Chapter 4

Stochastic Differential Equations.

4.1 Existence and Uniqueness.

Our goal in this chapter is to construct Markov Processes that are Diffusions in R^d corresponding to specified coefficients $a(t,x) = \{a_{i,j}(t,x)\}$ and $b(t,x) = \{b_i(t,x)\}$. Ito's method consists of starting from any $(\Omega, \mathcal{F}_t, P)$ and an adapted Brownian Motion $\beta(t,\omega) = \{\beta_i(t,\omega)\}$ relative to $(\Omega, \mathcal{F}_t, P)$, with values in R^d. That is to say β has almost surely continuous paths and

$$\exp\left[< \theta\,, \beta(t) > - \frac{t\|\theta\|^2}{2}\right]$$

is a martingale with respect to $(\Omega, \mathcal{F}_t, P)$ for all $\theta \in R^d$.

The basic assumption on a and b are the following.

H1. The symmetric positive semidefinite matrix $a(t,x)$ can be written as $a(t,x) = \sigma(t,x)\sigma^*(t,x)$ for some matrix $\sigma(t,x)$ that satisfies a Lipschitz condition in x.

$$\|\sigma(t,x) - \sigma(t,y)\| \le A|x-y|$$

H2. The coefficients $b_i(t,x)$ satisfy a similar condition.

$$\|b(t,x) - b(t,y)\| \le A|x-y|$$

H3. Growth conditions. For simplicity we will assume that for some constant C

$$\|\sigma(t,x)\| \le C \qquad \text{and} \qquad \|b(t,x)\| \le C$$

Note that the choice of σ is not unique. We only assume that there is a choice of σ that satisfies the Lipschitz condition. The bounds of course are really bounds on a.

Theorem 4.1. *3.2 Given $s_0 \geq 0$ and an $\mathcal{F}_{s_0}$ measurable, R^d valued square integrable function $\xi_0(\omega)$, there exists an almost surely continuous progressively measurable function $\xi(t) = \xi(t, \omega)$ for $t \geq s_0$ that solves the equation*

$$\xi(t) = \xi_0 + \int_{s_0}^{t} \sigma(s, \xi(s)) d\beta(s) + \int_{s_0}^{t} b(s, \xi(s)) ds \tag{4.1}$$

The solution is unique in the class of progressively measurable functions.

Proof. The existence and uniqueness follow very closely the standard Picard's method for constructing solutions to ODE. We define

$$\xi_0(t) \equiv \xi_0 \quad \text{for} \quad t \geq s_0$$

and define successively, for $k \geq 1$,

$$\xi_k(t) = \xi_0 + \int_{s_0}^{t} \sigma(s, \xi_{k-1}(s)) d\beta(s) + \int_{s_0}^{t} b(s, \xi_{k-1}(s)) ds \tag{4.2}$$

Let us remark that the iterations are well defined. They generate progressively measurable almost surely continuous functions at each stage and by induction they are well defined. In order to prove the convergence of the iteration scheme we estimate successive differences. Let us assume with out loss of generality that $s_0 = 0$ and pick a time interval $[0, T]$ in which we will prove convergence. Since T is arbitrary that will be enough. If we denote the difference $\xi_k(t) - \xi_{k-1}(t)$ by $\eta_k(t)$, we have

$$\begin{aligned} &\eta_{k+1}(t) \\ &= \int_0^t [\sigma(s, \xi_k(s)) - \sigma(s, \xi_{k-1}(s))] d\beta(s) + \int_0^t [b(s, \xi_k(s)) - b(s, \xi_{k-1}(s))] ds \\ &= \int_0^t \delta_k(s) db(s) + \int_0^t e_k(s) ds \end{aligned} \tag{4.3}$$

Because of the Lipschitz assumption

$$\|\delta_k(s)\| \leq A \|\eta_k(s)\| \qquad \text{and} \qquad \|e_k(s)\| \leq A \|\eta_k(s) \tag{4.4}$$

We can estimate

$$\sup_{0 \leq \tau \leq t} \|\eta_k(\tau)\| \leq \sup_{0 \leq \tau \leq t} \| \int_0^\tau \delta_k(s) db(s)\| + \int_0^t \|e_k(s)\| ds$$

