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**CONTINUOUS-TIME METHODS IN FINANCE:  
A REVIEW AND AN ASSESSMENT**

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# Continuous-Time Methods in Finance

## A Review and an Assessment

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### **Abstract**

I survey and assess the development of continuous-time methods in finance during the last 30 years. The subperiod 1969 to 1980 saw a dizzying pace of development with seminal ideas in derivatives securities pricing, term structure theory, asset pricing, and optimal consumption and portfolio choices. During the period 1981 to 1999 the theory has been extended and modified to better explain empirical regularities in various subfields of finance. This latter subperiod has seen significant progress in econometric theory, computational and estimation methods to test and implement continuous-time models. Capital market frictions, and bargaining issues are being increasingly incorporated in continuous-time theory.

## Introduction and Overview

The roots of modern continuous-time methods in finance can be traced back to the seminal contributions of Merton (1969, 1971, 1973b) in the late 1960s and early 1970s. Merton (1969) pioneered the use of continuous-time modeling in financial economics by formulating the intertemporal consumption and portfolio choice problem of an investor in a stochastic dynamic programming setting. Merton (1973b) also showed how such a framework can be used to develop equilibrium asset pricing implications, thereby significantly extending the asset pricing theory to richer dynamic settings and expanding the scope of applications of continuous-time methods to study problems in financial economics.<sup>2</sup> Within a span of about 30 years from the

publication of Merton's influential papers, continuous-time methods have become an integral part of financial economics. Indeed, in certain core areas in finance (such as asset pricing, derivatives valuation, term structure theory and portfolio selection, for example) continuous-time methods have proved to be the most attractive way to conduct research and gain economic intuition. The continuous-time approach in these areas has produced models with a rich variety of testable implications. The econometric theory for testing continuous-time models has made rapid strides in the last decade and has thus kept pace with the impressive progress on the theoretical front. One hopes that the actual empirical investigations and estimation using the new procedures will follow suit soon.

Undertaking a review of a field that is so vast is a daunting task. The continuous-time field in the last three decades has made such a strong impact in many core areas in finance and is starting to make significant inroads into other areas of finance in recent times that a comprehensive survey of the field is beyond the scope of this effort. In order to keep the task of this survey paper more tractable and to keep it within a reasonable number of pages of this journal, it is imperative that one rely on extant surveys and texts in this field. We are very fortunate in this respect: many scholarly surveys and texts are available with varying levels of technical sophistication to suit the palate of scholars with varied research interests. Once again in this context Merton (1990) serves as an excellent guidepost. Whereas the publication of Merton (1969) started this thriving field, a comprehensive survey and an illuminating perspective of the development of this field during the period 1969-1990 can be found in the book by Merton (1990). Several excellent texts and articles have surveyed this field or certain subsections of this field as applied to research in financial economics. The texts by Bhattacharya and Constantinides (1989), Harrison (1985), Malliaris (1982), Ingersoll (1987),

Dothan (1990), and Duffie (1988, 1996), Karatzas and Shreve (1988, 1998) in addition to Merton (1990) provide a detailed treatment of the developments in this field during the period 1969-1990 and also provide extensive references to research in this area. Moreover, there are excellent surveys of important subfields of continuous-time finance that are also available. Examples in this context are Bhattacharya and Constantinides (1989), Constantinides (1989) and Merton (1990). For this reason, my survey of the development of the field during the period 1969-1990 will be *relatively brief*. I will focus my attention instead mostly on the period 1990-1999.

It is useful to begin our survey with an overview of some of the major developments in the field. I will sketch the developments in the 1969-1980 period first and then those in the period 1981-1999 thereafter.

## 1969-1980 Period

The period from 1969 through 1980 saw most of the important breakthroughs in the field of continuous-time finance. The seminal contributions on options pricing by Black and Scholes (1973) and Merton (1973a) are unquestionably among the most influential papers in financial economics during this period. These papers changed the landscape of research in finance and the way in which finance research was viewed by the practitioners. Besides providing the first truly satisfactory model for the elusive problem of pricing options on equity, these papers also made the connection that many of the liabilities of the firm such as equity and debt can be thought of as contingent claims on the value of the assets of the firm<sup>3</sup>. This insight led to a whole new field of study which has come to be known as “contingent claims research” which has been the bedrock of much of the valuation questions in derivatives, corporate finance and the default risk literature until now. Simultaneously, exciting research developments were taking place in intertemporal asset pricing theory during this period. As noted earlier, the

papers by Merton (1969, 1971, 1973b) laid the foundations for the development of intertemporal asset pricing theory. Merton (1973b) extended in an intuitive way the insights of the static equilibrium asset pricing models of Sharpe (1964) and Lintner (1965) to an intertemporal setting. These formulations were cast in a partial equilibrium setting. An important insight of the Intertemporal CAPM [ICAPM] of Merton (1973b) was that the investor has hedging demands in the optimal portfolio to hedge against stochastic shifts in his or her opportunity set. This component of asset demand is unique to the intertemporal nature of the asset pricing problem, wherein the changing state of the economy over time can be satisfactorily specified. Merton showed that this in turn has pricing implications<sup>4</sup>. A significant plateau in partial equilibrium asset pricing theory was reached when Breeden (1979) showed that the intertemporal CAPM of Merton (1973b) with multiple betas can be collapsed into a single beta ICAPM when the beta is measured with respect to the consumption. During this period, general equilibrium asset pricing theory in a discrete-time setting was developed in a seminal paper by Lucas (1978) in an exchange economy setting. Cox, Ingersoll and Ross (1985a) developed a general equilibrium framework for asset pricing in a continuous-time setting production economy. It is important to note that while this paper was published in 1985, the central ideas of the published paper were available in a working paper form as early as 1977. As an application of their general equilibrium framework, Cox, Ingersoll and Ross (1985b) developed a general equilibrium model of the term structure of default-free securities. This period also saw the development of a “risk-neutral” pricing approach which was originally proposed by Cox and Ross (1976a, 1976b). They illustrated their approach with several explicit and constructive examples of options pricing with alternate stochastic processes. In a seminal paper, Harrison and Kreps (1979) later provided the conceptual

foundation for this approach. Their insight, which has come to be known as the Martingale representation theory, is now commonly used in many branches of financial economics. It provides the basis for much of the derivatives research today. This paper has also helped to make the connections between equilibrium asset pricing models and arbitrage-free models. This rapid pace of developments in the continuous-time field during this period has been responsible for the impact that this field has had in the finance profession (in academic as well as in industry) today.

### 1981-1999 Period

The major developments during this latter period have taken place around the following broad issues and questions:

1. Establishment of an isomorphic relationship between dynamic stochastic optimal control problems and static state space representation frameworks in complete markets: The papers by Cox and Huang (1989a) and Karatzas, Lehoczky and Shreve (1987) show how Martingale representation theory can be used to reduce the dynamic intertemporal problems into a static problem in a complete markets setting. This method has been particularly helpful in solving intertemporal portfolio selection problems and asset pricing problems wherein the investors are subject to certain constraints. Explicit solutions have been obtained to consumption and portfolio rules under constraints.

