Market Models

One of the principal disadvantages of short rate models, and HJM models more generally, is that they focus on unobservable instantaneous interest rates. The so-called market models that were developed[1] in the late 90's overcome this problem by directly modelling observable market rates such as LIBOR and swap rates. These models are straightforward to calibrate and have quickly gained widespread acceptance from practitioners. The first market models were actually developed in the HJM framework where the dynamics of instantaneous forward rates are used via Itô's Lemma to determine the dynamics of zero-coupon bonds. The dynamics of zero coupon bond prices were then used, again via Itô's Lemma, to determine the dynamics of LIBOR. Market models are therefore not inconsistent with HJM models. In these lecture notes, however, we will prefer to specify the market models directly rather than derive them in the HJM framework. In the process, we will derive Black's formulae for caplets and swaptions thereby demonstrating the consistency of these formulae with martingale pricing theory.

Throughout these notes, we will ignore the possibility of default or counterparty risk and treat LIBOR interest rates as the fundamental rates in the market. Zero-coupon bond prices are then computed using LIBOR rather than the default-free rates implied by the prices of government securities. This does result in a minor inconsistency in that we price derivative securities assuming no possibility of default yet the interest rates themselves that play the role of "underlying security", i.e. LIBOR and swap rates, implicitly incorporate the possibility of default. This inconsistency actually occurs in practice when banks trade caps, swaps and other instruments with each other, and ignore the possibility of default when quoting prices. Instead, the associated credit risks are kept to a minimum through the use of *netting* agreements and by counterparties limiting the total size of trades they conduct with one another. This approach can also be justified when counterparties have a similar credit rating and similar exposures to one another. Finally, we should mention that it is indeed possible[2], and sometimes necessary, to explicitly model credit risk even when we are pricing 'standard' securities such as caps and swaps. It goes without saying of course, that default risk needs to be modelled explicitly when pricing credit derivatives and related securities.

1 LIBOR, Swap Rates and Black's Formulae for Caps and Swaptions

We now describe two particularly important market interest rates, namely LIBOR and swap rates. We first define LIBOR and forward LIBOR, and then describe Black's formula for caplets. After defining LIBOR we then proceed to discuss swap rates and forward swap rates as well as describing Black's formula for swaptions. In practice, the "underlying security" for caps and swaptions are LIBOR and LIBOR-based swap rates. Therefore by modelling the dynamics of these rates directly we succeed in obtaining more realistic models than those developed in the short-rate or HJM framework.

LIBOR

The forward rate at time t based on simple interest for lending in the interval $[T_1, T_2]$ is given by[3]

$$F(t, T_1, T_2) = \frac{1}{T_2 - T_1} \left(\frac{Z_t^{T_1} - Z_t^{T_2}}{Z_t^{T_2}} \right) \tag{1}$$

[1] See Miltersen, Sandmann and Sondermann (1997), Brace, Gatarek and Musiela (1997), Jamshidian (1997) and Musiela and Rutkowski (1997).

[2] See chapter 11 of Cairns for a model where swaps are priced taking the possibility of default explictly into account.

[3] This follows from a simple arbitrage argument.

where, as before, Z_t^T is the time t price of a zero-coupon bond maturing at time T. Note also that if we measure time in years, then (1) is consistent with $F(t, T_1, T_2)$ being quoted as an annual rate.

LIBOR rates are quoted as **simply-compounded** interest rates, and are quoted on an annual basis. The accrual period or *tenor*, $T_2 - T_1$, is usually fixed at $\delta = 1/4$ or $\delta = 1/2$ corresponding to 3 months and 6 months, respectively. With a fixed value of δ in mind we can define the δ-year forward rate at time t with maturity T as

$$L(t, T) \; := \; F(t, T, T+\delta) \; = \; \frac{1}{\delta}\left(\frac{Z_t^T - Z_t^{T+\delta}}{Z_t^{T+\delta}}\right). \tag{2}$$

Note that the δ-year spot LIBOR rate at time t is then given by $L(t, t)$.

