Overview of Stochastic Calculus

These notes provide an overview of results from stochastic calculus that we will be using in this course. Most of the results should be familiar to you already, but others will be new. We will ignore most of the "technical" details and take the "engineering" approach to stochastic calculus.

We make the following assumptions throughout.

- There is a probability triple $(\Omega, \mathcal{F}, P)$ where
  - P is the "true" or *physical* probability measure
  - Ω is the universe of possible outcomes. We use $\omega \in \Omega$ to represent a generic outcome, typically a sample path(s) of a stochastic process(es).
  - the set[1] $\mathcal{F}$ represents the set of possible *events* where an event is a subset of Ω.
- There is also a *filtration*, $\{\mathcal{F}\}_{t\geq 0}$, that models the evolution of information through time. So for example, if it is known by time t whether or not an event, E, has occurred, then we have $E \in \mathcal{F}_t$. If we are working with a finite horizon, $[0, T]$, then we can take $\mathcal{F} = \mathcal{F}_T$.
- We also say that a stochastic process, X_t, is $\mathcal{F}_t$-adapted if the value of X_t is known at time t when the information represented by $\mathcal{F}_t$ is known. All the processes we consider will be $\mathcal{F}_t$-adapted so we will not bother to state this in the sequel.
- In the continuous-time models that we will study, it will be understood that the filtration $\{\mathcal{F}\}_{t\geq 0}$ will be the filtration *generated* by the Brownian motion(s), W_t, that are specified in the model description.

1 Martingales and Brownian Motion

Definition 1 *A stochastic process,* $\{W_t \; : \; 0 \leq t \leq \infty\}$, *is a* standard Brownian motion *if*

1. $W_0 = 0$
2. *It has continuous sample paths*
3. *It has independent, normally-distributed increments.*

Definition 2 *An* n*-dimensional process,* $W_t = (W_t^{(1)}, \ldots, W_t^{(n)})$, *is a* standard n-dimensional Brownian motion *if each* $W_t^{(i)}$ *is a standard Brownian motion and the* $W_t^{(i)}$*'s are independent of each other.*

Definition 3 *A stochastic process,* $\{X_t \; : \; 0 \leq t \leq \infty\}$, *is a* martingale *with respect to the filtration,* $\mathcal{F}_t$, *and probability measure,* P, *if*

1. $\mathrm{E}^P[|X_t|] < \infty$ *for all* $t \geq 0$
2. $\mathrm{E}^P[X_{t+s}|\mathcal{F}_t] = X_t$ *for all* $t, s \geq 0$.

[1]Technically, $\mathcal{F}$ is a σ-algebra.

Example 1 (Brownian martingales)

Let W_t be a Brownian motion. Then W_t , $W_t^2 - t$ and $\exp\left(\theta W_t - \theta^2 t/2\right)$ are all martingales.

The latter martingale is an example of an *exponential martingale*. Exponential martingales are of particular significance since they are positive and may be used to define new probability measures. ▮

Exercise 1 (Conditional expectations as martingales) *Let Z be a random variable and set $X_t := \mathrm{E}[Z|\mathcal{F}_t]$. Show that X_t is a martingale.*

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2 Stochastic Integrals

We now discuss the concept of a stochastic integral, ignoring the various technical conditions that are required to make our definitions rigorous. In this section, we write $X_t(\omega)$ instead of the usual X_t to emphasize that the quantities in question are stochastic.

Definition 4 *A* stopping time *of the filtration $\mathcal{F}_t$ is a random time, τ, such that the event $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t > 0$.*

In non-mathematical terms, we see that a stopping time is a random time whose value is part of the information accumulated by that time.

Definition 5 *We say a process, $h_t(\omega)$, is* elementary *if it is piece-wise constant so that there exists a sequence of stopping times $0 = t_0 < t_1 < \ \ldots \ < t_n = T$ and a set of $\mathcal{F}_{t_i}$-measurable[2] functions, $e_i(\omega)$, such that*

$$h_t(\omega) \ = \ \sum_i e_i(\omega) I_{[t_i, t_{i+1})}(t)$$

where $I_{[t_i, t_{i+1})}(t) = 1$ if $t \in [t_i, t_{i+1})$ and 0 otherwise.

Note that our definition of an elementary function assumes that the function, $h_t(\omega)$, is evaluated at the left-hand point of the interval in which t falls.