2. Efficiency results in continuous trading with a few securities: Duffie and Huang (1985) showed that continuous trading permits the implementation of Arrow-Debreu equilibrium with far fewer securities than the full complement of securities. This is one of the very few papers to address the issue of welfare consequences of continuous-trading opportunities in a few long-lived securities. Their contribution formalized the importance of dynamic trading opportunities from a welfare perspective. Their direction

is central in addressing more ambitious questions within the paradigm of continuous-time methods. [We return to this under the Challenges section of the review.]

3. Reconciling the received theory of continuous-time finance with some observed empirical regularities: Specific examples of such empirical regularities are the equity premium puzzle; countercyclical variations in equity premia; predictability of equity returns (in asset pricing); volatility smiles and skews (in derivatives valuation); the persistent negative correlation between the changes in the default premium and the changes in the default-free interest rates (in the credit risk literature); the inability of conventional single-factor models of term structure to explain the rich variations in the shape of the yield curve and its evolution over time (in the term structure theory); and the inability to explain the levels and variations in the spreads between corporate debt securities and treasury securities (default risk literature). The homebias in domestic and international portfolio selection has presented another challenge to the theorists in this field. The attempts by researchers to resolve the equity premium puzzle at least in part have led to the development of continuous-time models with richer set of preferences. Models of habit formation by Sundaresan (1989) and Constantinides (1990) have provided an impetus for empirical work in asset pricing, as evidenced in Campbell and Cochrane (1999). The generalizations of utility functions to the family of Stochastic Differential Utility was accomplished by Duffie and Epstein (1992).

4. The incorporation of market frictions into continuous-time theory has helped the theory to better explain some stylized facts: Examples in this regard include taxes, transactions costs, restricted market participation, incompleteness, informational asymmetries, etc. Researchers have started to expand the field to incorporate market frictions as in Back (1992, 1993),

Brennan and Xia (1999), Detemple (1986), He and Pearson (1991), and Veronesi (1999). Optimal contracting arrangements in a principal-agent setting have been derived as in Holmstrom and Milgrom (1987).

5. Calibration of models to data derived from the markets: How can the models of asset pricing be calibrated to market data to make them attractive enough for valuing financial claims in real life? Examples in this context are the default-free term structure models and the reduced-form approach to valuing securities that are subject to credit risk. An example is the development of arbitrage-free models of term structure that can be calibrated to the term structure of interest rates and the term structure of volatilities that are derived from market data. This research agenda was initiated by Ho and Lee (1986) and generalized in a conceptually satisfactory manner by Heath, Jarrow and Morton (1992).

6. In addition, as a logical progression of research, estimation of continuous-time models in finance has become an active area of research during this period. Indeed, most striking contributions in the continuous-time field during this period appear to have come from the econometric theory designed to develop moment restrictions in continuous-time models and parametric and non-parametric approaches to estimating continuous-time models in finance. The development of estimation techniques such as the simulated method of moments by Duffie and Singleton (1993), maximum likelihood estimation techniques by Ait-Sahalia (1999a), the characteristics function-based estimation strategies for affine processes by Singleton (1999), and the generation of moment implications by Hansen and Scheinkman (1995), are a few important examples of research in this area.

7. As Cox and Huang (1989b) predicted, there has been a stream of papers attempting to integrate game-theoretic and bargaining considerations into continuous-time models to enhance the intertemporal pricing richness



of the framework with the objective of making the contractual features endogenous. This has always been the weak link in the continuous-time field. Game theoretic and strategic considerations in models of securities valuation have been introduced in Fan and Sundaresan (1999), Mella Barral and Perraudin (1997), and Grenadier (1999).<sup>5</sup>

This is also the period which has seen the growth of the so-called real options literature which was initiated by Brennan and Schwartz (1985) and McDonald and Siegel (1986). The role of costs of reversibility in asset pricing has been investigated by Kogan (1998). This literature has also made a significant inroads into the mainstream economics literature thanks to some insightful papers by Dixit (1989a, 1989b, 1991) and the influential book by Dixit and Pindyck (1994).

### Survey and Assessment

For concreteness, I will classify the literature into the following subfields. Needless to say, these subfields are interrelated. A paper which is placed in one subfield can also be situated in one or more of the other subfields. For example, many developments in econometric theory and estimation procedures have been primarily directed towards either the term structure area or derivatives area. Likewise, papers in market frictions often attempt to explain stylized facts in the equity premium or asset pricing. Nonetheless, this classification allows us to get a perspective of how the field has evolved over time in related subfields and what insights we have been able to glean in these different subfields. The subfields that we will consider are the following:

1. Options and other derivatives valuation.
2. Term structure of interest rates.
3. Asset pricing.
4. Dynamic consumption and portfolio choice.

5. Default risk and credit spreads.
6. Real options applications.
7. Capital market frictions - transactions costs, lack of symmetric information, restricted market participation, taxes, incomplete markets, *etc.*
8. Estimation of continuous-time models.
9. International markets and exchange rate dynamics.

Together, these subfields span most but not all areas of finance in which continuous-time methods have made a significant impact. We will take up each of these areas in turn and review some of the major contributions and try to identify some of the open questions that are yet to be addressed in the literature. In a companion paper in this journal, John Campbell reviews the asset pricing literature in considerable detail. Hence in our survey we will try to complement Campbell's review by briefly reviewing the contributions of continuous-time methods in asset pricing rather than undertaking an independent survey of this important sub-field.

## I. Options and Other Derivatives Valuation

The seminal contributions of Black and Scholes (1973) and Merton (1973a) opened the flood gates in this area of research<sup>6</sup>. Since the publication of their papers, literally hundreds of papers have been written on the valuation of derivatives securities such as options on different underlying assets, forward contracts, futures contracts, swaps, etc. Their observation that many claims (such as debt, equity, guarantees, etc.) can be thought of as contingent claims has further fueled the growth of this subfield. Much of the research from 1990 to 1999 has attempted to develop models to value the dizzying array of derivative securities that are currently offered in the exchanges and in the dealer markets. There has also been a concerted effort to build models to reconcile the theory with documented empirical anomalies. This strand

of research can be classified into the following groups:

1. Models of valuation of complex derivative securities. Illustrative examples of such securities are mortgage-backed securities such as collateralized mortgage obligations, exotic options (barrier options, lookback options, Asian options, etc.), passport options, shout options, volatility swaps, options on swaps, etc. There is a very large fixed-income derivative market. The focus in this strand is to develop and implement models that can be used for valuing and hedging complex derivative instruments and transactions.

2. Numerical and computational advances to implement those models for which there are no closed-form solutions. Prime examples here would include almost all interesting American options pricing problems, valuation of tranches of CMOs, etc. In addition, the problem of managing and measuring the risk of large portfolios has assumed major importance. Models that attempt to measure market risk and credit risk and the possible interactions between the two are increasingly becoming relevant to industry. The risk management of large portfolios is another important area where numerical and computational procedures have become indispensable. The development of numerical procedures has kept pace with the development of the theory due to the free-boundary nature of some derivatives (such as American style options that can be optimally prematurely exercised) and due to the fact that some derivatives have payoffs that are path-dependent (such as Asian options or look-back options).

