Speculators will buy (sell) futures contracts only if they expect prices to increase (decrease). Hedgers, in comparison, are willing to pay a premium to unload unwanted risk onto speculators. They provide the market with liquidity, enabling hedgers to buy or sell large numbers of contracts without adversely disrupting the price [88].

If hedgers in aggregate are short, then speculators are net long, and the futures price will be set below the expected future spot price (backwardation). On the other hand, if hedgers are net long, speculators will be net short, and the futures price will be set above the expected future spot price (contango). There seems to be evidence that short hedging exceeds long hedging in most of the markets most of the time. The basic reason advanced for this asymmetry is that the net position in the spot market (hence the overall market as well) is positive. Now, if the hedgers are net short in the futures, the speculators must be net long. It has been theorized that speculators will only be net long if the futures price is expected to rise until it equals the spot price at maturity. Speculators therefore extract a risk premium from hedgers. This is Keynes's theory of normal backwardation. By this theory, the futures price underestimates the future spot price [420]. See [257, 422] for surveys on this long-standing subject.

16.2.3 Perfect and imperfect hedging

A number of factors make hedging with futures contracts less than perfect. The asset whose price is to be hedged may not be identical to the asset underlying the futures contract; the date when the asset is to be transacted may be uncertain; the hedge may require the futures contract to be closed out before its expiration date. These problems give rise to basis risk [422]. Of course, basis risk does not exist in situations where the spot price relative to the futures price always moves in predictable manners.

Consider an investor who plans on selling an asset in t years. To eliminate some price uncertainty, the investor sells futures contracts on the same asset with a delivery date in T years. After t years, the investor liquidates the futures position and sells the asset as planned. The cash flow at that time is

$$S_t - (F_t - F) = F + (S_t - F_t) = F + \textsf{basis}, \tag{16.1}$$

where S_t is the spot price at time t, F_t is the futures price at time t, and F is the original futures price. (We treat futures contracts as if they were forward contracts.) Note that we could achieve a **perfect hedge** if $t = T$, that is, when there is a futures contract with a matching delivery date. Observe that the investor has replaced the uncertainty of the price with the smaller uncertainty of the basis; as a consequence, risk has been reduced.

If, furthermore, the cost of carry as well as the convenience yield is predictable, then the cash flow can be anticipated with complete confidence according to (12.15). This holds even under a mismatched maturity $t \neq T$: We will show that a hedging strategy can still be set up to eliminate risk if (1) the interest rate r is known and (2) the cost of carry c and the convenience yield y are constants. In this case, $F_t = S_t e^{(r+c-y)(T-t)}$ by (12.13). Now, let h be the number of futures contracts sold initially. The cash flow at time t, after liquidating the futures position and selling the asset, is

$$S_t - h\,(F_t - F) = S_t - h\left(S_t e^{(r+c-y)(T-t)} - F\right).$$

To eliminate any uncertainty associated with S_t, pick

$$h = e^{-(r+c-y)(T-t)}$$

and the cash flow becomes a constant, hF. Note that $h = 1$ may not be the best choice. The above choice becomes $h = 1$ if there is no maturity mismatch, i.e., $t = T$. The number h is the hedge ratio.

Although the above arguments hint at choosing a futures contract with a delivery date close to the trading date, holding onto a futures position in the delivery month runs some risk such as taking delivery at a location that is inconvenient. The delivery risk can be avoided by canceling the positions before the settlement month begins. Hedgers usually choose a futures contract that has a slightly longer maturity than the holding period [650].

Cross hedge

Hedges may have either an asset mismatch or a maturity mismatch. A hedge that is established with a mismatched maturity, a mismatched asset, or both is referred to as a **cross hedge** [650]. When firms want to hedge against price movements in a commodity for which there are no futures contracts, they can use futures contracts on related commodities whose price movements closely correlate with the price to be hedged.