Remark 1 *LIBOR or the* London Inter-Bank Offered Rate*, is determined on a daily basis when the* British Bankers' Association *(BBA) polls a pre-defined list of banks with strong credit ratings for their interest rates. The highest and lowest responses are dropped and then the average of the remainder is taken to be the LIBOR rate. Because there is some credit risk associated with these banks, LIBOR will be higher than the corresponding rates on government treasuries. However, because the banks that are polled have strong credit ratings the spread between LIBOR and treasury rates is generally not very large and is often less than* 100 *basis points. Moreover, the pre-defined list of banks is regularly updated so that banks whose credit ratings have deteriorated are replaced on the list with banks with superior credit ratings. This has the practical impact of ensuring that forward LIBOR rates will still only have a very modest degree of credit risk associated with them.*

Black's Formula for Caplets

Consider now a caplet with payoff $\delta(L(T,T) - K)^+$ at time $T + \delta$. The time t price, C_t, is given by

$$\begin{aligned} C_t &= B_t \mathrm{E}_t^Q\left[\frac{\delta(L(T,T)-K)^+}{B_{T+\delta}}\right] \\ &= \delta Z_t^{T+\delta}\mathrm{E}_t^{P_{T+\delta}}\left[(L(T,T)-K)^+\right]. \end{aligned}$$

where (B_t, Q) is an arbitrary numeraire-EMM pair and $(Z_t^{T+\delta}, P_{T+\delta})$ is the forward measure-numeraire pair. The market convention is to quote caplet prices using Black's formula which equates C_t to a Black-Scholes like formula so that

$$C_t \;=\; \delta Z_t^{T+\delta}\left[L(t,T)\Phi\left(\frac{\log(L(t,T)/K)+\sigma^2(T-t)/2}{\sigma\sqrt{T-t}}\right) \;-\; K\Phi\left(\frac{\log(L(t,T)/K)-\sigma^2(T-t)/2}{\sigma\sqrt{T-t}}\right)\right] \tag{3}$$

where $\Phi(\cdot)$ is the CDF of a standard normal random variable. Note that (3) is what you would get for C_t if you assumed that

$$dL(t,T) \;=\; \sigma L(t,T)\; dW^{T+\delta}(t)$$

where $W^{T+\delta}(t)$ is a $P_{T+\delta}$-Brownian motion and σ is an 'implied' volatility that is used to quote prices.

Black's formula for caps is to equate the cap price with the sum of caplet prices given by (3) but where a common σ is assumed. Similar formulae exist for floorlets and floors.

Swap Rates

Consider a payer *forward start* swap where the swap begins at some fixed time T_n in the future and expires at time $T_M \geq T_n$. We assume the accrual period is of length δ. Since payments are made in arrears, the first payment occurs at $T_{n+1} = T_n + \delta$ and the final payment at $T_{M+1} = T_M + \delta$. Then martingale pricing implies that the time $t < T_n$ value, SW_t, of this forward start swap is

$$SW_t \;=\; \mathrm{E}_t^Q\left[\delta\sum_{j=n}^{M}\frac{B_t}{B_{T_{j+1}}}(L(T_j,T_j)-R)\right]$$

where R is the fixed rate (annualized) specified in the contract. A standard argument using the properties of floating-rate bond prices implies that

$$SW_{T_n} \;=\; 1 - Z_{T_n}^{T_{M+1}} \;-\; R\delta \sum_{j=n+1}^{M+1} Z_{T_n}^{T_j}.$$

This in turn easily implies (why?) that for $t < T_n$ we have

$$SW_t \;=\; Z_t^{T_n} - Z_t^{T_{M+1}} \;-\; R\delta \sum_{j=n+1}^{M+1} Z_t^{T_j}.$$

Definition 1 *The* forward swap rate *is the value $R = R(t, T_n, T_M)$ for which $SW_t = 0$. In particular, we obtain*

$$R = R(t, T_n, T_M) = \frac{Z_t^{T_n} - Z_t^{T_{M+1}}}{\delta \sum_{j=n+1}^{M+1} Z_t^{T_j}}. \tag{4}$$

The swap rate is then obtained by taking $t = T_n$ in (4).