3. Development of models that are motivated by stylized facts that are not easily explained by Black-Scholes models. Examples in this context are: The presence of an implied volatility smile or skew in options data. There seems to be a term structure of volatility smiles in the options data. In other words, the volatility smile effect appears to depend in a systematic way on the maturity structure of options. It also appears to be a lot stronger in

short-term options and less so in long-term options in many markets. This suggests a mis-specification in existing models of options pricing and points to the possible presence of skewness in the conditional distribution of returns. Recent empirical evidence has uncovered that equity returns are predictable. This may have important implications for derivatives research. In order to reconcile these observations, options theorists have tended to focus on two fronts: (a) models that take into account jumps in the underlying state variables, and (b) models that allow for volatility to be state-dependent or stochastic. The results have been mixed. It appears that the modeling of jumps risks and stochastic volatility improves our ability to fit the options data, but the term structure of implied volatilities still appears to have patterns that cannot be so easily reconciled. The development of options pricing models where the underlying equity returns are predictable has not had a big impact yet.

4. Effect of constraints on trading and transactions costs on derivatives hedging and pricing. The importance of constraints and transactions costs in the pricing and hedging has been explored by authors. Both utility-based approaches and no-arbitrage approaches have been used to identify useful bounds on options prices and implied volatility. Informational differences and their effects on options pricing is also a new area that is developing in the literature.

For readers interested in a more detailed development of this field several specialized sources are available. Recent journals that publish research in this area include: 1) *Journal of Derivatives*, 2) *Mathematical Finance*, 3) *Review of Derivatives Research*, 4) *Journal of Financial Engineering* (which is now merged with the *Journal of Derivatives*), 5) *Finance and Stochastics*, 6) *Applied Mathematical Finance*, 7) *Journal of Computational Finance*, and 8) the *RISK* magazine. Several texts that are exclusively devoted to this

topic are available. They are listed at the end of this review.

## The Valuation of Options

The basic insight in this sub-category has remained the same: with (dynamically) complete markets, it is possible to synthesize such derivative securities through dynamic portfolio strategies that are self-financing. By forming a portfolio of the underlying stock coupled with borrowing, and continuously rebalancing the portfolio, Black and Scholes (1973) and Merton (1973a) show that the payoffs of a call option can be replicated. Several papers have clarified the nature of the restrictions that are needed to preclude a free lunch in the process of creating such self-financed replicating portfolios. These restrictions take the form of constraints on portfolio choice or a non-negativity restriction on the wealth of the investor. The contribution by Dybvig and Huang (1988) stresses the role of non-negative wealth and portfolio constraints; other technical conditions are discussed in Cox and Huang (1989). Until 1976, the use of replicating portfolios was the basis for valuing options. Then, Cox and Ross (1976a, 1976b) introduced the idea of risk-neutral valuation wherein the drift of the underlying stock price process was replaced by the risk-free rate of interest. This risk-neutral valuation procedure pioneered by Cox and Ross (1976a) began to be used in the subsequent option valuation literature. It was only after the publication of Harrison and Kreps (1979) that a fully satisfactory theoretical framework was developed for risk-neutral pricing and its connection to no-arbitrage in models with continuous trading made clear<sup>7</sup>. In their paper, they show that the absence of arbitrage implies the existence of a risk-neutral probability measure. This technique is now extensively used for pricing options. A brief description of their main valuation result is provided next. We introduce some notation to state their main result.

The price  $p_t$  of a security at date  $t$  which pays an amount  $X_T$  (which is

stochastic) at date  $T > t$ , is

$$p_t = E^P [m_t(T)X_T] \quad (1)$$

where  $m_t(T)$  is the pricing kernel and  $E^P [.]$  is the expectations under the original probability measure  $P$ . In equilibrium models of asset pricing  $m_t(T)$  is the marginal rate of substitution between time  $t$  and time  $T$ . Let us denote by  $r_s$  the instantaneously risk-free rate of return. Under the assumption of complete markets, and using the money market account as the numeraire, the price of the security may also be written as

$$p_t = E^Q \left[ e^{-\int_t^T r_s ds} X_T \right] \quad (2)$$

where the  $E^Q [.]$  is the expectation under the risk-neutral probability measure  $Q$ . The money market account is defined by the accumulation factor  $B_t = e^{\int_0^t r_s ds}$ . Much of the early work chose the “money market account” as the numeraire and this is what we have used to illustrate the representation above. There is flexibility in the choice of the numeraire asset. Geman, El Karoui, and Rochet (1994) show that alternative numeraires can be chosen. If we choose an asset  $Y$  as the numeraire, then the associated probability measure  $Q^Y$  is defined by its Radon-Nikodym derivative with respect to  $Q$ . Associated with any numeraire there will be a probability measure. Depending on the choice of the numeraire asset, different pricing models can be developed. Discount bond prices, and forward rates with different maturities, etc., can serve as numeraire assets. Some important papers have exploited this to develop interesting models of derivatives valuation. The so-called LIBOR market model uses discretely compounded forward rates as the numeraire and this approach has led to theoretically consistent models for valuing caps, options on swaps, etc. Important papers that apply this idea are: Brace, Gatarek, and Musiela (1997), Miltersen, Sandmann, and

Sondermann (1997). Musiela and Rutkowski (1997) and Jamshidian (1989, 1991). They develop pricing models with different numeraire assets. The flexibility in the choice of the numeraire has resulted in interesting models of interest rate derivatives in the profession.

Current research in options pricing has focused on valuing options when the underlying asset has both diffusion and jump components. Amin (1993) and Scott (1997) value options with jump risk and stochastic volatility. The valuation of options with stochastic volatility also has been a growing part of the options valuation literature. The initial contribution in this area is by Merton (1976), who assumed that the jump risk is diversifiable. A number of papers have since addressed this issue: Hull and White (1987) provided a framework for valuing options with stochastic volatilities. Heston (1993) provided a closed-form solution for options with stochastic volatility. A number of recent papers, including those by Hobson and Rogers (1988), Kallsen and Taqqu (1998), Melino and Turnbull (1990), Bates (1996), Bakshi, Cao and Chen (1997) have stressed the importance of jump components and stochastic volatilities in option pricing.

### The Valuation of Other Derivatives

The valuation of other derivative securities paralleled the development of equity option pricing theory. In part, this was due to the evolution of markets such as financial futures contracts, forward contracts, options on futures contracts, swaps and more recently exotic options whose payoffs are path-dependent. Margrabe (1978) generalized the basic options pricing model to develop the valuation formula for options to exchange one asset with another. Geske (1979) provided a solution to the compound options pricing formula. Black (1976) developed a model for valuing options on futures contracts. Brenner, Courtadon and Subrahmanyam (1985) and Ramaswamy and Sundaresan (1985) extended the theory to value options on futures when the

interest rates are stochastic. Cox, Ingersoll and Ross (1981), Richard and Sundaresan (1981), and Jarrow and Oldfield (1981) developed models for valuing forward and futures contracts. Duffie and Stanton (1992) developed a model for pricing contingent claims that are continuously resettled. Much of the theory until this stage regarded the options and other contingent claims as being default-free. Hull and White (1995) explicitly recognized the impact of default risk in valuing options and other derivative securities. Valuation of swaps with and without credit risk has been provided by a number of researchers including, Sundaresan (1991), Cooper and Mello (1991), Duffie and Singleton (1997), and Duffie and Huang (1996). More recently, the valuation of exotic options has mushroomed into a big research area. In this field, the focus is on pricing options or other derivatives whose payoffs are path-dependent. It is impossible to do justice to this area in a survey of this general sort. I should note that there are a number of books that have been written on this topic. Books by Briys, Bellalah, Mai and De Varenne (1998), Cox and Rubinstein (1985), Hull (1999), Dempster and Pliska (1997), Musiela and Rutkowski (1998), Clewlow and Strickland (1997), Nielsen (1999) and Zhang (1999) are but a few examples which provide a detailed treatment of this subject. They also explore the numerical schemes that are used to implement the valuation of exotic securities. In Haug (1997) one can get an extensive collection of options pricing formulas for many exotic derivatives.