Example 16.2.1 To hedge a future purchase price of 10,000,000 (Dutch) guilders for which no guilder futures exist, one may proceed as follows. Suppose the current exchange rate is \$0.48 per guilder. At this rate, the dollar cost is \$4,800,000. A regression analysis is conducted between the daily change in the guilder rate and the daily change in the nearby German mark futures. Say the estimated slope is 0.95 with an R^2 of 0.92. Now that the guilder is highly correlated with the German mark, we decide to use German mark futures. The current exchange rate is \$0.55/DEM1; hence the commitment of 10,000,000 guilders translates to $4,800,000/0.55 = 8,727,273$ German marks. Since each futures contract controls 125,000 marks (review Fig. 12.3), the number of futures contracts to trade is $0.95 \times 8,727,273/125,000 \approx 66$. □

Example 16.2.2 A British firm expecting to pay DEM2,000,000 for purchases in three months would like to lock into the price in pounds. Besides the standard way of using mark futures contracts that trade in pounds, the firm can trade mark and pound futures contracts that trade in U.S. dollars as follows. Each mark futures contract controls 125,000 marks, and each pound futures contract controls 62,500 pounds. Suppose the payment date coincides with the last trading date of the currency futures contract at CME. Currently, the three-month mark futures price is \$0.7147 and the pound futures price is \$1.5734. The firm buys $2,000,000/125,000 = 16$ mark futures contracts, locking into a purchase price of \$1,429,400. To further lock into the price in pounds at the exchange rate of \$1.5734/£1, the firm shorts $1,429,400/(1.5734 \times 62,500) \approx 15$ pound futures contracts. The end result is a purchase price of

$$2,000,000 \times \frac{0.7147}{1.5734} = 908,478$$

pounds with the cross rate £0.45424/DEM1. □

16.2.4 Establishing hedge ratio

Up to now we have assumed a hedge ratio of one in the case of matched maturity. In general and under the assumption that the hedger seeks to minimize risk for a given time horizon, a hedge ratio other than one may result [422].

Let δ_S denote the standard deviation of S_t, δ_F the standard deviation of F_t, and ρ the correlation between S_t and F_t. For a short hedge, the cash flow at time t is

$$S_t - h\,(F_t - F),$$

while for a long hedge, it is

$$-S_t + h\,(F_t - F).$$

In either case, the variance V is given by

$$V = \delta_S^2 + h^2 \delta_F^2 - 2h\rho\delta_S\delta_F.$$

The optimal hedge ratio is the h that minimizes the variance of the cash flow of the hedged position. Since

$$\frac{\partial V}{\partial h} = 2h\delta_F^2 - 2\rho\delta_S\delta_F,$$

the h that minimizes V is

$$h = \rho\,\frac{\delta_S}{\delta_F}. \tag{16.2}$$

Example 16.2.3 Suppose the standard deviation of the change in the price per bushel of corn over a three-month period is 0.4, and the standard deviation of the change in the soybeans futures price over a three-month period is 0.3. Suppose further that the correlation between the three-month change in the price of corn and the three-month change in the soybeans futures price is 0.9. Consider a company expected to buy one 1,000,000 bushels of corn in three months. Assume, for some reason, the company decides to hedge by buying futures contracts on soybeans. The optimal hedge ratio is

$$0.9 \times \frac{0.4}{0.3} = 1.2.$$

Since the size of one soybeans futures contract is 5,000 bushels, the company should buy $1.2 \times (1000000/5000) = 240$ contracts. □

Estimating hedge ratio

Hedge ratio can be estimated as follows. Suppose $\{ S_1, S_2, \ldots, S_t \}$ and $\{ F_1, F_2, \ldots, F_t \}$ are the time series for the daily closing spot and futures prices. Define $\Delta S_i \equiv S_{i+1} - S_i$ and $\Delta F_i \equiv F_{i+1} - F_i$, respectively. Then ρ, δ_S, and δ_F can be estimated by (6.2) and (6.24).