Now consider the time t price[4] of a *payer-swaption* that expires at time $T_n > t$ and with payments of the underlying swap taking place at times $T_{n+1}, \ldots, T_{M+1}$. Assuming a fixed rate of $\widehat{R}$ (annualized) and a notional principle of \$1, the value of the option at expiration is given by

$$C_{T_n} \;=\; \left(1 - Z_{T_n}^{T_{M+1}} \;-\; \widehat{R}\delta \sum_{j=n+1}^{M+1} Z_{T_n}^{T_j}\right)^+ . \tag{5}$$

Substituting (4) at $t = T_n$ into (5) we find that

$$\begin{aligned} C_{T_n} \;&=\; \left(\delta \left[R(T_n, T_n, T_M) - \widehat{R}\right] \sum_{j=n+1}^{M+1} Z_{T_n}^{T_j}\right)^+ \\ &=\; \left(\delta \sum_{j=n+1}^{M+1} Z_{T_n}^{T_j}\right) \left[R(T_n, T_n, T_M) - \widehat{R}\right]^+ . \end{aligned} \tag{6}$$

Therefore we see that the swaption is like a call option on the swap rate. The time t value of the swaption, C_t, is then given by the Q-expectation of the right-hand-side of (6), suitably deflated by the numeraire.

Black's Formula for Swaptions

Market convention, however, is to quote swaption prices via Black's formula which equates C_t to a Black-Scholes-like formula so that

$$\begin{aligned} C_t \;=\; &\left(\delta \sum_{j=n+1}^{M+1} Z_t^{T_j}\right) \left[R(t, T_n, T_M)\Phi\left(\frac{\log(R(t, T_n, T_M)/\widehat{R}) + \sigma^2(T_n - t)/2}{\sigma\sqrt{T_n - t}}\right) - \right. \\ &\left. \widehat{R}\Phi\left(\frac{\log(R(t, T_n, T_M)/\widehat{R}) - \sigma^2(T_n - t)/2}{\sigma\sqrt{T_n - t}}\right)\right] \end{aligned} \tag{7}$$

where again σ is an 'implied' volatility that is used to quote prices.

Note that the expression in (7) is what we would obtain for the expectation of

$$\left(\delta \sum_{j=n+1}^{M+1} Z_t^{T_j}\right) \left[R(T_n, T_n, T_M) - \widehat{R}\right]^+$$

[4]Note that in (5) we have implicitly assumed that the strike is $k = 0$.

if $dR(t, T_n, T_M) = \sigma R(t, T_n, T_M)\ dW_t$.

It should be stated that Black's formulae for caps and swaptions did not originally correspond to prices that arise from the application of martingale pricing theory to some particular model. As originally conceived, they merely provided a framework for quoting market prices. The market models of these lecture notes will provide a belated justification for these formulae. We shall see that the justifications are mutually inconsistent, however, in that it is impossible for both formulae to hold simultaneously within the one model.

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2 The Term Structure of Volatility

The term structure of volatility[5] is a graph of volatility plotted against time to maturity, τ. There are of course many definitions of volatility and care is needed in specifying which definition is intended. Some commonly used definitions of the term structure of volatility at time t include:

1. The volatility of spot rates $Y_t^{t+\tau}$ as a function of τ. Depending on the model under consideration, this volatility may be available in closed form and the model calibrated to historical or implied rates.
2. The volatilty, $\sigma(t, t+\tau)$, of instantaneous forward rates, $f(t, t+\tau)$.
3. The implied volatility, σ, given by Black's formula for caplets. This will vary with time to maturity and can be computed at any time from market prices for caplets.
4. The implied volatility, σ, given by Black's formula for caps. Again this will vary with time to maturity and can be computed at any time from market prices for caps.

When calibrating term structure models it is common to calibrate using both market prices and the term structure of volatility. As a result we often want to work with models that allow for a rich variety of term structures of volatility as well of course, as a rich variety of term structures of interest rates.

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3 Numeraires and Zero-Coupon Bond Prices

While the cash account with $B_t := \exp\left(-\int_0^t r_s\ ds\right)$ has been the default numeraire to date, we will not work with the cash account as our numeraire in the context of market models. The reason is clear: in market models we take LIBOR rates (or swap rates) with a fixed tenor, δ, in mind, as our fundamental interest rates. It would therefore be very inconvenient (as well as defeating the purpose) if we had to determine the instantaneous short rate at each point in time. As a result we will generally work with other numeraire-EMM pairs as described below.