Definition 6 *The stochastic integral of an elementary function, $h_t(\omega)$, with respect to a Brownian motion, W_t, is defined as*

$$\int_0^T h_t(\omega)\ dW_t(\omega) \ := \ \sum_{i=0}^{n-1} e_i(\omega)\left(W_{t_{i+1}}(\omega) - W_{t_i}(\omega)\right). \tag{1}$$

For a more general process, $X_t(\omega)$, we have

$$\int_0^T X_t(\omega)\ dW_t(\omega) \ := \ \lim_{n \to \infty} \int_0^T X_t^{(n)}(\omega)\ dW_t(\omega)$$

where $X_t^{(n)}$ is a sequence of elementary processes that converges (in an appropriate manner) to X_t.

Definition 7 *We define the space $L^2[0, T]$ to be the space of processes, $X_t(\omega)$, such that*

$$\mathrm{E}\left[\int_0^T X_t(\omega)^2\ dt\right] < \infty.$$

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[2] A function $f(\omega)$ is $\mathcal{F}_t$ measurable if its value is known by time t.

Theorem 1 (Itô's Isometry) *For any $X_t(\omega) \in L^2[0,T]$ we have*

$$\mathrm{E}\left[\left(\int_0^T X_t(\omega)\ dW_t(\omega)\right)^2\right] \;=\; \mathrm{E}\left[\int_0^T X_t(\omega)^2\ dt\right].$$

Exercise 2 *Check that Itô's isometry holds when X_t is an elementary process.*

Theorem 2 (Martingale Property of Stochastic Integrals) *The stochastic integral, $Y_t := \int_0^t X_s(\omega)\ dW_s(\omega)$, is a martingale for any $X_t(\omega) \in L^2[0,T]$.*

Exercise 3 *Check that $\int_0^t X_s(\omega)\ dW_t(\omega)$ is indeed a martingale when X_t is an elementary process.*

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3 Stochastic Differential Equations

Definition 8 *An n-dimensional Itô process, X_t, is a process that can be represented as*

$$X_t \;=\; X_0 + \int_0^t a_s\ ds + \int_0^t b_s\ dW_s \tag{2}$$

where W is an m-dimensional standard Brownian motion, and a and b are n-dimensional and $n \times m$-dimensional $\mathcal{F}_t$-adapted[3] *processes, respectively*[4].

We often use the notation

$$dX_t \;=\; a_t\ dt + b_t dW_t$$

as shorthand for (2). An n-dimensional *stochastic differential equation* (SDE) has the form

$$dX_t \;=\; a(X_t,t)\ dt + b(X_t,t)\ dW_t; \quad X_0 = x \tag{3}$$

where as before, W_t is an m-dimensional standard Brownian motion, and a and b are n-dimensional and $n \times m$-dimensional adapted processes, respectively. Once again, (3) is shorthand for

$$X_t \;=\; x + \int_0^t a(X_s,s)\ dt + \int_0^t b(X_s,t)\ dW_s. \tag{4}$$

While we do not discuss the issue here, various conditions exist to guarantee existence and uniqueness of solutions to (4). A useful tool for solving SDE's is Itô's Lemma which we now discuss.

[3] a_t and b_t are $\mathcal{F}_t$-'adapted' if a_t and b_t are $\mathcal{F}_t$-measurable for all t. We always assume that our processes are $\mathcal{F}_t$-adapted.
[4] Additional technical conditions on a_t and b_t are also necessary.

4 Itô's Lemma

Itô's Lemma is the most important result in stochastic calculus, the "sine qua non" of the field.

Theorem 3 (Itô's Lemma for 1-dimensional Itô process) *Let X_t be a 1-dimensional Itô process satisfying the SDE*

$$dX_t \;=\; \mu_t\; dt + \sigma_t\; dW_t.$$

If $f(t,x) : [0,\infty) \times R \rightarrow R$ is a $C^{1,2}$ function and $Z_t := f(t, X_t)$ then

$$\begin{aligned} dZ_t &= \frac{\partial f}{\partial t}(t,X_t)\; dt + \frac{\partial f}{\partial x}(t,X_t)\; dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,X_t)\; (dX_t)^2 \\ &= \left(\frac{\partial f}{\partial t}(t,X_t) + \frac{\partial f}{\partial x}(t,X_t)\; \mu_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,X_t)\; \sigma_t^2\right)\; dt \;+\; \frac{\partial f}{\partial x}(t,X_t)\; \sigma_t\; dW_t \end{aligned}$$

Example 2

If a stock price, S_t, satisfies the SDE

$$dS_t \;=\; \mu_t S_t\; dt + \sigma_t S_t\; dW_t.$$

then we can use the substitution, $Y_t = \log(S_t)$ and Itô's Lemma to find

$$S_t \;=\; S_0 \exp\left(\int_0^t (\mu_s - \sigma_s^2/2)\; ds + \int_0^t \sigma_s\; dW_s\right). \qquad (5)$$

