4.3 Random Time Change and Uniqueness in One Dimension.

One of the properties of Martingales is Doob's stopping theorem. If $M(t)$ is a Martingale with respect to $(\Omega\,,\mathcal{F}_t\,,P)$ and $0\le\tau_1\le\tau_2\le C$ are two bounded stopping times, with the correspnding σ-fields $\mathcal{F}_{\tau_1}\subset\mathcal{F}_{\tau_2}$, then

$$E^P\left[M(\tau_2)|\mathcal{F}_{\tau_1}\right]=M(\tau_1)\qquad\text{a.e.}$$

In particular if τ_t is a family of bounded stopping times with $\tau_s\le\tau_t$ for $s\le t$, then with

$$N(t)=M(\tau_t)\quad\text{and}\quad\mathcal{G}_t=\mathcal{F}_{\tau_t}$$

$N(t)$ is a martingale with respect to $(\Omega\,,\mathcal{G}_t\,,P)$. If P is any Martingale solution on $\Omega=C[[0\,,\infty)\,,X]$ that corresponds to some L, then

$$f(x(t))-f(x(0))-\int_0^t(Lf)(x(s))ds$$

is a martingale with respect to $(\Omega\,,\mathcal{F}_t\,,P)$. We consider the stopping times $\{\tau_t\}$, defined by,

$$\int_0^{\tau_t(\omega)}\frac{ds}{V(x(s\,,\omega))}=t$$

where $V(\cdot)$ is a positive measurable function on X, satisfying

$$0<c\le V(x)\le C<\infty\tag{4.6}$$

Then it is clear that τ_t is well defined for $t\ge0$ with $\tau_0=0$ and $\tau_s<\tau_t$ for $s<t$ and τ_t is almost surely continuous in t. We can use τ_t to define a map Φ_V of $\Omega\to\Omega$ by

$$(\Phi_V\,\omega)(t)=x(\tau_t(\omega)\,,\omega)$$

Lemma 4.2. *For any two functions U and V, satisfying the bound (4.6),*

$$\Phi_U\,\Phi_V=\Phi_{UV}$$

Proof. The proof depends on the simple calculation

$$\frac{d\tau_t(\omega)}{dt}=V(x(\tau_t(\omega)))=V(y(t))$$

where $y(t)=x(\tau_t(\omega))=(\Phi_V\,\omega)(t)$. If σ_t solves

$$\int_0^{\sigma_t(\omega)}\frac{ds}{U(y(s))}=t$$

or

$$\frac{d\tau_{\sigma_t}}{dt}=\frac{d\tau_\sigma}{ds}|_{s=\sigma_t}\cdot\frac{d\sigma_t}{dt}=(VU)(y(\sigma_t))=(VU)(x(\tau_{\sigma_t}))$$

proving the composition rule. ∎

In particular Φ_V is invertible with $\Phi_{\frac{1}{V}} = [\Phi_V]^{-1}$. The σ-field $\sigma\{y(s) : 0 \le s \le t\} \subset \mathcal{F}_{\tau_t}$, and

$$f(y(t)) - f(y(0)) - \int_0^{\tau_t} (\mathcal{L}f)(x(s))ds$$

is an $(\Omega, \mathcal{F}_{\tau_t}, P)$ martingale. By change of variables

$$\int_0^{\tau_t} (\mathcal{L}f)(x(s))ds = \int_0^t V(y(s))(\mathcal{L}f)(y(s))ds$$

Therefore

$$f(y(t)) - f(y(0)) - \int_0^t V(y(s))(\mathcal{L}f)(y(s))ds$$

is a martingale with respect to $(\Omega, \mathcal{F}_{\tau_t}, P)$. In particular $Q = \Phi_V^{-1}P$ is a Martingale solution for $\tilde{\mathcal{L}}$ defined as

$$(\tilde{\mathcal{L}}f)(x) = V(x)(\mathcal{L}f)(x)$$

The steps are reversible so that existence or uniqueness for a Martingale solution for $\mathcal{L}$ and $\tilde{\mathcal{L}}$ are equivalent so long as V satisfies the bounds (4.6).