By Doob's inequality for martingales, the property of stochastic integrals and equation (4.4)

$$E\left[\sup_{0\le\tau\le t}\|\int_0^\tau \delta_k(s)db(s)\|^2\right] \le C_0 E\left[\|\int_0^t \delta_k(s)db(s)\|^2\right]$$
$$= C_1\int_0^t E\left[\|\delta_k(s)\|^2\right]ds$$
$$\le A^2C_1\int_0^t E\left[\|\eta_k(s)\|^2\right]ds$$

On the other hand we can also estimate for $t\le T$,

$$E\left[\left(\int_0^t \|e_k(s)\|ds\right)^2\right] \le TE\left[\int_0^t \|e_k(s)\|^2 ds\right]$$
$$\le A^2T\int_0^t E\left[\|\eta_k(s)\|^2\right]ds$$

Putting the two pieces together, if we denote by

$$\Delta_k(t) = E\left[\sup_{0\le\tau\le t}\|\eta_k(\tau)\|^2\right]$$

then, with $C_T = A^2C_1(1+T)$,

$$\Delta_k(t) \le C_T\int_0^t \Delta_{k-1}(s)ds$$

Clearly

$$\eta_1(t) = \int_{s_0}^t \sigma(s\,,\xi_0)d\beta(s) + \int_{s_0}^t b(s\,,\xi_0)ds$$

and

$$\Delta_1(t) \le C_T\, t$$

By induction

$$\Delta_k(t) \le \frac{C_T^k\, t^k}{k!}$$

From the convergence of $\sum_k[\frac{C_T^k T^k}{k!}]^{\frac{1}{2}}$ we conclude that

$$\sum_k E\big[\sup_{0\le t\le T}\|\eta_k(t)\|\big] < \infty.$$

By Fubini's theorem

$$\sum_k \sup_{0\le t\le T}\|\eta_k(t)\| < \infty \qquad \text{a.e.}\quad P.$$

In other words for almost all ω with respect to P,

$$\lim_{k\to\infty} \xi_k(t) = \xi(t)$$

exists uniformly in any finite time interval $[0,T]$. The limit $\xi(t)$ is easily seen to be progressively measurable solution of equation (4.1).

Uniqueness is a slight variation of the same method. If we have two solutions $\xi(t)$ and $\xi'(t)$, their difference $\eta(t)$ satisfies

$$\begin{aligned}\eta(t) &= \int_0^t [\sigma(s\,,\xi(s)) - \sigma(s\,,\xi'(s))]d\beta(s) + \int_0^t [b(s\,,\xi(s)) - b(s\,,\xi'(s))]ds\\ &= \int_0^t \delta(s)db(s) + \int_0^t e(s)ds\end{aligned}$$

with

$$\|\delta(s)\| \le A\|\eta(s)\| \qquad \text{and} \qquad \|e(s)\| \le A\|\eta(s)\|$$

Just as in the proof of convergence, for the quantity

$$\Delta(t) = E\left[\sup_{0\le s\le t} \|\eta(s)\|^2\right]$$

we can now obtain

$$\Delta(t) \le C_T \int_0^t \Delta(s)ds$$

We have the obvious estimate $\Delta(t) \le C_T$ and we obtain by iteration

$$\Delta(t) \le (C_T)^{k+1}\frac{t^k}{k!}$$

for every k. Therefore $\Delta(t) \equiv 0$ implying uniqueness. □

This uniqueness theorem has a special form. If two solutions of equation (4.1) are constructed on the same same space for the same Brownian motion with the same choice of σ then they are identical for almost all ω. This seems to leave open the possibility that somehow different choices of σ or constructions in different probability spaces could produce different results. That this is not the case is easily established. Before we return to this let us proceed with some comments.

Remark 4.1. We can start with a constant x for our initial value at some time s and construct a solution $\xi(t) = \xi(t\,;s\,,x)$ for $t \ge s$. If we define

$$p(s\,,x\,,t\,,A) = P\big[\xi(t\,;s\,,x) \in A\big]$$

then our solutions are Markov processes with transition probability $p(s\,,x\,,t\,,A)$.