16.2.5 Hedging with stock index futures

Stock index futures can be used to hedge a well-diversified portfolio of stocks. According to the Capital Asset Pricing Model, the relationship between the return on a portfolio of stocks and the return on the market can be described by a parameter β, called **beta**. This is the slope of the best fit line obtained when the excess return on the portfolio over the riskless rate is regressed against the excess return on the market over the riskless rate. Consider the value of a portfolio during a period of time. Then it is approximately true that

$$\Delta_1 = \alpha + \beta \times \Delta_2,$$

where Δ_1 is the change in the value of $1 during the period if it is invested in the portfolio, and Δ_2 is the change in the value of $1 during the same period if it is invested in the market index. Here, α is some constant. The change in the value of the portfolio during the period is therefore

$$S \times \Delta_1 = S \times \alpha + S \times \beta \times \Delta_2,$$

where S denotes the current value of the portfolio. The change in the value of one futures contract during this time is approximately $F \times \Delta_2$, where F is the current value of one futures contract. Note that the value of one futures contract is equal to the futures price multiplied by the contract size. For example, one futures contract on the S&P 500 Index is on $500 times the index; hence, if the futures price of the S&P 500 is 1000, the value of one contract would be $1000 \times 500 = 500,000$.

The uncertain component of the change in the value of the portfolio, $S \times \beta \times \Delta_2$, is approximately $\beta S/F$ times the change in the value of *one* futures contract, $F \times \Delta_2$. The optimal number of futures contracts to short when hedging the portfolio is thus $\beta S/F$ [422]. This strategy is called **portfolio immunization**. The same idea can be applied to change the beta of a portfolio to any desired level. Specifically, to changing the beta from β_1 to β_2 requires shorting

$$(\beta_1 - \beta_2)\,\frac{S}{F} \tag{16.3}$$

contracts. A portfolio, perfectly hedged, has a beta of zero by definition and corresponds to choosing $\beta_2 = \beta_1$.

Example 16.2.4 Hedging a well-diversified stock portfolio with the S&P 500 Index futures works as follows. Suppose the portfolio in question is worth \$2,400,000 with a beta of 1.25 against the returns on the S&P 500 Index. So, for every 1% advance in the index, the expected advance in the portfolio is 1.25%. With the current futures price at 1200,

$$1.25 \times \frac{2,400,000}{1200 \times 500} = 5$$

futures contracts need to be sold short. □

16.3 Hedging and Options

This section discusses dynamic hedging strategies for options and hedging strategies in connection with options.

16.3.1 Delta hedge

The delta (hedge ratio) of a derivative is defined as $\Delta \equiv \partial f / \partial S$. This means

$$\Delta f \approx \Delta \times \Delta S$$

for relatively small changes in stock price, ΔS, other things being equal. The delta of the underlying stock is of course one. A delta-neutral portfolio is hedged in the sense that it is immunized against instantaneous, small changes in the stock price. A trading strategy that dynamically maintains a delta-neutral portfolio is called a **delta hedge**.

Since delta changes with the stock price, a delta hedge needs to be rebalanced periodically in order to maintain delta neutrality. In the limit where the portfolio is adjusted continuously, perfect hedge is achieved and the strategy becomes self-financing [256]. See Fig. 16.1 for illustration. This was the gist of the Black-Merton-Scholes argument for the continuous-time model in §15.2.1.

For non-dividend-paying stocks, the delta-neutral portfolio hedges N short derivatives with $N \times \Delta$ shares of the underlying stock plus B borrowed dollars such that

$$0 = -N \times f + N \times \Delta \times S - B. \tag{16.4}$$

We call it the **self-financing condition** since the combined portfolio has zero value. At the time of rebalancing when the delta is Δ', buy $N \times (\Delta' - \Delta)$ shares to maintain a level of $N \times \Delta'$ shares with a total borrowing of $B' = N \times \Delta' \times S' - N \times f'$, where f' is the derivative's new price. This procedure is repeated at each rebalancing point.