Now when $d = 1$ we can prove existence and uniqueness of Martingale Solutions to

$$\mathcal{L} = \frac{a(x)}{2} D_x^2 + b(x) D_x$$

so long as a, b are bounded measurable with $0 < c \le a(x) \le C < \infty$. From Girsanov Formula we can assume without loss of generality that $b \equiv 0$. By random time change we can assume that $a(x) \equiv 1$. Now we are in the Brownian motion case, and we have existence and uniqueness. Of course once we have existence and uniqueness the Markov Property as well as the Strong Markov Property follow.

In the time dependent case it is more complicated. In one dimension we can improve the Lipschitz assumption on σ to a Hölder condition with exponent $\frac{1}{2}$.

Theorem 4.3. *Assume that b is Lipschitz but σ satifies*

$$|\sigma(t, x) - \sigma(t, y)| \le C|x - y|^{\frac{1}{2}}$$

Then any two solutions

$$x_i(t) = x_0 + \int_0^t \sigma(s, x_i(s))d\beta(s) + \int_0^t b(s, x_i(s))ds$$

are identical.

Proof. The proof involves the application of Ito's formula for the function

$$f(x_1(t), x_2(t)) = |x_1(t) - x_2(t)|.$$

Formally

$$\begin{aligned}
df(x_1(t)\,,x_2(t)) &= [\mathrm{sig}(x_1(t)-x_2(t))](\sigma(t\,,x_1(t))-\sigma(t\,,x_2(t)))d\beta(t)\\
&+\delta(x_1(t)-x_2(t))|\sigma(t\,,x_1(t))-\sigma(t\,,x_2(t))|^2dt\\
&+[\mathrm{sig}(x_1(t)-x_2(t))][b(t\,,x_1(t))-b(t\,,x_2(t))]dt
\end{aligned}$$

We will give an argument as to why the term with the delta-function δ is zero. Granting that, we have by the Lipschitz condition on b,

$$E[|x_1(t)-x_2(t)|]\le C\int_0^t E[|x_1(s)-x_2(s)|]ds$$

and this implies uniqueness. Let us approximate $|x|$ by $f_\epsilon(x)=\sqrt{(\epsilon^2+x^2)}$. Then

$$f_\epsilon''(x)=\frac{\epsilon^2}{(\epsilon^2+x^2)^{\frac{3}{2}}}$$

and

$$\begin{aligned}
|f_\epsilon''(x_1-x_2)|\sigma(t\,,x_1)-\sigma(t\,,x_2)|^2 &\le \frac{C\epsilon^2|x_1-x_2|}{(\epsilon^2+(x_1-x_2)^2)^{\frac{3}{2}}}\\
&\le C\sup_u\left[\frac{u}{(1+u^2)^{\frac{3}{2}}}\right]\\
&\le C'
\end{aligned}$$

We can now let $\epsilon\to 0$, use the dominated convergence theorem and pass to the limit to show that there is no contribution from the term involving δ. □

4.4 General comments on existence and uniqueness of the martingale solutions.

If we are given a $a(t\,,x)=\{a_{i,j}(t\,,x)\}$ and $b(t\,,x)=\{b_j(t\,,x)\}$ and are interested in proving existence and uniqueness of martingale solutions, we specifically wish to show that the set $C_{s,x}$ of probability measures P on $\Omega_s=C[[s\,,\infty)\,;R^d\,]$ such that

$$P[x(s)=x]=1$$

and

$$Z_f(t)=f(x(t))-f(x(s))-\int_s^t(\mathcal{L}_sf)(x(s)ds \tag{4.7}$$

are martingales with respect to $(\Omega_s\,,\mathcal{F}_t^s\,,P)$ for all smooth f, consists of exactly one probability measure.