Research in the valuation of exotic options has not broken any new conceptual ground. Mostly, the researchers apply the standard pricing theory to value the exotic security. New results are primarily in the numerical procedures that are used (we review this in the next subsection). Some papers have been able to derive closed-form solutions by imposing sufficient structure on the problem. The original paper that initiated the research



[when this area was not so fashionable] was by Goldman, Sosin, and Gatto (1979). Since then a number of papers have been written on the valuation of exotic options. Kemna and Vorst (1990) provide a pricing model for options which settle based on the average of the underlying asset value. Conze and Viswanathan (1991) value European path-dependent options which settle on the basis of the geometric average of the underlying asset. Other exotic options pricing models include the papers by Akahori (1995) and Dassios (1995). Geman and Yor (1993) provide a valuation method for Asian options using Bessel processes. Reiner (1992) provides valuation of quantity-adjusting (or quanto) derivatives, and Rubinstein (1991) has valuation formulas for many exotic options.

### Numerical Approaches

The development of numerical procedures paralleled the theory of valuing derivatives. Broadly the numerical procedures can be classified into (a) finite-difference approximations (with explicit and implicit schemes), (b) quasi-analytical procedures, and (c) Monte-Carlo simulation procedures. There are excellent surveys and texts (which are fairly up to date) on all these approaches. I can do no better than to cite them here first. The texts by Clewlow Strickland (1998), L. C. G. Rogers and D. Talay (1997) and the excellent survey paper by Boyle, Broadie and Glasserman (1997), and Brennan and Schwartz (1978). These papers provide a comprehensive survey of Monte-Carlo simulation approach and the finite-difference methods respectively. The early valuation models for American options required an efficient procedure for solving partial differential equations. Finite difference schemes for solving American put options were developed by Brennan and Schwartz (1977) and Schwartz (1977). In valuing American put options, several authors used analytical solutions for European puts to get good first approximations. In turn these approximations were used to obtain the

values of American put options using numerical schemes. Geske and Johnson (1984), MacMillan (1986) and Barone-Adesi and Whaley (1987) are examples of such quasi-analytical approaches to solving options valuation problems. The Monte Carlo methods have become quite popular in the industry. In part this is due to the growth of the mortgage-backed securities (MBS) markets where prepayments are typically path-dependent and therefore the valuation is typically more efficiently performed using simulation techniques. An excellent survey of Monte Carlo methods is to be found in Boyle, Broadie and Glasserman (1997a). The first paper in finance to have used this approach is Boyle (1977) who developed a Monte-Carlo simulation approach for valuing options. The American options pricing problem has received considerable attention during the last decade. Recently, Broadie and Detemple (1996) examine the American options pricing problem. They begin by establishing some new bounds on options prices. Using these bounds, they develop an interpolation procedure and implement a scheme for getting very accurate approximations to the values of American options. They also compare many numerical methods and analytical approximations. Karatzas (1988), Kim (1990), Jacka (1991), and Carr, Jarrow and Myneni (1992) provide interesting characterizations of the American options pricing problems. Ju (1998) shows that by approximating the early exercise boundary by multipiece exponential functions, it is possible to get excellent approximations for American options with even long-term maturity. Nelson and Ramaswamy (1990) show how simple binomial lattices can be constructed as diffusion approximations. They use this approach to value derivatives. In a series of papers, Broadie and Glasserman (1997a, 1997b, 1998) show how Monte-Carlo simulation methods may be used to value American style options pricing problems and in risk management applications. Other papers that have explored this problem include the paper by Longstaff and

Schwartz (1998). Anderson and Tu (1998) provide a numerical recipe for analyzing contingent claims valuation problems with strategic behavior.

### Transactions Costs and Frictions

Much of the impetus in this area came from Leland's important paper (1985) which studied the problem of replicating options with transactions costs. A number of papers have explored this question as well as valuing options with transactions costs or leverage constraints. They include Boyle and Vorst (1992), Broadie, Cvitanić and Soner (1998), and others. In a discrete-time setting Naik and Uppal (1993) studied the problem of hedging options when there are leverage constraints. They showed that the so-called "superreplication" strategies in which the values of hedge portfolios strictly dominate that of the option may be cheaper than exact replication. Broadie, Cvitanić and Soner (1998) extend the insights of Naik and Uppal (1993) to a continuous-time setting. They solve for the minimum cost portfolio, which "superreplicates" the payoff of a contingent claim when the replicating strategy is subject to convex constraints on portfolio weights. Cvitanic and Karatzas (1993, 1996) provide a framework based on Martingale representation approach for hedging and portfolio optimization under constraints and transactions costs. Detemple and Sundaresan (1999) explore the effect of non-tradability restrictions on options which are underlying assets that are subject to short-sale restrictions. Executive stock options are one class of assets that fall into this category. They show that the presence of such restrictions is conceptually equivalent to an unrestricted problem in which the drift of the underlying asset is reduced by an implicit dividend yield. This implies that such options may be optimally prematurely exercised even in the absence of actual dividend disbursements. Soner, Shreve and Cvitanic (1995) show that the minimal superreplication cost under transaction costs for the European Call option is simply the cost of buying one share of the

stock. Constantinides and Zariphopoulou (1999a, 1999b) provide a utility-based theory for obtaining bounds on options prices in the presence of transactions costs. In their first paper, Constantinides and Zariphopoulou (1999a) provided bounds for a European option when there is only one positive-net-supply security. In their second paper, they generalize the results (for a slightly weaker set of preferences) to include multiple assets, multiple derivatives which may have an early exercise feature or path-dependency. Merton (1990) and Boyle and Vorst (1992) have treated proportional transactions costs problems in a binomial setting.

The derivatives literature in the last decade has focussed more on implementation of realistic and fairly complex models of valuation. Nowhere this trend is more evident than in the area of the mortgage-backed securities market, where complex models of term structure are integrated with fairly intricate models of prepayments to produce valuation results and risk management inputs for MBS portfolios. This is also an area where industry is arguably ahead of the academics in many issues. The risk management area is another area in which the valuation of large portfolios of derivative securities is integrated with modeling issues, computational and estimation issues. The derivatives subfield is probably the one that has had most impact on the practitioners. Most financial services firms employ derivatives models of varying sophistication in their day-to-day activities of valuation and hedging. There has not been much of a conceptual breakthrough in this field since the seminal contributions in the period before 1980. Much of the insights have come in the areas of computational and implementation issues.