But first we will fix the maturities or *tenor dates* to which our market models will apply. At time t we could in principal have LIBOR rates, $L(t, T)$, available for all $T > t$. This is unnecessary, however, as the prices of most important securities, e.g. caps, floors, swaps, swaptions, Bermudan swaptions, etc., are determined by the rates (LIBOR or swap) applying to only a finite set of maturities. We therefore fix in advance a set of tenor dates[6]

$$0 := T_0 < T_1 < T_2 < \ \dots \ < T_M < T_{M+1} \quad \text{with}$$

$$\delta_i := T_{i+1} - T_i, \qquad i = 0, 1, \dots, M.$$

While the δ_i's are usually nominally equal, e.g. $1/4$ or $1/2$, day-count conventions will results in slightly different values for each δ_i. We let Z_t^n denote the time t price of a zero-coupon bond maturing at time $T_n > t$ for

[5]'Quants' in the fixed-income industry commonly refer to the 'term-structure of volatility' when discussing fixed-income derivatives and models. In this section we briefly give some possible definitions of the 'term-structure of volatility' but we will not need these definitions elsewhere in the course.

[6]The notation and setup in this section and the next will borrow heavily from Section 3.7 in *Monte Carlo Methods in Financial Engineering* by Glasserman.

$n = 1, \ldots, M$. Similarly, we use $L_n(t)$ to denote the time t forward rate applying to the period $[T_n, T_{n+1}]$ for $n = 0, 1, \ldots, M$. In particular, a simple arbitrage argument then implies

$$L_n(t) \;=\; \frac{Z_t^n - Z_t^{n+1}}{\delta_n Z_t^{n+1}}, \quad \text{for } 0 \le t \le T_n, \quad n = 0, 1, \ldots, M. \tag{8}$$

With some work we can invert (8) to obtain an expression for bond prices in terms of LIBOR rates. We find

$$Z_{T_i}^n \;=\; \prod_{j=i}^{n-1} \frac{1}{1 + \delta_j L_j(T_i)} \quad \text{for } n = i+1, \ldots, M+1. \tag{9}$$

Equation (9) only determines the bonds prices at the fixed maturity dates. However, for an arbitrary date t we can easily check that

$$Z_t^n \;=\; Z_t^{\phi(t)} \prod_{j=\phi(t)}^{n-1} \frac{1}{1 + \delta_j L_j(t)} \quad \text{for } 0 \le t \le T_n. \tag{10}$$

where we define $\phi(t)$ to be next tenor date after time t. That is,

$$\phi(t) \;:=\; \min_{i=1,\ldots,M+1} \{i \;:\; t < T_i\}.$$

Remark 2 *The presence of $Z_t^{\phi(t)}$ in (10) suggests that it may not be sufficient to model only the dynamics of the forward LIBOR rates, $L_n(t)$, when we specify a market model since they are not sufficient to determine $Z_t^{\phi(t)}$ at an arbitrary time t. However, as we shall see below, this will not prove to be a problem as the $\phi(t)$ factor vanishes upon deflating by the numeraire.*

Exercise 1 *Prove equations (9) and (10).*

Numeraire-EMM Pairs

The following numeraire-EMM pairs are commonly used in market models:

1. The spot measure, Q, assumes that B_t^* is the numeraire where B_t^* is defined as follows.
   - start with \$1 at $t = 0$ and then purchase $1/Z_0^1$ of the zero-coupon bonds maturing at time T_1
   - at time T_1 reinvest the funds in the zero-coupon bond maturing at time T_2
   - by continuing in this way, we see that at time t the spot numeraire will be worth

$$B_t^* \;=\; Z_t^{\phi(t)} \prod_{j=0}^{\phi(t)-1} [1 + \delta_j L_j(T_j)]. \tag{11}$$

2. The forward measure, P^T, takes the zero-coupon bond maturing at time T as numeraire. We have seen this numeraire-EMM pair already.

3. The swap measure, P^X, is useful for pricing swaptions analytically. It takes the numeraire to be $X_t = \delta \sum_{k=1}^{M} Z_t^k$, which is indeed a positive security price process.

Deflating Zero-Coupon Bond Prices by the Spot Numeraire

Equations (10) and (11) show that deflated[7] zero-coupon bond prices, D_t^n, satisfy

$$D_n(t) \;=\; \left(\prod_{j=0}^{\phi(t)-1} \frac{1}{1 + \delta_j L_j(T_j)} \right) \prod_{j=\phi(t)}^{n-1} \frac{1}{1 + \delta_j L_j(t)} \quad \text{for } 0 \le t \le T_n. \tag{12}$$

In particular, we see that the factor, $Z_t^{\phi(t)}$, has vanished.

[7] We will take the spot numeraire to be the default numeraire.