Note that S_t does not appear on the right-hand-side of (5) so that we have indeed solved the SDE. ∎

Example 3 (Ornstein-Uhlenbeck Process)

Let S_t be a security price and suppose $X_t = \log(S_t)$ satisfies the SDE

$$dX_t \;=\; [-\gamma(X_t - \mu t) + \mu]\; dt \;+\; \sigma dW_t.$$

Then we can apply Itô's Lemma to $Y_t := \exp(\gamma t)X_t$ to find that

$$X_t = X_0 e^{-\gamma t} + \mu t + \sigma e^{-\gamma t}\int_0^t e^{\gamma s}\; dW_s. \qquad (6)$$

Once again, note that X_t does not appear on the right-hand-side of (6) so that we have indeed solved the SDE. ∎

Exercise 4 *Compute $\mathrm{E}[X_t]$ and $\mathrm{Var}(X_t)$ in Example 3. How do your answers compare with the corresponding values for geometric Brownian motion?*

Theorem 4 (Itô's Lemma for n-dimensional Itô process) *Let X_t be an n-dimensional Itô process satisfying the SDE*

$$dX_t \;=\; \mu_t\; dt + \sigma_t\; dW_t.$$

where $X_t \in \mathbf{R}^n$, $\mu_t \in \mathbf{R}^n$, $\sigma_t \in \mathbf{R}^{n\times m}$ and W_t is a standard m-dimensional Brownian motion. If $f(t,x) : [0,\infty) \times \mathbf{R}^n \rightarrow \mathbf{R}$ is a $C^{1,2}$ function and $Z_t := f(t, X_t)$ then

$$dZ_t \;=\; \frac{\partial f}{\partial t}(t,X_t)\; dt + \sum_i \frac{\partial f}{\partial x_i}(t,X_t)\; dX_t^{(i)} + \frac{1}{2}\sum_{i,j}\frac{\partial^2 f}{\partial x_i \partial x_j}(t,X_t)\; dX_t^{(i)} dX_t^{(j)}$$

where $dW_t^{(i)} dW_t^{(j)} \;=\; dt\; dW_t^{(i)} = 0$ for $i \neq j$ and $dW_t^{(i)} dW_t^{(i)} = dt$.

Exercise 5 *Let X_t and Y_t satisfy*

$$\begin{aligned} dX_t &= \mu_t^{(1)}\ dt + \sigma_t^{(1,1)}\ dW_t^{(1)} \\ dY_t &= \mu_t^{(2)}\ dt + \sigma_t^{(2,1)}\ dW_t^{(1)} + \sigma_t^{(2,2)}\ dW_t^{(2)} \end{aligned}$$

and define $Z_t := X_t Y_t$. Apply the multi-dimensional version of Itô's Lemma to find the SDE satisfied by Z_t.

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5 The Martingale Representation Theorem

The martingale representation theorem is an important result that is particularly useful for constructing replicating portfolios in financial models.

Theorem 5 *Suppose M_t is an $\mathcal{F}_t$-martingale where $\{\mathcal{F}_t\}_{t\geq 0}$ is the filtration generated by the n-dimensional standard Brownian motion, $W_t = (W_t^{(1)}, \ldots, W_t^{(n)})$. If $\mathrm{E}[M_t^2] < \infty$ for all t then there exists a unique[5] n-dimensional adapted stochastic process, ϕ_t, such that*

$$M_t \;=\; M_0 + \int_0^t \phi_s^T\ dW_t \quad \text{for all} \;\; t \geq 0.$$

Exercise 6 *Let $F = W_T^3$ and define $M_t = \mathrm{E}_t[F]$. Show that*

$$M_t \;=\; 3\int_0^t (T - s + W_s^2)\ dW_s$$

which is consistent with the Martingale Representation theorem.

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6 Gaussian Processes

Definition 9 *A process X_t, $t \geq 0$, is a Gaussian process if $(X_{t_1}, \ldots, X_{t_n})$ is jointly normally distributed for every n and every set of times $0 \leq t_1 \leq t_2 \leq \ldots \leq t_n$.*

If X_t is a Gaussian process, then it is determined by its *mean function*, $m(t)$, and its covariance function, $\rho(s,t)$, where

$$\begin{aligned} m(t) &= \mathrm{E}\left[X_t\right] \\ \rho(s,t) &= \mathrm{E}\left[(X_s - m(s))(X_t - m(t))\right]. \end{aligned}$$

In particular, the joint moment generating function (MGF) of $(X_{t_1}, \ldots, X_{t_n})$ is given by

$$M_{t_1,\ldots,t_n}(\theta_1, \ldots, \theta_n) \;=\; \exp\left(\theta^T m(\mathbf{t}) + \frac{1}{2}\ \theta^T \Sigma \theta\right) \tag{7}$$

where $m(\mathbf{t}) = (m(t_1)\ \ldots\ m(t_n))^T$ and $\Sigma_{i,j} = \rho(t_i, t_j)$.