## II. Term Structure of Interest Rates

The term structure of default-free interest rates is yet another area where continuous-time methods have made a tremendous impact. An early precursor to equilibrium models of term structure is the paper by Merton (1975)

in which Merton studies a stochastic growth model. A side result of this paper shows that the instantaneous risk-free interest rate follows a nonlinear diffusion which is the basis for equilibrium term structure models. In this paper Merton did not pursue the term structure implications. In a later contribution, expanding on the insights of Merton, Sundaresan (1984) explores a two-sector model and its consequences for the term structure. The so-called arbitrage-free models of term structure in the 70s include the papers by Vasicek (1977), Richard (1978), Dothan (1978) and Brennan and Schwartz (1979). Back (1997) notes that a distinction has to be made between the “traditional arbitrage-free approach” to term structure and the absence of arbitrage opportunities. Absence of arbitrage-free opportunities implies the existence of a risk-neutral measure and in this sense the equilibrium models of term structure such as the CIR (1985b) are equivalent to risk-neutral pricing. Back (1997) and CIR (1985b) have pointed out that the so-called arbitrage-free models of term structure models in the late 1970s and early 1980s may in fact be prone to arbitrage opportunities. Part of the difficulty with the earlier models was the fact that the factor risk premia were specified exogenously. This may not always be consistent with absence of arbitrage opportunities. Recently there is a resurgence in the theoretical and empirical work on default-free term structure. The basic pricing equation for a pure discount bond whose price at time  $t$  [denoted by  $b(t, T)$ ] for the delivery of \$1 at time  $T > t$  is given below.

$$b(t, T) = E_t^P [m_t(T)] \quad (3)$$

where  $m_t(T)$  is the marginal rate of substitution or the pricing kernel. With a change of measure under the martingale representation method, the pure discount bond price can be written as:

$$b(t, T) = E_t^Q \left[ e^{-\int_t^T r_s ds} \right] \quad (4)$$

The theoretical work in this area has led to several empirical tests of single-factor models. The following single-factor models have been tried by scholars in attempting to specify a satisfactory pricing formula for a pure discount bond. Let the univariate diffusion process be specified as:

$$dr = \mu(r)dt + \sigma(r)dW_t \quad (5)$$

where,  $\{W_t, t \geq 0\}$  is a standard Brownian motion process. Various specifications that have been used in the field are provided in table I below which is reproduced from Aït-Sahalia (1998).

**Table I: Alternative Specifications of the Spot Interest Rate Process**

**Single – Factor Models**

$\mu(r)$	$\sigma(r)$	Stationary	References
$\beta(\alpha - r)$	$\sigma$	Yes	Vasicek (1977)
$\beta(\alpha - r)$	$\sigma r^{\frac{1}{2}}$	Yes	CIR (1985b), Brown & Dybvig (1986) and Gibbons and Ramaswamy (1993)
$\beta(\alpha - r)$	$\sigma r$	Yes	Courtadon (1982)
$\beta(\alpha - r)$	$\sigma r^\lambda$	Yes	Chan et.al
$\beta(\alpha - r)$	$\sqrt{\sigma + \gamma r}$	Yes	Duffie and Kan (1993)-Single factor specialization.
$\beta r(\alpha - \ln r)$	$\sigma r$	Yes	Brennan and Schwartz (1979)
$\beta r + \alpha r^{-(1-\delta)}$	$\sigma r^{\frac{\delta}{2}}$	Yes	Marsh and Rosenfeld (1983)
$\alpha + \beta r + \gamma r^2$	$\sigma + \gamma r$	Yes	Constantinides (1992)
$\beta$	$\sigma$	No	Merton (1973)
0	$\sigma r$	No	Dothan (1978)
0	$\sigma r^{\frac{3}{2}}$	No	Cox (1975) and CIR (1980)

Single-factor models are elegant and are tractable from an analytical point of view. But they are too parsimonious to explain the rich variability that one sees in the default-free yield curve over time and across maturities. These models have not performed well in empirical tests. Brown and Dybvig (1986) were the first to conduct a formal test of the CIR model. Gibbons and Ramaswamy (1993) used the GMM method to test the CIR model. Pearson and Sun (1994) made a few generalizations to the CIR model and used the conditional density to perform a maximum-likelihood test of the CIR model. Chan, et al., (1992) provided a comprehensive empirical analysis of single-factor term structure models. The test results generally reject the validity of single factor models as a sound basis for understanding the default-free term structure. Aït-Sahalia (1996) uses a non-parametric estimation procedure to examine single factor models with linear drifts. He concludes that every parametric single factor model is rejected by the data. Stanton (1997) uses the non-parametric approach and concludes that there are non-linearities in the drift coefficient of single factor models. The mean-reversion appears to play a strong role only at higher levels of short rates. Even before this empirical work, several scholars have explored multiple factor models of term structure. One of the earlier models is by Langetieg (1980), who extends the Vasicek (1977) model by assuming that the short rate is the sum of  $n$  state variables, each of which follows the process specified for the short rate in the Vasicek (1977) paper. Brennan and Schwartz (1979) considered a two-factor model with the short rate as one factor and the consol rate as the other. Richard (1978) proposed a two factor model with the real rate and the expected inflation rate as the state variables. Cox, Ingersoll and Ross (1985) propose a family of two factor models as well. Schaefer and Schwartz (1984) proposed a two-factor model in which the short-term rate and the spread between the long rate and the short rate as the state variables. Several



papers including Longstaff and Schwartz (1992), Fong and Vasicek (1992), Hull and White (1994), and Chen (1996), etc. have proposed multi-factor models of term structure.

Rebonato (1996) contains a discussion of multi-factor models of term structure and explores interest rate models, numerical issues and implementation. We can summarize the multi-factor models in terms of the state variables chosen by the authors as follows:

Table II: **Multi-Factor Models**

State Variables	Relationship Between State Variables	Analytical	References
Many unobserved State Variables	Short rate is the sum of all state variables	Yes	Langsetieg (1980)
Long rate and Short rate	Both rates are correlated	No	Brennan and Schwartz (1979)
Real rate and Expected rate of inflation	Restricted	Yes	Richard (1978)
Short rate and Volatility	Correlated	Yes	Longstaff and Schwartz (1992)
Unspecified	Correlated-flexible	Yes	Duffie and Kan (1994) - special cases of affine models
Short rate and the drift of the short rate	Correlated	Yes	Hull and White (1994)
Short rate, Drift of Short rate and the Volatility	Restricted	Yes	Chen (1996)
Short rate and Volatility	Correlated	Yes	Fung and Vasicek (1992)