4 Arbitrage-Free LIBOR Dynamics

Dynamics under the Spot Measure

We assume that the dynamics of the LIBOR rates satisfy

$$dL_n(t) \;=\; \mu_n(t)L_n(t)\;dt + L_n(t)\sigma_n(t)^T\;dW(t), \quad 0 \le t \le T_n, \quad n = 1, \ldots, M \tag{13}$$

where $W(t)$ is a d-dimensional Brownian motion, and $\mu_n(t)$ and $\sigma_n(t)$ are adapted processes that may depend on the current vector of interest rates $L(t) := (L_1(t), \ldots, L_M(t))$. The assumption of no arbitrage and the positivity of deflated bond prices implies the existence of an R^d-valued process $\nu_n(t)$ such that

$$dD_n(t) \;=\; D_n(t)\nu_n^T(t)\;dW(t). \tag{14}$$

We could apply Itô's Lemma directly to our expression for $D_n(t)$ in (12) but this would be awkward. Instead we will apply Itô's Lemma to $Y_n(t) := \log D_n(t)$. We see from (14) that

$$dY_n(t) \;=\; -\frac{1}{2}||\nu_n(t)||^2\;dt + \nu_n^T(t)\;dW(t) \tag{15}$$

We can also find an alternative expression for $dY_n(t)$ using (12). In particular, noting that the first factor in (12) is constant between maturities, we obtain via Itô's Lemma

$$\begin{aligned} dY_n(t) \;&=\; -\sum_{j=\phi(t)}^{n-1} d\log\left(1+\delta_j L_j(t)\right) \\ &=\; -\sum_{j=\phi(t)}^{n-1}\left(\frac{\delta_j\mu_j(t)L_j(t)}{1+\delta_j L_j(t)} - \frac{\delta_j^2 L_j(t)^2\sigma_j^T(t)\sigma_j(t)}{2\left(1+\delta_j L_j(t)\right)^2}\right)dt \;-\; \left(\sum_{j=\phi(t)}^{n-1}\frac{\delta_j L_j(t)\sigma_j^T(t)}{1+\delta_j L_j(t)}\right)dW(t). \end{aligned} \tag{16}$$

Comparing the volatility terms in (15) and (16) then gives us

$$\nu_n(t) \;=\; -\sum_{j=\phi(t)}^{n-1}\frac{\delta_j L_j(t)\sigma_j(t)}{1+\delta_j L_j(t)}. \tag{17}$$

We would now like to find an expression for the μ_j's. Towards this end, we could compare the drift terms in (15) and (16), and this is easy to do when $n = 2$ and $\phi(t) = 1$. After some straightforward algebra, we easily find[8]

$$\mu_1(t) = -\sigma_1^T(t)\nu_2(t), \qquad 0 \le t \le T_1.$$

More generally, we obtain

$$\mu_n(t) = -\sigma_n^T(t)\;\nu_{n+1}(t) \;=\; \sum_{j=\phi(t)}^{n}\frac{\delta_j L_j(t)\sigma_n^T(t)\sigma_j(t)}{1+\delta_j L_j(t)}. \tag{18}$$

We could have obtained (18) by again comparing the drift terms in (15) and (16) but this appears to be very cumbersome. Exercise 2 instead provides a more elegant approach.

Exercise 2 *Use induction to establish that the drifts, $\mu_n(t)$, must satisfy (18) under the no-arbitrage assumption. In particular, first assume $\mu_1, \ldots, \mu_{n-1}$ have been chosen consistent with the Q-martingale assumption on $D_1, \ldots, D_n$. Show that D_{n+1} is a martingale if and only if L_nD_{n+1} and then apply Itô's Lemma to obtain[9] (18).*

[8] Note that $L_1(t)$, and therefore $\mu_1(t)$, do not have any meaning for $t > T_1$.

[9] See Glasserman, page 170.