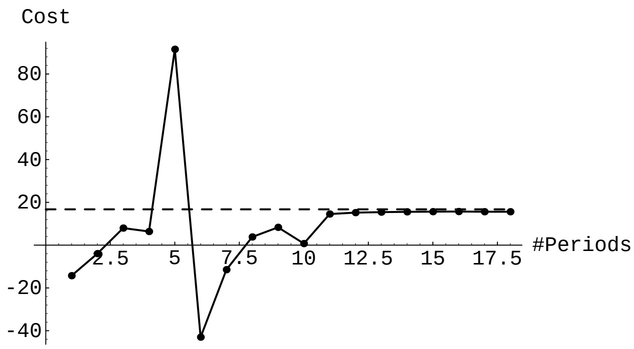

Figure 16.1: TRACKING ERROR OF DELTA HEDGE. Each point is the result of a random sample path of stock prices starting from the current price, \$100. At only 12 partitions per year, the tracking error is already within 10% of the analytical value, 16.7375, which is plotted as a horizontal line for reference. The option is identical to the one used in Fig. 15.1.

Weeks to expiration	Stock price S (1)	Option value f (2)	Delta Δ (3)	Gamma Γ (4)	Change in delta (3)−(3″) (5)	No. shares bought $N\times$(5) (6)	Cost of shares (1)×(6) (7)	Cumulative cost FV(8″)+(7) (8)
4	50	1.76791	0.538560	0.0957074	—	538,560	26,928,000	26,928,000
3	51	2.10580	0.640355	0.1020470	0.101795	101,795	5,191,545	32,150,634
2	53	3.35087	0.855780	0.0730278	0.215425	215,425	11,417,525	43,605,277
1	52	2.24272	0.839825	0.1128601	−0.015955	−15,955	−829,660	42,825,960
0	54	4.00000	1.000000	0.0000000	0.160175	160,175	8,649,450	51,524,853

Figure 16.2: DELTA HEDGE. The cumulative cost reflects the part of buying stocks to maintain delta neutrality—or, equivalently, to replicate the options. The total number of shares ends up $N = 1,000,000$ at expiration since it is assumed that trading takes place at expiration, too. A doubly primed number refers to the entry from the *previous* row.

It must be emphasized that delta hedge is a discrete-time analogue of the continuous-time limit and will not be self-financing except by coincidence. In other words, the proceeds from the sale of stock may be insufficient or more than enough to maintain the net position of borrowing B' dollars, and funds may be injected or withdrawn. Delta hedge aims to minimize this variation, not to generate a huge profit in the end.

A numerical example

Let us illustrate the procedure with the European call. The delta is positive and increases as the stock price increases by (10.1). In maintaining delta neutrality, the hedger buys stock if the stock price rises and sells stock if it falls. It is very important to look at the following example from both the viewpoints of option replication (discussed below under cumulative cost) and hedging (discussed below under net borrowing). Equivalent as these two views are, covering both has the benefit of illuminating the key ideas more clearly.

Consider a trader who is short 10,000 calls. Since an option covers 100 shares of stock, $N = 1,000,000$. This option's expiration is four weeks away, and its strike price is \$50. The underlying stock has an annual volatility of 30%, and the annual riskless rate is 6%. Assume the trader adjusts the portfolio weekly. We will track how the portfolio changes under the stock price scenario in Fig. 16.2.

At the current stock price of \$50, the option value is $f = 1.76791$ with $\Delta = 0.538560$.

To set up the delta hedge, $N \times \Delta = 538,560$ shares are purchased at a total cost of

$$538,560 \times 50 = 26,928,000$$

dollars financed by borrowing

$$B = N \times \Delta \times S - N \times f = 26,928,000 - 1,767,910 = 25,160,090$$

dollars net. This makes the portfolio delta-neutral with zero net value.

The stock price rises to \$51 at three weeks to expiration. Since the new option value is $f' = 2.10580$, the portfolio is worth

$$-N \times f' + 538,560 \times 51 - B \times e^{0.06/52} = 171,622 \tag{16.5}$$

before rebalancing. That this number, call it R, is not zero demonstrates the point made earlier that delta hedge does not replicate the option perfectly; it is not self-financing as \$171,622 can be withdrawn. The magnitude of this error—the variation in the net portfolio value—can be mitigated if adjustments are made more frequently, say daily instead of weekly. It can be proved that R is positive about 68% of the time even though its expected value is essentially zero [98].