A parallel development in this area has been the growth of affine models of term structure. Brown and Schaefer (1994) were the first to explore affine specifications of term structure. The contribution by Duffie and Kan (1996) significantly expanded the use of this approach, wherein the equilibrium (or arbitrage-free) short rate is an affine function of some underlying state variables of the economy. For example, the default-free short rate  $r_t$  may be given by

$$r_t = \varrho_0 + \varrho_1 \cdot X_t \quad (6)$$

where  $X_t$  is a multi-dimensional affine jump diffusion (AJD) process that can have both diffusion and jump components and  $\varrho_0$  and  $\varrho_1$  are constants. This process is *linearly related* to underlying state variables. Under this setup, the default-free bond prices can be written in closed form by evaluating the following expectations under the risk-neutral measure:

$$b(t, T) = E_t \left[ e^{-\int_t^T r_s ds} \right] \quad (7)$$

Duffie and Kan (1996) show how to compute the yield curve and prove that the yields are also affine functions of the vector of state variables. Thus,

$$y_t(\tau) = a(\tau) + b(\tau)X_t \quad (8)$$

where  $y_t(\tau)$  is the yield at time  $t$  of a  $\tau$  – *period* bond. *Many of the single factor models that we described earlier are in fact special cases of the affine models of term structure.* These models have now been extended to include the possibility of default as well (more will be said about this later in the paper). These models have mixed results. On the one hand, this family appears to be the only one for which tractable multi-factor term structure modeling seems feasible. This family also holds out great promise in terms of one’s ability to estimate parameters and nests several interesting term

structure models. On the other hand, this approach has difficulty matching conditional moments and correlations between adjacent sectors of the yield curve. The admissability conditions in affine models also preclude the state variables from reaching certain values so that the variance-covariance matrix is always positive definite. With a sufficient number of bonds in affine models the state vector is fully identified. Dai and Singleton (2000) contains a thorough analysis of affine models of term structure, and its advantages and limitations. Duffee (1998) reports that affine models do not have much of a predictive power in forecasting yields. Affine models imply that conditional moments are linear in state variables and lagged variables should be of no consequence in explaining yield changes. Kimmel (1999) provides convincing evidence that this implication is rejected by data. This also casts some doubt on the empirical validity of Markovian models of term structure.

An important development in this subfield is the growth of models that can be calibrated to the market data. These models began with the publication of Ho and Lee (1986). They showed in the context of an arbitrage-free model how one can construct a lattice of interest rates such that the model prices are consistent with the ones that we observe in the market. Once again, it is useful to recall the observation of Back (1997) that the absence of arbitrage in this family of “calibrated” models is in the sense that there is a risk-neutral probability. It should also be emphasized that one can always add free [time-dependent] parameters to term structure models to fit the current yield curve, as CIR (1985b) note.<sup>8</sup> Since the publication of this paper, there have been many variations of this basic idea: papers by Ho and Lee (1986), Black, Derman and Toy (1990), Hull and White (1990), Black and Karasinski (1991), etc. have provided alternate models for calibration to the data derived from the market. We provide a brief summary of the calibrated models in the next table. In all these cases, the short rate dy-

namics may be written using time-varying drifts and diffusion coefficients. They depend on the initial forward curve,  $F(0, t)$ , and other parameters.

**Table III: Calibrated Models**

$$dr = \mu(r, t)dt + \sigma(r, t)dz$$

Or,

$$dy = \mu(y, t)dt + \sigma(y, t)dz, \text{ where } y = \ln(r)$$

$\mu(r, t)$ or $\mu(y, t)$	$\sigma(r, t)$ or $\sigma(y, t)$	References
$F_t(0, t) + \sigma^2 t$	$\sigma$	Ho and Lee (1986)
$F_t(0, t) + aF(0, t) + \frac{\sigma^2}{2a}(1 - e^{-2at}) - ar$	$\sigma$	Hull and White (1990)
$\theta(t) + \frac{\sigma'(t)}{\sigma(t)}y$	$\sigma(t)$	Black, Derman and Toy (1990)
$\theta(t) + a(t)y$	$\sigma(t)$	Black and Karasinski (1991)

Perhaps the most significant development in this subfield during the 90s is the paper by Heath, Jarrow and Morton (1992) (hereafter referred to as HJM). They make several contributions. First, they show that the absence of arbitrage imposes a restriction on the process for the forward rates of interest. *Indeed, the drift of the forward rates process is entirely pinned down by the volatility structure.* This is one of their main insights. Since volatility is something that we can potentially estimate using the actual data, this method is positive in its approach. They also show that the basic idea can be generalized to multiple factors. *Indeed, most existing models of term structure can be thought of as subsets of the HJM framework.* One of the shortcomings of the HJM model is that in its more general settings the model implies a short-rate process that is non-Markovian. The implementation requires the use of non-recombining trees. As a consequence, the implementation of the model is time consuming. But with the development of computing technology, this should be less of a concern in the not-too-distant-future. Since the publication of their influential paper, one of the important developments in the term structure literature are the so-called LIBOR market models which are based on discretely compounded forward rates as the numeraire. As noted earlier, papers by Brace, Gatarek, and Musiela (1997), and Miltersen, Sandmann and Sondermann (1997) have shown that alternate numeraires [other than the money market account] can lead to tractable models of interest rate derivatives. These models are able to provide a theoretically consistent framework for the Black's model for caps and swaptions. In a recent paper Jin and Glasserman (1999) show that it is possible to construct an underlying equilibrium model to support HJM models.

These models have many free parameters or “undetermined constants” that can be used to calibrate the market prices to model prices. These

models often have to be “recalibrated” from time to time with the result that the parameter stability is often a serious issue. A modeling strategy that was advanced by Kennedy (1994) proposes infinitely many state variables with a few parameters. Santa-Clara and Sornette (1998) apply this approach to valuing interest rate instruments.

The default-free term structure is another success story in the application of continuous-time methods in financial economics. Not only are these models widely used in the financial services industry in a variety of different applications in the construction of term structure models in the industry and in the pricing of mortgage-backed securities, but they are also used in the valuation of interest rate derivatives as well. There is a delightful interplay of valuation, estimation and computational issues that define and shape the questions in this area. In the last decade, many of the estimation techniques that have been developed are designed to address term structure issues. We will review them later in the paper.

### III. Asset Pricing

In a companion review paper in this issue of the Journal, John Campbell has a paper which is devoted *exclusively to the asset pricing literature*. Hence my survey of this important sub-field will be very brief and focussed only on the developments in asset pricing in a continuous-time setting. Much of these developments are motivated by the empirical regularities that are present in the data on asset prices. The key regularities in the data are:

1. Equity premium is far too high in relation to the levels that can be supported by the existing asset pricing models without recourse to implausible levels of risk aversion. The so-called equity premium puzzle was first set forth by Mehra and Prescott (1985). It is countercyclical.
2. There is long-term memory in asset returns and the default risk premium helps to predict equity returns.



3. The risk-free rate is relatively less volatile and models that appear to be successful in reconciling equity risk premium appear to generate implausible levels of fluctuations in risk-free rates, leading to what is now known as the “risk-free rate” puzzle.

Campbell (2000) in his lucid survey has articulated several additional stylized facts and has examined the extent to which the current asset pricing theory has been able to come to terms with these regularities. For my purposes, I will survey how the asset pricing scholars have responded to modeling in the continuous-time field.