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[5] To be precise, additional integrability conditions are required of ϕ_s in order to claim that it is unique.

Example 4 (Brownian motion)
Brownian motion is a Gaussian process with $m(t) = 0$ and $\rho(s, t) = \min(s, t)$ for all $s, t \geq 0$. ∎

Theorem 6 (Integration of a deterministic function w.r.t. a Brownian motion) *Let W_t be a Brownian motion and suppose*

$$X_t = \int_0^t \delta_s \, dW_s$$

where δ_s is a deterministic function. Then X_t is a Gaussian process with $m(t) = 0$ and $\rho(s, t) = \int_0^{\min(s,t)} \delta_s^2 \, ds$.

Proof: (Sketch)
(i) First use Itô's Lemma to show that

$$\mathrm{E}\left[e^{uX_t}\right] = 1 + \frac{1}{2}u^2 \int_0^t \delta_s^2 \, \mathrm{E}\left[e^{uX_s}\right] \, ds. \tag{8}$$

If we set $y_t := \mathrm{E}\left[e^{uX_t}\right]$ then we can differentiate across (8) to obtain the ODE

$$\frac{dy}{dt} = \frac{1}{2}u^2 \delta_t^2 y.$$

This is easily solved to obtain the MGF for X_t,

$$\mathrm{E}\left[e^{uX_t}\right] = \exp\left(\frac{1}{2}u^2 \int_0^t \delta_s^2 \, ds\right) \tag{9}$$

which, as expected, is the MGF of a normal random variable with mean 0 and variance $\int_0^t \delta_s^2 \, ds$.
(ii) We now use (9) and similar computations to show that the joint MGF of $(X_{t_1}, \ldots, X_{t_n})$ has the form given in (7) with $m(\mathbf{t}) = 0$ and $\rho(s, t) = \int_0^{\min(s,t)} \delta_s^2 \, ds$. (See Shreve's *Stochastic Calculus for Finance II* for further details.) ■

The next theorem again concerns Gaussian processes and is often of interest[6] when studying short-rate models.

Theorem 7 *Let W_t be a Brownian motion and suppose δ_t and ϕ_t are deterministic functions. If*

$$X_t := \int_0^t \delta_u \, dW_u \quad \text{and} \quad Y_t := \int_0^t \phi_u X_u \, du$$

then Y_t is a Gaussian process with $m(t) = 0$ and

$$\rho(s, t) = \int_0^{\min(s,t)} \delta_v^2 \left(\int_v^s \phi_y \, dy\right) \left(\int_v^t \phi_y \, dy\right) \, dv.$$

Proof: The proof is tedious but straightforward. (Again, see Shreve's *Stochastic Calculus for Finance II* for further details.) ■

[6] See, for example, Hull and White's one-factor model.

7 Feynman-Kac Formula

Suppose X_t is a stochastic process satisfying the SDE $dX_t = \mu(t, X_t)\ dt + \sigma(t, X_t)\ dW_t$. Now consider the function, $f(x,t)$, given by

$$f(x,t) = \mathrm{E}_t^x \left[\int_t^T \phi_s^{(t)} h(X_s, s)\ ds + \phi_T^{(t)} g(X_T) \right]$$

where

$$\phi_s^{(t)} = \exp\left(- \int_t^s r(X_u, u)\ du \right)$$

and the notation $\mathrm{E}_t^x[\cdot]$ implies that the expectation should be taken conditional on time t information with $X_t = x$. Note that $f(x,t)$ may be interpreted as the time t price of a security that pays dividends at a continuous rate, $h(X_s, s)$ for $s \geq t$, and with a terminal payoff $g(X_T)$ at time T. (Of course $\mathrm{E}[\cdot]$, $r(\cdot,\cdot)$ and X also need to be interpreted appropriately.)