The proof is based on the following argument. Because of uniqueness the solution starting from time 0 can be solved upto time s and then we can start again at time s with the initial value equal to the old solution, and we should not get anything other than the solution obtainable in a single step. In other words

$$\xi(t\,;s\,,\xi(s\,,0,x)) = \xi(t\,;0\,,x)$$

Since the solution $\xi(t\,;s\,,\xi(s\,,0,x))$ only depnds on $\xi(s\,,0,x)$ which is $\mathcal{F}^s$ measurable and increments $d\beta$ of the Brownian paths over $[s,t]$ that are independent of $\mathcal{F}_s$, the conditional distribution

$$\begin{aligned}
P\left[\xi(t) \in A|\mathcal{F}_s\right] &= P\left[\xi(t\,;s\,,\xi(s)) \in A|\mathcal{F}_s\right] \\
&= P\left[\xi(t\,;s\,,z) \in A\right]|_{z=\xi(s)} \\
&= p(s\,,\xi(s)\,,t\,,A)
\end{aligned}$$

establishing the Markov property.

Remark 4.2. A similar argument will yield the strong Markov property. We use the fact that the after a stopping time τ the future increments of the Brownian motion are still independent of the σ-field $\mathcal{F}_\tau$. There are some details to check about restarting the SDE at a stopping time. But this is left as an exercise.

Remark 4.3. If we have two solutions on two different spaces of the same equation with the same constant (i.e. non random) initial value, i.e. with the same σ and b that satisfy our assumptions, then they have the same distributions as stochastic processes. If we notice our construction, each iteration $\xi_k(t)$ was a well defined function of ξ_{k-1} and the Brownian incremets. The iteration scheme is the same in both. At each stage they are identical functions of different Brownian motions. Therefore they have the same distribution. Pass to the limit.

Remark 4.4. If $\xi(t)$ is any solution anywhere for any choice $\bar{\sigma}$ of the square root, then ξ is a diffusion corresponding to the coefficients $a = \bar{\sigma}\bar{\sigma}^*$, b and can be represented, by enlarging the space if necessary, as a solution of equation(4.1) with any arbitrary choice of the square root σ. In particular if one is available with the Lipschitz property and b is also Lipschitz we are back in the old situation. Therefore if there is a Lipschitz choice available then the distribution of any solution with any choice of the square root is identical to the one coming from the Lipschitz choice. In particular the distribution of any two Lipschitz choices are identical.

4.2 Some Examples. A Discussion of Uniqueness.

Ornstein-Uhlenbeck Process. The stochastic differential equation

$$dx(t) = \sigma\beta(t) - ax(t)dt\,; \qquad x(0) = x_0 \tag{4.5}$$

has an explicit solution

$$x(t) = e^{-at}x_0 + \sigma e^{-at}\int_0^t e^{as}d\beta(s)$$

which has a Gaussian distribution with mean $e^{-at}x_0$ and variance given by

$$\sigma^2(t) = \sigma^2 e^{-2at}\int_0^t e^{2as}ds = \frac{\sigma^2}{2a}(1-e^{-2at})$$

This is a Markov Process with stationary Gaussian transition probablity densities:

$$p(t\,,x\,,y) = \frac{1}{\sqrt{2\pi}\,\sigma(t)}\exp\left[-\frac{(y-e^{-at}x)^2}{2\sigma^2(t)}\right]$$

This is particularly interesting when $a > 0$, which is the stable case, and then

$$\lim_{t\to\infty}\sigma^2(t) = \theta = \frac{\sigma^2}{2a}$$

and

$$\lim_{t\to\infty}p(t\,,x\,,y) = \frac{1}{\sqrt{2\pi\theta}}\exp\left[-\frac{y^2}{2\theta}\right]$$

Geometric Brownian Motion: The function $x(t) = x_0\exp\big[\sigma\beta(t)+\mu t\big]$ satisfies according to Ito's formula the equation

$$dx(t) = \sigma x(t)d\beta(t) + (\sigma\mu + \frac{\sigma^2}{2})x(t)dt\,; \qquad x(0) = x_0$$

so that a solution of

$$dx(t) = \sigma\,x(t)d\beta(t) + \mu x(t)dt\,; \qquad x(0) = x_0$$

is provided by

$$x(t) = x_0\exp\big[\sigma\beta(t) + (\mu - \frac{\sigma^2}{2})t\big]$$

Notice the behavior

$$\frac{1}{t}\log x(t) \simeq (\mu - \frac{\sigma^2}{2}) \quad \text{a.e.}$$

as well as

$$\frac{1}{t}\log E[x(t)] \simeq \mu$$

The explanation is that the larger expectation is accounted for by certain very large values with very small probabilities.