The most actively worked area is the generalization of utility functions. One early direction was based on the notion of habit-forming utility functions. Typically, the utility functions take the following form:

$$U(c) = E \left[ \int_t^T \Psi(c_t, z_t, t) \right] \quad (9)$$

where,

$$z_t = e^{-\alpha t} z_0 + \delta \int_0^t e^{-\alpha(t-s)} c_s ds, \text{ and } z_0, \alpha, \delta \geq 0 \quad (10)$$

The utility function  $\Psi(c_t, z_t, t)$  can be additive or multiplicative in habits. [The consumption flow rate is denoted by  $c_t$  and the habits by  $z_t$ . The weights attached to past consumption levels are controlled by the factor  $\alpha$ ]. The papers by Sundaresan (1989) and Constantinides (1990) have stressed the importance of the fact that in such specifications the optimal consumption decision now takes into account the fact that the agent is setting a “standard of consumption” for the future. This leads to interesting predictions for optimal consumption behavior and asset pricing. In these papers, the utility function depended on the surplus of current consumption rate to the standard. These are so-called “internal habit” models where the agent’s decisions affect the standard of consumption. Detemple and Zapatero

(1991) study habit formation in the context of an exchange economy and conclude that the equity premium and the risk-free rates are affected in the same manner by habits. In models of “external habits” such as Abel (1990, 1999) and Campbell and Cochrane (1999), the consumption standard is unaffected directly by agent’s decisions, but only on aggregate consumption. This line of research appears to have had some success in recent empirical work. Campbell and Cochrane (1999) construct a model with “external habits” and show that it is capable of generating a high equity risk premium. Furthermore, they are able to keep the real risk-free rate a constant thereby avoiding the “risk-free rate puzzle.” Heaton (1993, 1995) provides empirical tests of models of asset pricing with intertemporally dependent preferences and studies the effects of time aggregation.

Another area of research is to consider the durability of consumption goods which produces a flow of services thereby establishing an intertemporal link. The effect of durability was studied by Dunn and Singleton (1993) in the context of term structure, and later by Hindy and Huang (1992, 1993) and by Hindy, Huang and Zhu (1997). A number of authors have also pursued general utility specifications such as stochastic differential utility. Duffie and Epstein (1992) and Duffie and Skiadas (1994) provide a general class of intertemporal utility functions that can accommodate a wide variety of utility functions of interest.

They consider a large class of utility functionals that satisfy the integral equation shown below:

$$U_t = E \left[ \int_t^T f_s(Z_s(C), U_s(C)) ds \mid F_t \right] \quad (11)$$

where  $C_t$  denotes the cumulative consumption up to time  $t$ . Duffie and Epstein (1992) study the asset pricing consequences of such a family of utility functions.<sup>9</sup>

Another category of models explores the implications of incomplete markets on asset pricing. Models in this vein include the papers by Constantinides and Duffie (1996), who explore the heterogeneity of consumers in the economy which causes individual consumption behavior to differ from aggregate consumption behavior. In their model individual income shocks are assumed to be permanent precluding any insurance possibilities. By appropriately specifying the individual income their model can generate the required aggregate consumption series and security returns. Another form of incompleteness is the nature of the information structure in the economy. Grossman and Shiller (1982) demonstrate that uninsurable risk has no effect on asset pricing if all individuals' consumption processes are governed by diffusions.

Basak and Cuoco (1998) explore the consequences of restricted stock market participation on equity premium and asset pricing. They motivate their work by appealing to the paper by Mankiw and Zeldes (1991) which reports that more than seventy percent of the households in the representative sample from 1984 Panel Study of Income Dynamics hold no stocks. They consider a pure exchange economy with two types of agents. One type is prevented from holding stocks [presumably because of informational costs] while the other type can invest in stocks which are claims to a stream of exogenously specified dividends process. In their model restricted agents choose consumption which has zero covariation with the stock market leaving the unrestricted agents to absorb the risks. Their numerical calibration is able to produce risk premium levels that are consistent with the historical levels, even with a small relative risk aversion coefficient.

The asset pricing area promises to be a major subfield with a number of stylized facts yet to be satisfactorily resolved. The continuous-time methods and models are expected to be in the forefront of asset pricing literature.

Despite this impressive success challenges do remain. We have not developed a satisfactory general equilibrium model of default in a continuous-time setting. Empirical evidence suggests that the default premium plays an important role in explaining and predicting the equity premium. In this context, default premium refers to the spread between corporate debt and otherwise identical government debt. Chang and Sundaresan (1999) provide a model of default risk in an asset pricing context. Understanding how default risk influences asset prices in the economy may provide clues to issues such as limits to arbitrage, financial contagion, etc. Much more work remains to be done in this area.

## IV. Dynamic Consumption & Portfolio Choice

The initial thrust for this literature came from Merton (1969, 1971). Later, in an important break-through, Cox and Huang (1989) and Karatzas, Lehoczky and Shreve (1986, 1987) showed that the martingale representation theory can be applied to reduce the stochastic dynamic programming problem to a static problem in complete markets. They were able to reformulate Merton (1971) and incorporate explicit non-negativity restrictions and solve in closed form for optimal consumption and portfolio rules. See Pliska (1986) for a treatment of the optimal portfolio selection problem. In much of the literature, optimal consumption and portfolio rules were derived assuming no labor income or default. Basak (1999) characterizes optimal consumption in the presence of labor and human capital. In a representative consumer setting Basak introduces consumption and leisure choices. He identifies cases wherein the presence of labor causes consumption to be smoother than the stock market. Sethi (1997) has compiled in a text a number of contributions in which the optimal consumption and investment policy is characterized under bankruptcy risk that is exogenously specified. In the

presence of optimal endogenous default, Chang and Sundaresan (1999) characterize the optimal consumption policy. They show that the consumption is much more sensitive to wealth as the wealth approaches an optimally chosen default boundary: consumers reduce their consumption rate to avoid the costs associated with default. Olney (1999) presents evidence that this was the case in the U.S. economy during the great depression.

On the portfolio selection front, recent papers have succeeded in incorporating portfolio insurance strategies in an equilibrium setting. The main papers in this area are by Basak (1995) and Grossman and Zhou (1996). One of the limitations of the approaches by Merton (1971) was that hedging demand in the optimal portfolios under a stochastic opportunity set could not be characterized in closed form. Typically, this has to be obtained by solving a non-linear partial differential equation. But working with a non-stochastic opportunity set leads to a counter-factual implication that the intertemporal hedging demand is zero. Recent research shows that the optimal portfolio weights do indeed depend on the investment horizon when the stock returns are predictable. Papers by Barberis (1999), Kim and Omberg (1996), Brennan, Schwartz and Lagnado (1996), and Campbell and Viceira (1999, 2000) are examples of this line of work. Using a log-linear approximation, Campbell and Viceira are able to characterize the portfolio demand under a stochastic opportunity set. One of their results is that the ratio of the proportion of bonds to stocks in the optimal portfolio increases with risk aversion. Sundaresan and Zapatero (1997) show that the asset allocation policies in which indexed (stochastic) liability is funded will exhibit systematic time variation depending on how close the market value of the assets are relative to the indexed liability. Liu (1998) derives a closed-form solution for the optimal portfolio weights in a stochastic opportunity setting when the default-free short rate follows the square

root diffusion process introduced by Cox, Ingersoll and Ross (1985b). An important innovation of the model by Liu (1998) is that the stock returns exhibit stochastic volatility or predictability and he is able to consider incomplete markets explicitly. Wachter (1999) uses Martingale methods to characterize the consumption and portfolio strategies in complete markets when stock returns are predictable. Chacko and Viceira (1999) develop portfolio and consumption rules under an incomplete market setting with stochastic volatility. They rely on an approximation scheme to solve the Bellman equation in their general applications. In only one special case are they able to find the exact solution. Nonetheless, these results are useful in terms of shedding some light on portfolio and consumption policies in more realistic settings. The robustness of the results of all these models to a general equilibrium closure is open to question, however. Kogan and Uppal (1999) provides approximation methods for solving consumption and portfolio problems in a continuous-time setting. They show applications drawn from both partial equilibrium and general equilibrium formulations.