We can show[7] that $f(\cdot,\cdot)$ satisfies the following PDE

$$\begin{aligned} \frac{\partial f}{\partial t}(t, X_t) + \frac{\partial f}{\partial x}(t, X_t)\ \mu_t(t, X_t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, X_t)\ \sigma_t^2(t, X_t) - r(x,t) f(x,t) + h(x,t) &= 0, \quad (x,t) \in \mathbf{R} \times [0,T) \\ f(x,T) &= g(x), \quad x \in \mathbf{R} \end{aligned}$$

Exercise 7 *Derive the Feynman-Kac PDE by using the martingale property of conditional expectations,* $M_t := \mathrm{E}_t[F]$*, where* F *is a given random variable.*

Exercise 8 *Assuming martingale pricing, apply the Feynman-Kac formula to find the PDE satisfied by the price of a European call option in the Black-Scholes model.*

Remark 1 *The Feynman-Kac result generalizes easily to the case where* X_t *is an* n*-dimensional Itô process driven by an* m*-dimensional standard Brownian motion.*

8 Change of Probability Measure

Most applications in financial engineering price securities using the EMM, Q, that corresponds to taking the cash account, B_t, as numeraire. Sometimes, however, it is particularly useful to work with another numeraire, N_t, and its corresponding EMM, P_N say. We now describe how to create new probability measures and how to switch back and forth between these measures.

Let Q be a given probability measure and M_t a strictly positive Q-martingale such that $\mathrm{E}^Q[M_t] = 1$ for all $t \in [0,T]$. We may then define a new equivalent probability measure, P^M, by defining

$$P_M(A) = \mathrm{E}^Q \left[M_T 1_A \right].$$

Note that

(i) $P_M(\Omega) = 1$ and

(ii) the nullsets of Q and P_M coincide so P_M is indeed an equivalent probability measure.

Expectations with respect to P_M then satisfy

$$\mathrm{E}_0^{P_M}[X] = \mathrm{E}_0^Q [M_T X]. \tag{10}$$

[7] Additional technical conditions on μ, σ, r, h, g and f are required.

Exercise 9 *Verify (10) in the case where $X(\omega) = \sum_{i=1}^n c_i I_{\{\omega \in A_i\}}$ where the A_i's form a partition of Ω, i.e. $\bigcup A_i = \Omega$ and $A_i \bigcap A_j = \emptyset$ for $i \neq j$, and M_T is constant on each A_i.*

When we define a measure change this way, we use the notation dP_M/dQ to refer to M_T so that we often write

$$\mathrm{E}_0^{P_M}[X] \;=\; \mathrm{E}_0^Q\left[\frac{dP_M}{dQ}X\right].$$

The following result explains how to switch between Q and P_M when we are taking conditional expectations. In particular, we have

$$\mathrm{E}_t^{P_M}[X] \quad = \quad \frac{\mathrm{E}_t^Q\left[\frac{dP_M}{dQ}X\right]}{\mathrm{E}_t^Q\left[\frac{dP_M}{dQ}\right]} \quad = \quad \frac{\mathrm{E}_t^Q\left[\frac{dP_M}{dQ}X\right]}{M_t}$$

since M_t is a Q-martingale.

Exercise 10 *Show that if X is $\mathcal{F}_t$-measurable, i.e. X is known by time t, then $\mathrm{E}_0^{P_M}[X] = \mathrm{E}_0^Q[M_t X]$.*

Remark 2 *Since M_T is strictly positive we can set $X = I_A/M_T$ in (10) where I_A is the indicator function of the event A. We then obtain $\mathrm{E}_0^{P_M}[I_A/M_T] = \mathrm{E}_0^Q[I_A] = Q(A)$. In particular, we see that dQ/dP_M is given by $1/M_T$.*

Remark 3 *In the context of security pricing, we can take M_t to be the deflated time t price of a security with strictly positive payoff, normalized so that its expectation under Q is equal to 1. For example, let Z_t^T be the time t price of a zero-coupon bond maturing at time T, and let[8] B_t denote the time t value of the cash account. We could then set $M_T := 1/(B_T Z_0^T)$. The resulting measure, denoted by P_T, is sometimes called the T-forward measure. Note that we have implicitly assumed (why?!) that in this context, Q refers to the EMM when we take the cash account as numeraire. We discuss P_T in further detail in Section 11.*

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9 Girsanov's Theorem

Consider the process

$$L_t \;:=\; \exp\left(-\int_0^t \eta_s\, dW_s - \frac{1}{2}\int_0^t \eta_s^2\, ds\right) \tag{11}$$

where η_s is an adapted process. Using Itô's Lemma we can check that $dL_t \;=\; L_t\eta_t\, dW_t$ so L_t is a positive martingale[9] with $\mathrm{E}^P[L_t] = 1$ for all t.