With a higher delta of $\Delta' = 0.640355$, the trader purchases $N \times (\Delta' - \Delta) = 101,795$ more shares to increase the number of shares to $N \times \Delta' = 640,355$. The cost is \$5,191,545. The cumulative cost,

$$26,928,000 \times e^{0.06/52} + 5,191,545 = 32,150,634,$$

is recorded in Fig. 16.2. The net borrowed amount, however, is

$$B' = 640,355 \times 51 - N \times f' = 30,552,305.$$

(The same number could have been arrived at via $B \times e^{0.06/52} + 5,191,545 + 171,622 = 30,552,305$.) The portfolio is delta-neutral with zero value. The remaining steps are tabulated in Fig. 16.2.

At expiration, the trader has 1,000,000 shares, which are exercised against by the in-the-money calls for \$50,000,000. The trader is left with an obligation of

$$51,524,853 - 50,000,000 = 1,524,853,$$

the cost of replicating the options. It is close to the future value of the call premium,

$$1,767,910 \times e^{0.06 \times 4/52} = 1,776,088,$$

and the net gain is $1,776,088 - 1,524,853 = 251,235$. Indeed, as Δt goes to zero, the final obligation should converge to the future value of the option premium (see Fig. 16.3). The alternative, yet equivalent, view looks at how closely the options can be hedged by tracking, instead of the cumulative cost, the net borrowing B and the weekly tracking errors thereof (see Exercise 16.3.3).

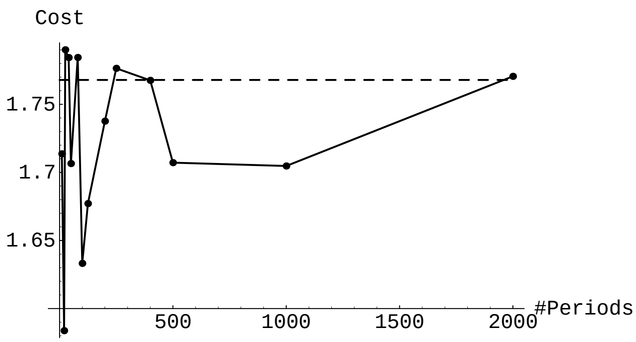

Figure 16.3: CONVERGENCE OF DELTA HEDGE. Each point is the result of an equidistributed sampling on the same stock price sample path. The present value of the cost of delta hedge is compared against the analytical value, 1.76791 per option, plotted as a horizontal line for reference.

In summary, had we started with

$$1,524,853 \times e^{-0.06 \times 4/52} = 1,517,831$$

dollars of our own money and borrowed \$26,928,000 to purchase 538,000 shares of stock, then the replication would have generated no net gain/loss at expiration. Although this conclusion holds only for the particular sample path of stock prices, the amount of money we started with in order to achieve perfect replication should converge to \$1,767,910 as Δt goes to zero.

16.3.2 Delta-gamma hedge

Delta hedge is based on the first-order approximation to changes in the derivative price due to small changes in the stock price ΔS. When ΔS is not small, the second-order term gamma Γ can help. Recall that gamma is defined as $\partial^2 f / \partial S^2$. The gamma of stock is zero.

Delta-gamma hedge is identical to delta hedge except that we now maintain gamma neutrality as well, meaning zero portfolio gamma. To meet this extra condition besides self-financing and delta neutrality, an extra security needs to be brought in. It should be emphasized that gamma neutrality addresses the practical concern that one cannot trade continuously.

A numerical example

The hedging procedure will be illustrated under the same scenario as in Fig. 16.2. A second option, called the hedging option, is added to the hedging process. Its specifications are listed in Fig. 16.4.