The existence part is simple under fairly general conditions. If a and b are smooth we can have Lipschitz σ and b and Ito's theory of SDE provides us, as we saw, both existence and uniqueness. If we only assume that a and b are just bounded and continuous we can prove existence along the following lines. We take $s = 0$ with out loss of generality and approximate a, b by smoother a_n, b_n that converge as $n \to \infty$ to a, b. The convergence can be assumed to be uniform over compact subsets of R^d, and we can also assume that a_n as well as b_n are uniformly bounded by some constant M. For some x, let $P_{n,x}$ be the unique solution starting at time 0 from the point x, corresponing to a_n, b_n. We will prove that $P_{n,x}$ is a totally bounded sequence of probability measures on Ω, and that if P is any weak limit, then P is a solution starting at time 0 from x for the limiting coefficients and therefore we have existence.

Lemma 4.4. *The sequence P_n satisfies the following. For any $T < \infty$ and any $\epsilon > 0$, there exists $A(T\,, \epsilon)$ depending only on the bound M such that*

$$P_n\left[\omega : \sup_{0\le s\le t} \frac{|x(s)-x(t)|}{|t-s|^{\frac{1}{4}}} \le A(T\,,\epsilon)\right] \ge 1-\epsilon$$

In particular the sequence is totally boundeded.

Proof. Let P be a diffusion corresponding to some a, b that are bounded by M. We remark that we can write

$$x(t) = y(t) + \int_0^t b(x(s))ds$$

Clearly the difference $|x(t) - y(t)|$ is uniformly Lipschitz with a bound of M for the Lipschitz constant and $y(t)$ is such that

$$\exp\left[< \theta\,, y(t) - y(0) > -\frac{1}{2}\int_0^t < \theta\,, a(s\,,\omega)\theta > ds\right]$$

is a martingale with respect to $(\Omega\,, \mathcal{F}_t\,, P)$. From this we deduce the following bound

$$E^P\left[\exp[< \theta\,, y(t) - y(s) >]\right] \le \exp[\frac{M(t-s)}{2}\|\theta\|^2]$$

or

$$E^P\left[\exp[< \theta\,, \frac{y(t)-y(s)}{\sqrt{t-s}} >]\right] \le \exp[\frac{M}{2}\|\theta\|^2].$$

It is easy to conclude now that

$$E^P\left[\frac{|y(t)-y(s)|^4}{|t-s|^2}\right] \le CM^2$$

for a universal constant C. From Garsia-Rodemich-Rumsey lemma we get our estimte and by Prohorov's theorem we get the total boundedness of the sequence P_n of probability measures. $\square$

We take a weak limit along a subsequence and call it P. We might as well assume that $P_n \to P$ weakly.

Lemma 4.5. *The limit P is a martingale solution for a, b.*

Proof. With $Z_f(t)$ as in the expression (4.7) we need to establish

$$\int_A Z_f(t)dP = \int_A Z_f(s)dP$$

for $A \in \mathcal{F}_s$. It is sufficient to prove

$$\int \Phi(\omega)Z_f(t)dP = \int \Phi(\omega)Z_f(s)dP \tag{4.8}$$

for bounded continuous (in the topology of uniform convergence on bounded time intervals) functions Φ that are $\mathcal{F}_s$ measurable. For such a Φ clearly

$$\int \Phi(\omega)Z_f^n(t)dP_n = \int \Phi(\omega)Z_f^n(s)dP_n \tag{4.9}$$

where

$$Z_f^n(t) = f(x(t)) - f(x(s)) - \int_s^t (\mathcal{L}_s^n f)(x(s))ds$$

and

$$\mathcal{L}_s^n = \frac{1}{2}\sum_{i,j} a_{i,j}^n(s\,,x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_j b_j^n(s\,,x)\frac{\partial}{\partial x_j}$$

We can let $n \to \infty$ and from the weak convergence of P_n to P, the convergence of $Z_f^n(t)$ to $Z_f(t)$, (uniformly on compact subsets of Ω), and the uniform boundedness of $Z_f^n(t)$ we can let $n \to \infty$ in equation (4.9) to conclude that equation (4.8) holds. We are done. □

Uniqueness is a much harder issue. Clearly we have it in the Lipschitz case. But the uniqueness cannot be done by approximation. The following general Markovian Principle works. Assume existence.