Theorem 8 (Girsanov's Theorem) *Define an equivalent probability measure, Q^h, by setting*

$$Q^\eta(A) := \mathrm{E}^P[L_T 1_A]. \tag{12}$$

Then $\widehat{W}_t := W_t + \int_0^t \eta_s ds$ is a standard Q^η-Brownian motion. Moreover, $\widehat{W}_t$ has the martingale representation property under Q^η.

Remark 4 *The Girsanov Theorem generalizes easily to the case where W_t is an n-dimensional Brownian motion.*

[8] We assume the zero-coupon bond has face value \$1 and $B_0 = \$1$.

[9] In fact we need η_s to have some additional properties before we can claim L_t is a martingale. A sufficient condition is Novikov's Condition which requires $\mathrm{E}\left[\exp\left(\frac{1}{2}\int_0^T \eta_s^2\, ds\right)\right] < \infty$.

Exercise 11 *Let $dX_t = \mu_t\ dt + \sigma_t\ dW_t$. Find a process, η_s, such that X_t is a Q^η-martingale.*

Remark 5 *Note that Girsanov's Theorem enables us to compute Q^h-expectations directly without having to switch back to the original measure, P.*

We can get some intuition for the Girsanov Theorem by considering a random walk, $\mathbf{X} = \{X_0, X_1, \ldots, X_n\}$ with the interpretation that X_i is the value of the walk at time iT/n. In particular, X_n corresponds to the value of the random walk at time T. We assume that $X_{t+1} - X_t \sim N(0, T/n)$ under P and is independent of $X_0, \ldots, X_t$. Note that X is an approximation to Brownian motion on $[0, T]$.

Suppose now that we want to compute $\theta := \mathrm{E}_0^Q[h(\mathbf{X})]$ where Q denotes the probability measure under which $X_{t+1} - X_t \sim N(\mu, T/n)$, again independently of $X_0, \ldots, X_t$.
With a slight abuse of notation, let us write $h(\mathbf{X}) = h(\mathbf{Y})$ with $\mathbf{Y} = (Y_0 \ldots, Y_n)$ and $Y_i := X_i - X_{i-1}$ (with the understanding that $Y_{-1} := 0$). This formulation is convenient as the Y_i's are IID $\sim N(0, T/n)$ under P and IID $\sim N(\mu, T/n)$ under Q. Let $f(\cdot)$ and $g(\cdot)$ denote the PDF's of $N(\mu, T/n)$ and $N(0, T/n)$ random variables, respectively.

If we set $\mu := -T\eta/n$, we then have

$$
\begin{aligned}
\theta \;=\; \mathrm{E}_0^Q\left[h(\mathbf{Y})\right] &= \int_{R^n} h(y_1, \ldots, y_n) \left(\prod_{i=1}^n f(y_i)\right)\ dy_1 \ldots dy_n \\
&= \int_{R^n} h(y_1, \ldots, y_n) \prod_i \left(\frac{f(y_i)}{g(y_i)} g(y_i)\right)\ dy_1 \ldots dy_n \\
&= \\
&= \int_{R^n} h(y_1, \ldots, y_n) \prod_i \left(\frac{f(y_i)}{g(y_i)}\right) \left(\prod_i g(y_i)\right)\ dy_1 \ldots dy_n \\
&= \mathrm{E}_0^P\left[h(y_1, \ldots, y_n) \prod_i \left(\frac{f(y_i)}{g(y_i)}\right)\right] \\
&= \mathrm{E}_0^P\left[h(y_1, \ldots, y_n) \prod_i \exp\left(-\eta \sum_i y_i - \frac{\eta^2 T}{2}\right)\right]
\end{aligned}
$$

which is consistent with our statement of Girsanov's Theorem in (11) and (12) above.

Remark 6 **(i)** *As in the statement of the Girsanov Theorem itself, we could have chosen μ (and therefore η) to be adapted, i.e. to depend on prior events, in the random walk.*

(ii) *Note that Girsanov's Theorem allows the drift, but not the volatility of the Brownian motion, to change under the new measure, Q^η. It is interesting to see that we are not so constrained in the case of the random walk. Have you any intuition for why this is so?*

The multidimensional version of Girsanov's Theorem is a straightforward generalization of the one-dimensional version. In particular let W_t be an n-dimensional standard P-Brownian motion and define

$$L_t \;:=\; \exp\left(-\int_0^t \eta_s\ dW_s - \frac{1}{2}\int_0^t \eta_s \cdot \eta_s\ ds\right)$$

for $t \in [0, T]$. Then[10] $\widehat{W}_t := W_t + \int_0^t \eta_s ds$ is a standard Q^η-Brownian motion where $dQ^\eta/dP = L_T$.