Again, our trader is short 10,000 European call options; hence $N = 1,000,000$. With the stock price at \$50, the option has value $f = 1.76791$, delta $\Delta = 0.538560$, and gamma $\Gamma = 0.0957074$. The hedging option has value $f_2 = 1.99113$, delta $\Delta_2 = 0.543095$, and gamma $\Gamma_2 = 0.085503$. (The deltas and gammas are drawn from Figs. 16.2 and 16.4.) To

Weeks to expiration	Stock price S	Option value f_2	Delta Δ_2	Gamma Γ_2
4	50	1.99113	0.543095	0.085503
3	51	2.35342	0.631360	0.089114
2	53	3.57143	0.814526	0.070197
1	52	2.53605	0.769410	0.099665
0	54	4.08225	0.971505	0.029099

Figure 16.4: SECOND OPTION USED IN DELTA-GAMMA HEDGE. This option is the same as the one used in Fig. 16.2 except that the expiration date is one week later.

set up a delta-gamma hedge, solve

$$\begin{aligned} 0 &= -N \times f + n_1 \times 50 + n_2 \times f_2 - B \quad &\text{(self-financing)} \\ 0 &= -N \times \Delta + n_1 + n_2 \times \Delta_2 - 0 \quad &\text{(delta neutrality)} \\ 0 &= -N \times \Gamma + 0 + n_2 \times \Gamma_2 - 0 \quad &\text{(gamma neutrality)} \end{aligned}$$

The solutions are $n_1 = -69,351$, $n_2 = 1,119,346$, and $B = -3,006,695$. This means we should short $69,351$ stocks, buy $1,119,346$ options, and lend $3,006,695$ dollars. The cost of shorting the stock and buying options is

$$n_1 \times 50 + n_2 \times f_2 = -1,238,787.$$

A week later, the stock climbs to \$51 a share. Since the new option values are $f' = 2.10580$ and $f_2' = 2.35342$, respectively, the portfolio is worth

$$-N \times f' + n_1 \times 51 + n_2 \times f_2' - B \times e^{0.06/52} = 1,757$$

before rebalancing. This number is not zero, confirming that the strategy is not self-financing. Nevertheless, it is substantially smaller than the corresponding number with delta hedge: 171,622 in (16.5). Now, we solve for

$$\begin{aligned} 0 &= -N \times f' + n_1' \times 51 + n_2' \times f_2' - B' \\ 0 &= -N \times \Delta' + n_1' + n_2' \times \Delta_2' - 0 \\ 0 &= -N \times \Gamma' + 0 + n_2' \times \Gamma_2' - 0 \end{aligned}$$

The solutions are $n_1' = -82,633$, $n_2' = 1,145,129$, and $B' = -3,625,138$. The trader purchases $n_1' - n_1 = -82,633 + 69,351 = -13,282$ shares and $n_2' - n_2 = 1,145,129 - 1,119,346 = 25,783$ options at a total cost of $-13,282 \times 51 + 25,783 \times f_2' = -616,704$ dollars. The cumulative cost is

$$-1,238,787 \times e^{0.06/52} - 616,704 = -1,856,921.$$

Our portfolio is again delta-neutral and gamma-neutral with zero value. The remaining steps are tabulated in Fig. 16.5.

At expiration, the trader owns 1,000,000 shares, which are exercised against by the in-the-money calls for \$50,000,000. The trader is left with an obligation of

$$51,832,134 - 50,000,000 = 1,832,134,$$

Weeks to expiration	Stock price S	No. shares bought $n_1' - n_1$	Cost of shares (1)×(2)	No. options bought $n_2' - n_2$	Cost of options (4)×f_2	Net borrowing B	Cumulative cost FV(7″)+(3)+(5)
	(1)	(2)	(3)	(4)	(5)	(6)	(7)
4	50	−69,351	−3,467,550	1,119,346	2,228,763	−3,006,695	−1,238,787
3	51	−13,282	−677,382	25,783	60,678	−3,625,138	−1,856,921
2	53	91,040	4,825,120	−104,802	−374,293	810,155	2,591,762
1	52	−39,858	−2,072,616	92,068	233,489	−1,006,346	755,627
0	54	1,031,451	55,698,354	−1,132,395	−4,622,720	50,000,000	51,832,134

Figure 16.5: DELTA-GAMMA HEDGE. The cumulative cost reflects the part of buying stocks and replicating options to maintain delta-gamma neutrality. The net borrowing column shows the values of B mentioned in the text. The total number of shares ends up $N = 1,000,000$ at expiration, while the total number of hedging options ends up zero.

compared with the future value of the call premium $1,776,088. The net loss is therefore $56,046, better than the gain of $251,235 with delta hedge in terms of lower variation.