Lemma 4.6. *If there exists a family $\mu_{s,x,t}(\cdot)$ of probability measures such that, for any $P \in C_{s,x}$,*

$$P[x(t) \in A] = \mu_{s,x,t}(A)$$

then P is a Markov Process with $\mu_{s,x,t}(A)$ as transition probabilities and is therefore unique.

Proof. We proved a general princilpe that the conditional probabilty distribution $P_{t,\omega}$ of any solution $P \in C_{s,x}$ give $\mathcal{F}_t$ is in $C_{t,x(t)}$ almost surely. Therefore for $s < t < u$

$$P\left[x(u) \in A|\mathcal{F}_t\right] = \mu_{t,x(t),u}(A)$$

a.e. P, proving the Markov property and the lemma. □

Determining $P[x(t) \in A]$ for $P \in C_{s,x}$ can be done through solving certain partial differential equations. We know that

$$u(t\,,x(t)) - u(s\,,x(s)) - \int_s^t (\frac{\partial}{\partial \sigma} + \mathcal{L}_\sigma u)(\sigma\,,x(\sigma))d\sigma$$

is a martingale. Therefore for any smooth u and $P \in C_{s,x}$,

$$u(s\,,x) = E^P\left[f(x(t)) + \int_s^t g(\sigma\,,x(\sigma))d\sigma\right] \tag{4.10}$$

where

$$g(\sigma\,,\cdot) = -(\frac{\partial u}{\partial \sigma} + \mathcal{L}_\sigma u)(\sigma\,,\cdot) \tag{4.11}$$

and

$$u(t\,,\cdot) \equiv f(\cdot) \tag{4.12}$$

The relation holds for for every smooth u and every $P \in C_{s,x}$.

Lemma 4.7. *If u satisfies equations (4.11) and (4.12) with $g \geq 0$ and $f \geq 0$, and $C_{s,x}$ is nonempty then the maximum principle holds, i.e. $u(s\,,x) \geq 0$.*

Proof. Obvious from equation (4.10). □

We are actually interested in going in the converse. Suppose either

1. Suppose equation (4.11) is solvable for sufficiently many g satisfying (4.12) with $f \equiv 0$

or

2. Equation (4.11) is solvable with $g \equiv 0$ satsfying (4.12) for sufficiently many f, then

$$E^P\left[\int_s^t g(\sigma\,,x(\sigma))d\sigma\right]$$

or

$$E^P\left[f(x(t))\right]$$

are determined for sufficiently many g or f as the case may be. This can then be used to determine $P[x(t) \in A]$ for $P \in C_{s,x}$. What we mean by sufficiently many depends on the circumstances. We need either enough g's to recover the the measures $\{\mu_\sigma\}$ from the integrals

$$\int_s^t \int_{R^d} f(y)\mu_\sigma(dy)d\sigma$$

or enough f's to determine the measure from the integrals

$$\int_{R^d} f(y)\mu(dy)$$

If we know some thing about $P \in C_{s,x}$, like for instance

$$\mu_{s\,,x}(d\sigma, dy) = P_{s,x}[x(\sigma) \in dy]d\sigma$$

is always in some $L_p([s\,,t] \times R^d)$, then sufficiently many can be just any dense set in $L_q([s\,,t] \times R^d)$, where $\frac{1}{p} + \frac{1}{q} = 1$. Similarly if we know that for any $P \in C_{s,x}$, the measure $P[x(t) \in dy]$ is in $L_p[R^d]$ it is enough to solve f from a dense subset of $L_q[R^d]$. These remarks are quite pertinent especially when the coefficients are discontinuous.

4.5 Time dependent diffusions in one dimension.

Let us look at $d = 1$ and consider the equation

$$\frac{\partial u}{\partial \sigma} + \frac{1}{2}a(\sigma\,,y)\frac{\partial^2 u}{\partial y^2} = g(\sigma\,,y)$$

Suppose a is just measurable. If we insist on u being $C^{1,2}$, and g being continuous $u_t\,,u_{yy}$ and g are continuous and unless $u \equiv c$, there will be no solutions. For nonsmooth coefficients we have to deal with non-classical solutions.