[10]Again it is necessary to make some further assumptions in order to guarantee that L_t is a martingale. Novikov's condition is sufficient.

10 Martingale Pricing Theory

We use S_t to denote the time t price of a risky asset and as usual, B_t is the time t value of the cash account. We assume the risky asset does not[11] pay dividends. Let $\phi_t^{(s)}$ and $\phi_t^{(b)}$ denote the number of units of the security and cash account, respectively, that is held in a portfolio at time t. Then the value of the portfolio at time t is given by $V_t = \phi_t^{(s)} S_t + \phi_t^{(b)} B_t$.

Definition 10 *We say* $\phi_t := (\phi_t^{(s)}, \phi_t^{(b)})$ *is* self-financing *if*

$$dV_t \ = \ \phi_t^{(s)} \ dS_t \ + \ \phi_t^{(b)} \ dB_t.$$

Note that this definition is consistent with our definition for discrete-time models. Our definitions of *arbitrage*, *numeraire securities*, *equivalent martingale measures* and *complete* markets is unchanged from the discrete-time setup. We state[12] without proof the principal results of martingale pricing theory in continuous-time models. These results mirror those from the discrete-time theory.

Theorem 9 *There is no arbitrage if and only if there exists an EMM, Q.*

A consequence of Theorem 9 is that in the absence of arbitrage, the *deflated* value process, V_t/N_t, of any self-financing trading strategy is a Q-martingale. This implies that the deflated price of any attainable security can be computed as the Q-expectation of the terminal deflated value of the security.

Theorem 10 *Assume there exists a security with strictly positive price process and that there are no arbitrage opportunities. Then the market is complete if and only if there exists exactly one risk-neutral martingale measure, Q.*

This result will only play a background role in this course for several related reasons. First, we will generally assume that we are working with complete markets. This assumption is motivated in part by the assumption that zero-coupon bonds of every maturity are traded in the market. Second, when working with term-structure models we often choose to work directly under an EMM, Q, which is then calibrated to market data. This approach bypasses the issue of completeness which then only arises when we discuss hedging strategies.

The following example is particularly useful in many financial engineering applications.

Example 5 (Wealth Dynamics and Hedging)
We know B_t satisfies $dB_t = r_t B_t \ dt$ and suppose in addition that S_t satisfies

$$dS_t \ = \ \mu_t S_t \ dt \ + \sigma_t S_t dW_t. \tag{13}$$

Then for a portfolio $(\phi_t^{(s)}, \phi_t^{(b)})$, the portfolio value at time t is $V_t := \phi_t^{(s)} S_t + \phi_t^{(b)} B_t$. If $\phi_t := (\phi_t^{(s)}, \phi_t^{(b)})$ is self-financing, then we have

$$\begin{aligned} dV_t &= \phi_t^{(s)} \ dS_t \ + \ \phi_t^{(b)} \ dB_t \\ &= \phi_t^{(s)} \mu_t S_t \ dt + \phi_t^{(s)} \sigma_t S_t \ dW_t \ + \ \phi_t^{(b)} r_t B_t \ dt \\ &= V_t \left[\frac{\phi_t^{(s)} S_t}{V_t} \mu_t \ + \ \frac{\phi_t^{(b)} B_t}{V_t} r_t \right] \ dt \ + \ \frac{\phi_t^{(s)} S_t}{V_t} \sigma_t V_t \ dW_t \\ &= V_t \left[r_t \ + \ \theta_t (\mu_t - r_t) \right] \ dt \ + \ \theta_t \sigma_t V_t \ dW_t \end{aligned} \tag{14}$$

[11] We could easily adapt our definition of a self-financing trading strategy to accommodate securities that pay dividends. However, in this course we will generally take S_t to be the price(s) of a zero-coupon bond(s) that of course does not pay dividends (coupons).

[12] Additional technical conditions are generally required to actually prove these results.

where θ_t and $(1-\theta_t)$ are the *fractions* of time t wealth, V_t, invested in the risky asset and cash account, respectively, at time t. ∎

Exercise 12 *Show that the Q-dynamics of any traded asset in an arbitrage-free model must have a drift coefficient equal to the short rate, r_t. (This assumes that the cash account is the numeraire security.)*

Exercise 13 *Equation (14) gives the wealth dynamics for a self-financing portfolio and this is very useful for constructing hedging strategies in continuous-time models. Can you see why this might be the case?*