16.3.3 Delta-gamma-vega hedge

If volatility changes, delta-gamma hedge will not work well. An enhancement is the delta-gamma-vega hedge, in which we add the condition of vega neutrality, meaning zero portfolio vega. As before, to accomplish this, one extra security has to be brought into the process. Since this strategy does not involve new insights, we shall leave it to the reader. In practice, delta-vega hedge performs substantially better than delta hedge [37].

Programming assignment 16.3.1 Write a program to implement delta-gamma-vega hedge for a single option. See if decreasing Δt does lead to better approximation in that the cumulative cost converges to the future value of the option premium. ◊

16.3.4 Options on stock index and stock index futures

Three different kinds of derivatives are traded on stock indexes: index options (§11.5), index futures (§12.3.4), and stock index futures options (§12.4.5). A few stock indexes like the S&P 500 Index and the Dow Jones Industrial Average cover all three types of derivatives.

Hedging an index option is theoretically simple: Trade the underlying assets. This strategy is difficult to implement in practice because of the large numbers of stocks comprising the index. An alternative is to trade only a representative sample of the underlying assets. But it introduces tracking errors. A more attractive alternative is to use an index futures contract. This strategy is popular in hedging the OEX and SPX options. Serious issues remain, however. Perfect hedge is available only for *European* index options that are cash settled at the *same* time as the futures contract of the *same* index. This is strictly true only for certain indexes such as XMI and DJIA. Absent these conditions, we would be confronted with two problems: The futures contract may deviate from its fair value (basis problem), and the index underlying the futures contract may deviate from the index underlying the option contract (tracking error problem). In any case, exercise risk still remains.

16.3.5 Portfolio insurance

Portfolio insurance is a trading strategy that aims to protect a portfolio from market declines but without losing the opportunity to participate in market rallies. It is, in a word, a protective put. The idea is due to Leland and Rubinstein [672].

Using puts to protect a stock portfolio from falling below a given floor value is a simple example of **static portfolio insurance** scheme. To purchase a put on each of the stocks would not be economical if an option against the portfolio is available (see Theorem 8.8.1). Alternatives to static schemes in establishing a floor are dynamic trading strategies that create synthetic options. These strategies shift the portfolio between stocks and riskless bonds. More stocks are assumed when the value of the portfolio rises, while more riskless bonds are secured with funds from stock sales when the opposite happens. This is consistent with the intuition that more insurance is required as the portfolio heads toward the floor.

The dynamic strategy has the disadvantage of high transactions costs. This problem was answered by the introduction of stock index futures. So, instead of trading a portfolio's underlying assets, we can buy or sell futures with much lower transactions costs to achieve the appropriate mixture of the risky and riskless assets (review §12.4.6). Needless to say, the market is counted upon to supply the liquidity. The Crash of 1987 provides a vivid warning that such liquidity might not be available at times of extreme market movements. As prices began to fall, portfolio insurers sold stock index futures as dictated by the strategy. This activity in the futures market led to more selling in the cash market as program traders attempted to arbitrage the spreads between the cash and futures markets. Further price declines led to more selling by portfolio insurers, and so on [496, 559].

Implementation

In order to implement portfolio insurance, we start by specifying some minimum value X we wish to prevent the portfolio from penetrating. This minimum value is the floor and corresponds to the strike price in a protective put strategy.