Let us now illustrate the method with the problem of constructing solutions for the one dimensional problem with $0 < c \le a(t\,,x) \le C < \infty$ and $b \equiv 0$. We start with

Theorem 4.8. *Let us consider a stochastic integral with repect to the Brownian Motion on some probability space*

$$\xi(t) = x_0 + \int_0^t k(s\,,\omega)d\beta(s)$$

for some $k(s\,,\omega)$ satisfying

$$0 < c \le |k(s\,,\omega)|^2 \le C < \infty$$

Then there is a constant M depending only on c, C and T such that

$$\left|E\left[\int_0^T g(s\,,\xi(s))ds\right]\right| \le M\|g\|_{L_2([0,T]\times R)}$$

Proof. The key estimate is the following: Consider a function g with compact support on $(-\infty\,,\infty) \times R$. Define

$$u(s,x) = \int_s^\infty \int_R \frac{1}{\sqrt{2C\pi(t-s)}}g(t\,,y)\exp[-\frac{(x-y)^2}{2C(t-s)}]dtdy \tag{4.13}$$

If g is smooth it is easy to verify that

$$\frac{\partial u}{\partial s} + \frac{C}{2}u_{xx} = -g(s\,,x)$$

Taking Fourier transform $\widehat{u}$ of u in x and s,

$$i\tau\widehat{u}(\tau\,,\eta) - \frac{C\eta^2}{2}\widehat{u}(\tau\,,\eta) = -\widehat{g}(\tau\,,\eta)$$

or

$$\widehat{u}(\tau\,,\eta) = \frac{1}{\frac{C\eta^2}{2} - i\tau}\widehat{g}(\tau\,,\eta)$$

and

$$\widehat{u_{xx}}(\tau\,,\eta) = \frac{\eta^2}{i\tau - \frac{C}{2}\eta^2}\widehat{g}(\tau\,,\eta)$$

Therefore using the isometry of the Fourier transform

$$\|u_{xx}\|_{L_2} \leq \frac{2}{C}\|g\|_{L_2}$$

Now to prove the theorem there is no loss of generality in assuming that k is simple. With uniform bounds we can pass to the limit. We define the linear functional

$$\Lambda(g) = E\left[\int_0^T g(s\,,\xi(s))ds\right]$$

Clearly if k is simple, then ξ is piecewise Brownian Motion and the transition probability $p_{\sigma^2}(s\,,x\,,t\,,y)$ of the Brownian motion is in $L_2[[0\,,T]\times R]$ uniformly in s and x provided $0 < c \leq \sigma^2 \leq C < \infty$. It is now easy to get a bound

$$|\Lambda(g)| \leq M\|g\|_{L-2}$$

with a constant M that depends on the number of intervals over which k is constant. We want to improve our bound to make it depend only on c, C and T. If we take g that vanishes for $t \geq T$, construct u as in equation (4.13), then $u(T\,,\cdot) \equiv 0$ and by Ito's formula

$$\begin{aligned}
u(0\,,x) &= -E^P\left[\int_0^T u_s + \frac{k^2(s\,,\omega)}{2}u_{xx}(s\,,\xi(s))ds\right] \\
&= -E^P\left[\int_0^T (u_s + \frac{C}{2}u_{xx})(s\,,\xi(s))ds\right] \\
&\qquad + E^P\left[\int_0^T \frac{C - k^2(s\,,\omega)}{2}u_{xx}(s\,,\xi(s))ds\right] \\
&= E^P\left[\int_0^T g(s\,,\xi(s))ds\right] + E^P\left[\int_0^T \frac{C - k^2(s\,,\omega)}{2}u_{xx}(s\,,\xi(s))ds\right]
\end{aligned}$$

or

$$E^P\left[\int_0^T g(s\,,\xi(s))ds\right] = u(0\,,x) - E^P\left[\int_0^T \frac{C - k^2(s\,,\omega)}{2}u_{xx}(s\,,\xi(s))ds\right]$$

and therefore

$$E^P\left[\int_0^T g(s\,,\xi(s))ds\right] \le |u(0\,,x)| + \frac{C-c}{2}\,E^P\left[\int_0^T |u_{xx}(s\,,\xi(s))|ds\right]$$