Exercise 14 *Recall that when we price securities it is necessary to work with an EMM, Q. If the numeraire is the cash account, then the drift, μ_t, is replaced by r_t in (14) and the P-Brownian motion, W_t, is replaced by the Q-Brownian motion, $\widehat{W}_t$. Use Girsanov's Theorem to verify this statement. In particular explain why $d\widehat{W}_t = dW_t + \eta_t\, dt$ where $\eta_t = (\mu_t - r_t)/\sigma_t$.*

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11 The Forward Measure

As usual we let Z_t^τ denote the time t price of a zero-coupon bond maturing at time $\tau \geq t$ with face value $\$1$, and let Q be the EMM corresponding to taking the cash account, B_t, as numeraire. We assume $B_0 = \$1$ and now use Z_t^τ to define a new probability measure, P^τ, that we call the *τ-forward probability measure*. To do this, set

$$\frac{dP^\tau}{dQ} = \frac{1}{B_\tau Z_0^\tau}. \tag{15}$$

Exercise 15 *Check that (15) does indeed define an equivalent probability measure.*

Now let C_t denote the time t price of a contingent claim that expires at time τ. We then have

$$\begin{aligned} C_t &= B_t \mathrm{E}_t^Q\left[\frac{C_\tau}{B_\tau}\right] & (16)\\ &= \frac{B_t \mathrm{E}_t^{P^\tau}\left[\frac{C_\tau}{B_\tau}\, B_\tau Z_0^\tau\right]}{\mathrm{E}_t^{P^\tau}\left[B_\tau Z_0^\tau\right]} \\ &= \frac{B_t Z_0^\tau \mathrm{E}_t^{P^\tau}\left[C_\tau\right]}{\mathrm{E}_t^Q\left[1\right]/\mathrm{E}_t^Q\left[1/B_\tau Z_0^\tau\right]} \\ &= Z_t^\tau \mathrm{E}_t^{P^\tau}[C_\tau]. & (17) \end{aligned}$$

We can now find C_t, either through equation (17) or through equation (16) where we use the cash account as numeraire. Computing C_t through (16) is our "usual method" and is often very convenient. When pricing equity derivatives, for example, we usually take interest rates, and hence the cash account, to be deterministic. This means that the factor $1/B_\tau$ in (16) can be taken outside the expectation so only the Q-distribution of C_τ is needed to compute C_t.

When interest rates are stochastic we cannot take the factor $1/B_\tau$ outside the expectation in (16) and we therefore need to find the *joint* Q-distribution of (B_τ, C_τ) in order to compute C_t. On the other hand, if we use equation (17) to compute C_t, then we only need the P^τ-distribution of C_τ, regardless of whether or not interest rates are stochastic. Working with a univariate-distribution is generally much easier than working with a bivariate-distribution so if we can easily find the P^τ-distribution of C_τ, then it can often be very advantageous to work with this distribution. The forward measure is therefore particularly useful[13] when studying term-structure models.

Interestingly, the forward-measure approach is not so useful for lattice models, the reason being that the Markovian feature of the lattice is lost when we work under P^τ and so the advantages of a recombining-lattice are also lost.

Exercise 16 *We have seen how easy it is to price a contingent claim in a lattice model using the EMM,* Q *(with the cash account as numeraire) and backwards induction. Convince yourself by looking at Example* 4 *of* Introduction and Binomial Lattice Models *that if we price under* $P^{\tau=2}$*, then we cannot take advantage of the recombining property of the binomial lattice.*

Exercise 17 *Consider an equity model with two securities,* A *and* B*, whose price processes,* $S_t^{(a)}$ *and* $S_t^{(b)}$ *respectively, satisfy the following SDE's*

$$\begin{aligned} dS_t^{(a)} &= rS_t^{(a)}\,dt + \sigma_1 S_t^{(a)}\,dW_t^{(1)} \\ dS_t^{(b)} &= rS_t^{(b)}\,dt + \sigma_2 S_t^{(b)}\,\left(\rho\,dW_t^{(1)} + \sqrt{1-\rho^2}\,dW_t^{(2)}\right) \end{aligned}$$

where $(W_t^{(1)},\ W_t^{(2)})$ *is a* 2*-dimensional* Q*-standard Brownian motion. We assume the cash account is the numeraire security corresponding to* Q *(which is consistent with the* Q*-dynamics of* $S_t^{(a)}$ *and* $S_t^{(b)}$*) and that the continuously compounded interest rate,* r*, is constant. Use the change of numeraire technique to compute the time* 0 *price,* C_0 *of a European option that expires at time* T *with payoff* $\max\,(0,\ S_T^{(a)} - S_T^{(b)})$.

[13]Switching to a different numeraire can also be advantageous in other circumstances, even when interest rates are deterministic.