We therefore obtain that the arbitrage free Q-dynamics of the forward LIBOR rates are given by

$$dL_n(t) \;=\; \left(\sum_{j=\phi(t)}^{n} \frac{\delta_j L_j(t)\sigma_n^T(t)\sigma_j(t)}{1+\delta_j L_j(t)}\right) L_n(t)\; dt + L_n(t)\sigma_n(t)^T\; dW(t), \quad 0 \le t \le T_n, \quad n = 1, \ldots, M. \quad (19)$$

Dynamics under the Forward Measure

Consider now the case where we use the forward measure, P_{M+1}, and the associated numeraire, Z_t^{M+1}. We now find that deflated zero-coupon bond prices are given by

$$D_n(t) \;=\; \prod_{j=n}^{M} (1+\delta_j L_j(t)). \quad (20)$$

We would like to find the market-price-of-risk process, $\eta^{M+1}(t) \in R^d$, that relates the Q-Brownian motion $W(t)$ to the the P_{M+1} Brownian motion, $W^{M+1}(t)$, so that

$$dW(t) \;=\; dW^{M+1}(t) - \eta(t)\; dt. \quad (21)$$

There are a number of ways to do this but perhaps the easiest is the approach we followed with the Vasicek model when we switched to the forward measure. Equation (20) implies $D_M(t) = 1 + \delta_M L_M(t)$ so that

$$dD_M(t) = \delta_M\; dL_M(t). \quad (22)$$

We now substitute for $dL_M(t)$ in (22) using (19) evaluated at $n = M$, and then substitute for $W(t)$ using (21). Since $D_M(t)$ is a P_{M+1}-martingale we find that

$$\eta(t) = \sum_{j=\phi(t)}^{M} \frac{\delta_j L_j(t)\sigma_j(t)}{1+\delta_j L_j(t)}.$$

In particular, we obtain the arbitrage-free P_{M+1}-dynamics of the forward LIBOR rates are given by

$$dL_n(t) \;=\; -\left(\sum_{j=n+1}^{M} \frac{\delta_j L_j(t)\sigma_n^T(t)\sigma_j(t)}{1+\delta_j L_j(t)}\right) L_n(t)\; dt + L_n(t)\sigma_n(t)^T\; dW^{M+1}(t), \quad 0 \le t \le T_n, \quad n = 1, \ldots, M. \quad (23)$$

Black's Formula for Caplets

We are now in a position to derive Black's formula (see (3)) for caplet prices. If we take $n = M$ in (23), then we obtain

$$dL_M(t) \;=\; L_M(t)\sigma_M(t)^T\; dW^{M+1}(t) \quad (24)$$

implying in particular[10] that $L_M(t)$ is a P_{M+1}-martingale. If we assume that $\sigma_M(t)$ is a deterministic function, then we easily see that $L_M(t)$ is lognormally distributed. In particular, we obtain

$$\log L_M(t) \;\sim\; \mathsf{N}\left(\log(L_M(0)) - \frac{1}{2}\int_0^t ||\sigma_M(s)||^2\; ds\;,\; \int_0^t ||\sigma_M(s)||^2\; ds\right).$$

We can now obtain (3) if we let $T_M = T$ and reinterpret σ appropriately.

Note also that there is no problem when we take $\sigma_M(t)$ to be deterministic in (24) which contrasts with the HJM framework. This is because while the numerators in the drift of (19) are quadratic in $L_j(t)$, the $1+\delta_j L_j(t)$

[10]Subject, as usual, to technical conditions.

term in the denominator ensures that there is no possibility of explosion in the SDE. This is a further advantage of the market model framework where we model simple LIBOR rates rather than *instantaneous* forward rates.

BGM's Approximation for Swaption Prices

In their original paper, Brace, Gatarek and Musiela (BGM) succeeded in deriving Black's formula for caplets and thereby demonstrated its consistency with martingale pricing. Their framework did not enable them to derive Black's formula for swaptions, however. Instead they provided an analytic approximation for swaption prices that we will not describe[11] here. It is worth mentioning, however, that their approximation works well in practice and provides swaption prices that are very close to those obtained via Monte Carlo simulation.

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5 A Swap Market Model for Pricing Swaptions

Consider a *payer-swaption* that expires at time $T_n > t$ and with payments of the underlying swap taking place at times $T_{n+1}, \ldots, T_{M+1}$. Assuming a fixed rate of $\widehat{R}$ (annualized) and a notional principle of \$1, we showed in (6) that the time T_n price of the swaption is given by

$$C_{T_n} \;=\; \left(\delta \sum_{j=n+1}^{M+1} Z_{T_n}^{T_j}\right)\left[R(T_n, T_n, T_M) - \widehat{R}\right]^+ . \tag{25}$$