In other words

$$|\Lambda(g)| \le |u(0\,,x)| + \Lambda(|u_{xx}|) \tag{4.14}$$

Taking supremum in equation (4.14) over g with $\|g\|_{L_2} \le 1$, and denoting it by M, we get

$$M \le \sup_{g:\|g\|\le 1} |u(0\,,x)| + \frac{C-c}{2}\frac{2}{C}M = \sup_{g:\|g\|\le 1} |u(0\,,x)| + (1-\frac{c}{C})M$$

Since

$$\sup_{g:\|g\|\le 1} |u(0\,,x)| = \left[\int_0^T \int_R \left[\frac{1}{\sqrt{2\pi Ct}}\exp[-\frac{y^2}{2Ct}]\right]^2 dydt\right]^{\frac{1}{2}} = A(T\,,C) < \infty$$

we have

$$M \le \frac{C}{c}A(T\,,C) = A(T\,,C\,,c)$$

and the theorem is proved. □

Remark 4.11. An immediate consequence of the estimate is that any stochastic integral of the form

$$\xi(t) = \int_0^t k(s\,,\omega)d\beta(s)$$

with

$$0 < c \le |k(s\,,\omega)|^2 \le C < \infty$$

has a distribution $q(t\,,dy)$ that has a density $q(t\,,y)dy$ in y for almost all t, with the bound

$$\int_0^T \int_R |q(t\,,y)|^2 dtdy \le [A(T\,,C\,,c)]^2$$

Remark 4.12. In particular if $p(s\,,x\,,t\,,dy)$ is the transition probability for a diffusion with smooth coefficients $a = a(t\,,x)$, and $b = 0$, with

$$0 < c \le a(t\,,x) \le C < \infty$$

it has a density $p(s\,,x\,,t\,,y)$ for almost all t and,

$$\sup_{\substack{x\\ s\le t}} \int_t^T |p(s\,,x\,,t\,,y)|^2 dtdy \le [A(T-t\,,C\,,c)]^2$$

We will now use the above theorem to prove existence as well as uniqueness of martingale solutions. We assume $0 < c \le a(t\,,x) \le C < \infty$. We can construct a_n satisfying the same bounds that are smooth and we can have $a_n \to a$ almost everywhere in t and x. We have from Lemma 4.4, the total boundeness of the measures P_n for the approximating smooth coefficients. But now the expressions

$$Z_f^n(t) = f(x(t)) - f(x(0)) - \int_0^t \frac{a_n(s\,,x(s))}{2} f_{xx}(x(s))ds$$

do not converge uniformly on compact subsets of Ω to

$$Z_f(t) = f(x(t)) - f(x(0)) - \frac{1}{2}\int_0^t a(s\,,x(s)) f_{xx}(x(s))ds$$

But, given any $\epsilon > 0$, we can find a_n^ϵ and a^ϵ such that $a_n^\epsilon \to a^\epsilon$ uniformly on compact subsets of $[0\,,\infty) \times R$ and

$$\int_0^T \int_{|x|\le \ell} [|a_n^\epsilon - a_n|^2 + |a^\epsilon - a|^2] dx\,dt \le \delta_\epsilon(T\,,\ell)$$

for some $\delta_\epsilon(T\,,\ell)$ such that $\delta_\epsilon(T\,,\ell) \to 0$ as $\epsilon \to 0$ for each T and ℓ. Now

$$Z_f^{n,\epsilon}(t) = f(x(t)) - f(x(0)) - \frac{1}{2}\int_0^t a_n^\epsilon(s\,,x(s)) f_{xx}(x(s))ds$$