Remark 4.5. **ODE and SDE.** The solution $x(t) = x_0 \exp\left[\sigma\beta(t) + (\mu - \frac{\sigma^2}{2})t\right]$ of

$$dx(t) = \sigma\, x(t) d\beta(t) + \mu x(t) dt\,; \qquad x(0) = x_0$$

is nice smooth map of Brownian paths and makes sense for all functions f

$$x(t\,, f) = x_0 \exp\left[\sigma f(t) + (\mu - \frac{\sigma^2}{2})t\right]$$

and for smooth functions as well. If we replace β by a smooth path f, it solves

$$dx(t) = \sigma\, x(t) df(t) + (\mu - \frac{\sigma^2}{2}) x(t) dt\,; \qquad x(0) = x_0$$

The Ito map satisfies the wrong equation on smooth paths. This is typical.

There are various ways of constructing a solution that correspond to a Diffusion with coefficients $a(t\,,x) = \{a_{i,j}(t\,,x)\}$ and $b(t\,,x) = \{b_i(t\,,x)\}$. For a square root σ satisfying $\sigma\sigma^* = a$ we can attempt to solve the SDE

$$dx(t) = \sigma(t\,, x(t)) d\beta(t) + b(t\,, x(t)) dt\,; x(0) = x_0$$

on the Wiener space and get a map $\beta(\cdot) \to x(\cdot)$. Such a solution if it exists will be called a **strong solution**. A **Matingale Solution** is a measure P on $\Omega = C[[0\,,\infty)\,; R^d]$ such that $P[x(0) = x_0] = 1$ and for each smooth f the expression

$$f(x(t)) - f(x(0)) - \int_0^t (\mathcal{L}_s\, f)(x(s) ds$$

is a martingale with respect to $(\Omega\,, \mathcal{F}_t\,, P)$. If we can construct on some probability space $(\Omega\,, \mathcal{F}_t\,, \mu)$ a Brownian motion $\beta(\cdot)$ and an $x(\cdot)$ that satisfy

$$x(t) = x_0 + \int_0^t \sigma(s\,, x(s)) d\beta(s) + \int_0^t b(s\,, x(s)) ds$$

then we call $x(\cdot)$ a **Weak Solution** to the SDE. We make the following remarks.

Remark 4.6. A strong solution is a weak solution, and if σ is Lipschitz, then any weak solution is a strong solution. In particular two weak solutions on the same space involving the same Brownian Motion are identical.

Remark 4.7. The distribution P of any Weak Solution is a Martingale Solution and conversely any Martingale Solution is the distribution of some Weak Solution.

Remark 4.8. For a given square root σ if we deifine the $2d \times 2d$ matrix $\tilde{a}$ as the 2×2 matrix of $d \times d$ blocks

$$\tilde{a} = \begin{pmatrix} a & \sigma \\ \sigma^* & I \end{pmatrix}$$

and $\tilde{b}$ as $(b, 0)$, then a weak solution of $\sigma\,, b$ is the same as a Martingale Solution of $\tilde{a}\,, \tilde{b}$.

Remark 4.9. Any two weak solutions on different probability spaces can be put on the space space with the same Brownian Motion.

This needs an explanation. What we mean is the following: Let P_1 and P_2 be two martingale solutions for $\tilde{a}\,,\tilde{b}$. Then we can construct a Q which is a martingale solution for the $3d$ dimensional problem with coordinates $x\,,y\,,z$ for $\widehat{a}\,,\widehat{b}$ where in blocks of $d \times d$

$$\widehat{a} = \begin{bmatrix} a(x) & \sigma(x)\sigma(y)^* & \sigma(x) \\ \sigma(y)\sigma(x)^* & a(y) & \sigma(y) \\ \sigma(x)^* & \sigma(y)^* & I \end{bmatrix}$$

while $\widehat{b}$ is given by $[b(t\,,x)\,,b(t\,,y)\,,0]$ which has the following two additional properties:

1. The distribution of x, z coordinates is P_1 and that of the y, z coordinates P_2.

2. Given the z cordinate the x and y coordinates are conditionally independent.

We start with P the Wiener measure, P_i^ω the conditional of '$x(\cdot)$' given the Brownian Motion under P_i and define

$$Q = P(d\omega) \otimes [P_1^\omega \times P_2^\omega]$$

i.e. we build in conditional independence. We can check that Q is a Martingale Solution for the $3d$ dimensional problem.

This construction allows us to make the following remark.

Remark 4.10. If it is true that for some $\sigma\,,b$ any two weak solutions on the same space with the same Brownian Motion are identical, then any weak solution is a strong solution and in such a context the Martingale solution is unique.