LIBOR Interest Rates

The forward rate at time t based on simple interest for lending in the interval $[T_1, T_2]$ is given by[1]

$$F(t, T_1, T_2) \;=\; \frac{1}{T_2 - T_1}\left(\frac{Z_t^{T_1} - Z_t^{T_2}}{Z_t^{T_2}}\right) \tag{1}$$

where as before Z_t^T is the time t price of a zero-coupon bond maturing at time T. Note also that if we measure time in years, then (1) is consistent with $F(t, T_1, T_2)$ being quoted as an annual rate.

LIBOR rates are quoted as simply-compounded interest rates, and are quoted on an annual basis. The accrual period, $T_2 - T_1$, is usually fixed at $\delta = 1/4$ or $\delta = 1/2$ corresponding to 3 months and 6 months, respectively. With a fixed value of δ in mind we can define the δ-year forward rate at time t with maturity T as

$$L(t, T) \;:=\; F(t, T, T+\delta) \;=\; \frac{1}{\delta}\left(\frac{Z_t^T - Z_t^{T+\delta}}{Z_t^{T+\delta}}\right). \tag{2}$$

Note that the δ-year spot LIBOR rate at time t is then given by $L(t, t)$.

Black's Formula for Caplets

Consider now a caplet with payoff $\delta(L(T, T) - K)^+$ at time $T + \delta$. The time t price, C_t, is given by

$$C_t \;=\; B_t \mathrm{E}_t^Q\left[\frac{\delta(L(T, T) - K)^+}{B_{T+\delta}}\right].$$

The market convention is to quote caplet prices using Black's Formula which equates C_t to a Black-Scholes like formula so that

$$C_t \;=\; \delta Z_t^{T+\delta}\left[L(t, T)\Phi\left(\frac{\log(L(t, T)/K) + \sigma^2(T-t)/2}{\sigma\sqrt{T-t}}\right) \;-\; K\Phi\left(\frac{\log(L(t, T)/K) - \sigma^2(T-t)/2}{\sigma\sqrt{T-t}}\right)\right] \tag{3}$$

where $\Phi(\cdot)$ is the CDF of a standard normal random variable. Note that (3) is what you would get for C_t if you assumed that $Z_t^{T+\delta} = B_t/B_{T+\delta}$ (which is true when interest rates are deterministic) and that

$$dL(t, T) \;=\; \sigma L(t, T)\; dW_t.$$

where W_t is a Q-Brownian motion.

Black's formula for caps is to equate the cap price (which is a sum of caplets) with the corresponding sum of terms analogous to the right-hand-side of (3), assuming a common σ. Similar formulae exist for floorlets and floors.

Black's Formula for Swaptions

Consider the time t price of a payer-swaption that expires at time $T_n > t$ and with payments of the underlying swap taking place at times $T_{n+1}, \ldots, T_{M+1}$. Assuming a fixed rate of R (annualized) and a notional principle of \$1, the value of the option at expiration is given[2] by

$$SW_{T_n} \;=\; \left(1 - Z_{T_n}^{T_{M+1}} \;-\; R\delta \sum_{j=n+1}^{M+1} Z_{T_n}^{T_j}\right)^+. \tag{4}$$

[1] This follows from a simple arbitrage argument.
[2] Assignment 1 asks you to show both (4) and (5).

The forward-swap-rate is given by

$$R(t, T_n, T_M) = \frac{Z_t^{T_n} - Z_t^{T_{M+1}}}{\delta \sum_{j=n+1}^{M+1} Z_t^{T_j}} \tag{5}$$

so substituting (5) at $t = T_n$ into (5) we find that

$$\begin{aligned} SW_{T_n} &= \left(\delta \left[R(T_n, T_n, T_M) - R \right] \sum_{j=n+1}^{M+1} Z_{T_n}^{T_j} \right)^+ \\ &= \left(\delta \sum_{j=n+1}^{M+1} Z_{T_n}^{T_j} \right) \left[R(T_n, T_n, T_M) - R \right]^+ . \end{aligned} \tag{6}$$

Therefore we see that the swaption is like a call option on the swap rate. The time t value of the swaption, SW_t, is then given by the Q-expectation of the right-hand-side of (6), suitably deflated by the cash-account.

Market convention, however, is to quote swaption prices via Black's formula which equates SW_t to a Black-Scholes like formula so that

$$\begin{aligned} SW_t = \delta \sum_{j=n+1}^{M+1} Z_t^{T_j} \Bigg[& R(t, T_n, T_M) \Phi \left(\frac{\log(R(t, T_n, T_M)/R) + \sigma^2 (T_n - t)/2}{\sigma \sqrt{T_n - t}} \right) - \\ & R \Phi \left(\frac{\log(R(t, T_n, T_M)/R) - \sigma^2 (T_n - t)/2}{\sigma \sqrt{T_n - t}} \right) \Bigg] . \end{aligned}$$

It should be stated that Black's formulae for caps and swaptions do not necessarily correspond to prices that arise from the application of martingale pricing theory to some particular model. As originally conceived, they merely provided a framework for quoting market prices. More recently, however, separate (and mutually inconsistent) justifications for these formulae have been provided in the context of LIBOR market models.