Suppose that the value of the index is S and each option is on 100 times the index. Consider a well-diversified portfolio which has a beta of β. If for each $100 \times S$ dollars in the portfolio, one put option contract is purchased with exercise price X, the value of the portfolio is protected against the possibility of the index falling below X. Our goal is to design a strategy to implement the above protective put synthetically. The steps to follow shall show that, to protect *each dollar* of a portfolio against falling below W at time T, simply buy β put contracts for each $100 \times S$ dollars in the portfolio. In addition, the strike price X shall be the index value when the value of the portfolio reaches W. Recall that $100 \times S$ is the size of the contract.

Let r be the interest rate and q the dividend yield for the period in question. Suppose the index reaches S_T at time T. The excessive return from the index over riskless interest rate is $(S_T - S)/S + q - r$, and the excessive return from the portfolio over riskless interest rate is

$$\beta \left(\frac{S_T - S}{S} + q - r \right).$$

The return from the portfolio is therefore

$$\beta\left(\frac{S_T - S}{S} + q - r\right) + r,$$

and the increase in the portfolio value (net of the dividends) is

$$\beta\left(\frac{S_T - S}{S} + q - r\right) + r - q.$$

So the portfolio value per dollar of the original value is

$$1 + \beta\left(\frac{S_T - S}{S} + q - r\right) + r - q = \beta\,\frac{S_T}{S} + (\beta - 1)(q - r - 1). \tag{16.6}$$

We shall choose X to be the S_T that makes (16.6) equal W, in other words,

$$X = (W + (q - r - 1)(1 - \beta))\,\frac{S}{\beta}.$$

Why does it work? From (16.6), it is clear that the portfolio value is less than W by $\beta \Delta S/S$ if the index value is less than X by $(\beta \Delta S/S)(S/\beta) = \Delta S$. Exercising the option induces a matching cash inflow of

$$\beta\,\frac{\Delta S}{100 \times S} \times 100 = \beta\,\frac{\Delta S}{S}.$$

Note that the total number of put options bought is $\beta V/(S \times 100)$, where V is the current value of the portfolio. The cost of the strategy is hence $P\beta V/(S \times 100)$, where P is the put premium with strike price X.

Clearly, a higher strike price provides a higher floor of WV dollars at a greater cost. This tradeoff between the cost of the insurance and the level of protection is true of any insurance. The total wealth of course has a floor of

$$WV - \beta\,\frac{V}{S \times 100}\,P.$$

Example 16.3.1 Start with $S = 1000$, $\beta = 1.5$, $q = 0.02$, and $r = 0.07$ for a period of one year. We have the following table between the index value and the portfolio value (per dollar of the original value).

Index value in a year	1200	1100	1000	900	800
Portfolio value in a year	1.275	1.125	0.975	0.825	0.675

For example, if the portfolio starts at \$1 million and the insured value is \$0.825 million, then 15 $(= \frac{1.5 \times 1{,}000{,}000}{100 \times 1{,}000})$ put contracts with a strike price of 900 should be purchased. □

16.3.6 Static hedging

A dynamic trading strategy can perfectly replicate the payoffs of derivatives by creating synthetic securities. However, such schemes invariably incur huge trading activities and, in practice, transactions costs. Furthermore, if the gamma of the replicated security is high, periodic rebalancing may have high tracking error as well. Error from wrong estimation of the volatility of the underlying asset is directly proportional to the option's vega, which can be high.

It would be nice if the derivative can be replicated using a *static* strategy in the sense that we never need to trade except when certain events, which happen rarely, occur. This hope has been realized for hedging European barrier options and lookback options with standard options [138, 140, 232]. One interesting thing to note is that, although the above-mentioned options are path-dependent, they are hedged by standard path-independent options [136]. One disadvantage of static hedging over dynamic hedging is the relative illiquidity of the options market as compared to the market for the underlying assets.

Additional Reading

The literature on hedging is vast. Consult [422, 458, 650] for general references and [499] for more examples. The paper [712] adopts a broader view and considers a wider range of instruments than derivatives. The papers [303, 422, 496, 557, 598] contain additional information on portfolio insurance. See [791] for mathematical programming techniques in risk management, which are important when trading constraints are added. For further information on financial engineering and risk management, consult [323, 326, 331, 557, 559].