This implies that the time t price of the swaption, C_t, satisfies

$$C_t \;=\; X_t \, \mathrm{E}_t^{P_x}\left[\frac{\left(\delta \sum_{j=n+1}^{M+1} Z_{T_n}^{T_j}\right)\left[R(T_n, T_n, T_M) - \widehat{R}\right]^+}{X_{T_n}}\right] \tag{26}$$

where X_t is the time t price of the chosen numeraire security and P_x is the corresponding EMM. A particularly convenient choice of numeraire that we will adopt is the portfolio[12] consisting of δ units of each of the zero-coupon bonds maturing at times $T_{n+1}, \ldots, T_{M+1}$. Then $X_t = \delta \sum_{j=n+1}^{M+1} Z_t^{T_j}$ and we find

$$C_t \;=\; \left(\delta \sum_{j=n+1}^{M+1} Z_t^{T_j}\right) \; \mathrm{E}_t^{P_x}\left[\left[R(T_n, T_n, T_M) - \widehat{R}\right]^+\right] \tag{27}$$

Jamshidian (1997) developed a term structure framework where at any time t the current term structure was given in terms of the forward swap rates, $R(t, T_i, T_M)$ for $i = \phi(t), \ldots, M$. In particular, he showed that it was possible to assume that the P_x-dynamics of $R(t, T_n, T_M)$ satisfy

$$dR(t, T_n, T_M) \;=\; R(t, T_n, T_M)\sigma(t)^T \, dW^x(t) \tag{28}$$

where $\sigma(t)$ is a deterministic vector of volatilities. This implies that the forward swap rate is lognormally distributed so we can obtain[13] Black's formula for swaption prices (7).

Remark 3 *When we model swap rates directly as in (28) we say that we have a* swap *market model. This contrasts with the* LIBOR *market models of Section 4.*

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[11] See Chapter 9 of Cairns for a derivation.

[12] There is no difficulty taking a portfolio of securities rather than a fixed individual security as the numeraire. More generally in fact, we could take a dynamic self-financed portfolio as the numeraire security, assuming of course that it has strictly positive value at all times.

[13] Of course we need to reinterpret σ in (7) in terms of the deterministic function $\sigma(t)$ in (28).

Remark 4 *The advantage of Black's swaption formula is that it is elegant and exact, whereas the BGM formula is cumbersome and only an approximation. However, the BGM approximation is consistent with Black's formulae for caplets and caps whereas Black's swaption formula is not. Indeed, it may be shown*[14] *that if forward LIBOR rates have deterministic volatilities then it it is not possible for swap rates to also have deterministic volatilities. Therefore Black's formulae for caplets and swaptions cannot both hold within the same model. That said, within the LIBOR market framework with deterministic volatilities, it can be argued that forward swap rates are* approximately *lognormally distributed.*

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6 Monte-Carlo Simulation

While it is possible to price many commonly traded derivative securities such as caps, floors and swaptions in the market model framework, it is in general necessary to use Monte Carlo methods to price other securities. Indeed, if our market model has stochastic volatility functions then it will typically be necessary to also use Monte Carlo methods to price even caps, floors and swaptions.

The typical approach is to use some discretization scheme such as the Euler scheme when performing the Monte Carlo simulation. This does not create too much of a computational burden as we will only need to simulate the SDE's describing the forward LIBOR dynamics for a *finite* number of maturities. This contrasts with the HJM framework where we had infinitely many maturities which meant it was practically infeasible to use a very fine discretization. This in turn prompted the development of the discrete-time HJM framework with the resulting discrete-time arbitrage-free restriction on the drift.

It is also possible to develop discrete-time arbitrage-free market models in a manner that is analogous to our discrete-time HJM development. As described above, however, the need to do so is not as urgent as it is practically feasible to simulate the market model SDE's on a sufficiently fine grid and this is what is typically done in practice.

Nonetheless, Glasserman's *Monte Carlo Methods for Financial Engineering* describes how to build discrete-time arbitrage-free market models. It turns out to be inconvenient to choose the LIBOR rates as the fundamental variables that we choose to discretize. Instead it is more convenient to directly model deflated bond prices as discrete-time Q-martingales[15] and to define LIBOR rates in terms of these bond prices. Other choices of discretization variable are also possible. As usual, we can choose to simulate under any EMM that we prefer and all of the usual variance reduction techniques may be employed.

[14]This is done by applying Itô's Lemma to the forward swap rate given in (4).
[15]This ensures the discrete-time model is arbitrage-free.