converges nicely to

$$Z_f^\epsilon(t) = f(x(t)) - f(x(0)) - \frac{1}{2}\int_0^t a^\epsilon(s\,,x(s)) f_{xx}(x(s))ds$$

and

$$\int \Phi(\omega) Z_f^{n,\epsilon}(t) dP_n \to \int \Phi(\omega) Z_f^\epsilon(t) dP_n$$

for for smooth f and bounded continuous $\mathcal{F}_s$ measurable functons Φ. Since we now have a bound of the form

$$\begin{aligned}
&\sup_n \Big| \int \Phi(\omega)[Z_f^{n,\epsilon}(t) - Z_f^n(t)] dP_n \Big| \\
&\le C_1 \sup_n E^{P_n}\left[\int_0^T |a_n^\epsilon(t\,,x(s)) - a_n(t\,,x(t))| dt\right] \\
&\le 2CC_1 \sup_n P_n\left[\sup_{0\le t\le T} |x(t)| \ge \ell\right] \\
&\qquad + C_1 \int_0^T \int_{-\ell}^{\ell} |a_n^\epsilon(t\,,x) - a^\epsilon(t\,,x)| p_n(0\,,x\,,t\,,y) dy \\
&\le CC_1 \Delta(\ell) + C_1 \sqrt{\delta_\epsilon(T\,,\ell)} A(T\,,C\,,c)
\end{aligned}$$

with the a similar estimate for $Z_f^\epsilon - Z_f$

$$|\int \Phi(\omega)[Z_f^\epsilon(t) - Z_f(t)]dP| \leq CC_1\Delta(\ell) + C_1\sqrt{\delta_\epsilon(T\,,\ell)}A(T\,,C\,,c)$$

We can now interchange $n \to \infty$ limit and $\epsilon \to 0$ limit and we can conclude that

$$\lim_{n\to\infty} \int \Phi(\omega)Z_f^n(t)dP_n = \int \Phi(\omega)Z_f(t)dP$$

for all $t \geq s$, and therefore, from

$$\int \Phi(\omega)Z_f^n(t)dP_n = \int \Phi(\omega)Z_f^n(s)dP_n$$

it follows that

$$\int \Phi(\omega)Z_f(t)dP = \int \Phi(\omega)Z_f(s)dP$$

proving that P is a martingale solution for $[a(\cdot\,,\cdot), 0\,]$

Now we turn to proving uniqueness. We will attempt to solve the equations (4.11) and (4.12) with a function u of the form

$$u(s\,,x) = \int_s^T \int_R h(t\,,y)\,p_C(s\,,x\,,t\,,y)dy$$

Then as we saw earlier

$$\begin{aligned} u_s(s\,,x) + \frac{a(s\,,x)}{2}u_{xx}(s\,,x) &= u_s(s\,,x) + \frac{C}{2}u_{xx}(s\,,x) + \frac{a(s\,,x)-C}{2}u_{xx}(s\,,x) \\ &= -g(s\,,x) + [Bg](s\,,x) \\ &= -([I-B]g)(s\,,x) \end{aligned}$$

where

$$Bg(s\,,x) = \frac{a(s\,,x)-C}{2}u_{xx}(s\,,x)$$

and

$$\|Bg\|_{L_2} \leq (1 - \frac{c}{C})\|g\|_{L_2}$$

If we have two martingale solutions P^i, $i = 1, 2$ in $C_{s\,,x}$ and μ_t^i are their marginal distributions at times $t \geq s$, then they have densities $q^i(t\,,y)$ for almost all t, and

$$u(s\,,x) = \int_s^T [(I-B)g](t\,,y)\,q^1(t\,,y)dy = \int_s^T [(I-B)g](t\,,y)\,q^2(t\,,y)dy$$

Since we know that

$$\int_0^T \int_R |q^i(t\,,y)|^2 dtdy < \infty$$

in order to establish that $q^1 \equiv q^2$, it sufficient to show that the set of functions of the form $(I-B)g$ as g ranges over C^∞ functions is dense. Because $\|B\| < 1$ this is indeed true.