## V. Default Risk and Credit Spreads

Leland (1998) contains a lucid treatment of the development of research in this area and it is a must read for scholars interested in gaining an insight into the key research issues in default risk. In the area of corporate debt valuation and credit derivatives, we have the following empirical challenges. First, the contingent claims pricing models based on Merton (1974) are not able to deliver the levels of spreads between corporate debt yields and otherwise identical Treasury yields. Second, financial distress is not costless and frequently the borrowers and lenders engage in negotiations to avoid costly default. One observes negotiated debt reductions in both bank loans as well as in corporate debt. Bankruptcy code, its existence or absence and its perceived “friendliness” to borrowers or lenders is a matter of

significance in these markets. Yet, we have very few pricing theories that have explicitly addressed this as a structural issue in the determination of spreads. Third, there seems to be persistent negative correlation between the changes in default-free interest rates and the changes in credit spreads. Finally, even very short-term default-risky securities appear to have significant spreads over their Treasury counterparts: even assuming that a fraction of this spread is due to taxes and liquidity, there is reason to suspect that default related spreads are significant. Models of default risk have difficulty in reconciling these stylized facts.

The literature on corporate debt pricing falls into three categories. The first category of papers model the lower reorganization boundary and the allocation of residual values upon liquidation exogenously. This strand is the so-called structural models of default. The pioneering work of Black and Scholes (1973) and Merton (1974) provide the basic framework here. Black and Cox (1976) use this idea to explicitly model indenture provisions such as safety covenants. Brennan and Schwartz (1980) and Ingersoll (1977) use this approach to price convertible and callable corporate liabilities. John (1993) provides a survey of this literature. Kim, Ramaswamy and Sundaresan (1993) extend the analysis to include cash flow-based covenants and stochastic interest rates. In these models the absolute priority rules (APR) are strictly enforced. Longstaff and Schwartz (1995) consider stochastic interest rates and deviations from absolute priority rules. The structural models require the knowledge of the underlying asset value and its volatility. While these are not easily estimated, practitioners have succeeded in doing so. The KMV corporation uses the equity data and equity volatility to “back out” the estimates of the underlying asset value and its volatility. Then using this information, they are able to estimate the probability of default. Duffie and Lando (1999) construct a model with imperfect information. In their

formulation, they assume that the forcing variable (the value of an unlevered firm) is not observable but that only a noisy process may be observed. If there is no other source [such as the equity of the firm] from which investors can recover the forcing variable, then the intensity process can be derived from a structural framework. They show that the intensity depends on the lower reorganization boundary. Their paper is a significant extension of Leland (1994).



**Table IV- Structural Models of Default Risk  
with No Negotiations or Strategic Behaviour**

Lower Boundary	Reorganization	Nature of Default	References and Comments
Exogenous		Default occurs upon the firm value reaching an exogenously specified level at the maturity of debt. Liquidation costlessly enforced. Single layer of debt.	Merton (1974) - Risk is due to the fluctuations in the issuing firm's value.
Exogenous		Cash flows insufficient to meet coupons. Liquidity-induced default.	Kim, Ramaswamy and Sundaresan (1992). Stochastic interest rates that are correlated with the firm's value.
Endogenous		Equity is issued to fund contractual coupons until the equity value is driven to zero.	Leland (1994), Leland and Toft (1996)
Exogenous		Default may occur before maturity of debt	Longstaff and Schwartz (1995): Stochastic interest rates.
Exogenous		Default is a "surprise"	Zhou (1996): Jump-diffusion process

The second category of papers focuses on exogenous specification of default outcomes and recovery rates based on an arbitrage-free valuation. These models assign probabilities of default and recovery rates exogenously, but derive pricing formulae that can be calibrated to data. A number of papers in this vein have studied default risk including Artzner and Delbaen (1995), Jarrow and Turnbull (1995), Lando (1997, 1998), Madan and Unal (1998), and Duffie, Schroder and Skiadas (1996). Duffie and Huang (1996) and Duffie and Singleton (1999) have applied this approach to the valuation of derivatives, corporate bonds and swaps. The key variable is the time to default. This is typically assumed to be governed by a Poisson process where the intensity of the process can depend on some exogenously specified state variables.<sup>10</sup> By its very nature, default is a surprise event in the reduced form models. This is to be contrasted with the structural models in which default occurs as the forcing process approaches a reorganization boundary. Generally, the reduced form models specify a process for the time to default denoted by  $\tau$ . Then, a risky zero coupon bond paying \$1 in the absence of default before maturity and an amount  $Z_\tau$  [recovery rate] at the time of default in the event that default occurs before maturity, can be priced as follows:

$$p(t, T) = E_t^Q \left[ e^{-\int_t^T r_s ds} 1_{[\tau > T]} \right] + 1_{[\tau > t]} E_t^Q \left[ e^{-\int_t^\tau r_s ds} Z_\tau \right] \quad (12)$$

Models of default in reduced form setting tend to differ on how they specify the process for the default time and how they model the recovery rates. Jarrow and Turnbull (1995) assume that the default time is the first jump of a Poisson process and that it is independent of the short-term rates of interest. Jarrow, Lando and Turnbull (1997) allow the arrival rate to be dependent on a Markov chain which can be the credit rating information suitably summarized. Madan and Unal (1998) present another model which

assumes that the default time is independent of short-term interest rates. The intensity is allowed to depend on equity value. Duffie and Singleton (1999) show that under some restrictive assumptions, it is possible to write the default-risky zero coupon bond price as<sup>11</sup>

$$p(t, T) = E_t^Q \left[ e^{-\int_t^T R_s ds} \right], \quad (13)$$

where  $R_s = (r_s + \lambda_t L_t)$  can be thought of as the discounted rate adjusted by the local expected loss rate. The parameter  $\lambda_t$  is the intensity and  $L_t$  is the loss rate.

The approach of Duffie and Singleton (1999) generalizes the standard approach to the valuation of default-free securities for the valuation of claims that are subject to credit risk.

**Table V: Reduced Form Models of Default Risk**

Highlights of Models	References and Comments
Default event is modeled as a Poisson Process, the recovery rate is a fraction of the default-free bond.	Jarrow and Turnbull (1995) - Interest rate process is independent of the process driving default.
Hazard rate of default is a function on equity price and volatility. Multiple layers of debt allowed with APR.	Madan and Unal (1998) - Interest rate process is independent of the process driving default.
Recovery is a fraction of the market value, simple modification of the risk-free rate to get a default-adjusted short rate.	Duffie and Singleton (1999).
Allows recovery rates and default times to depend on common state variables.	Das and Tufano (1995)

One problem with the structural approach is that it is unable to reconcile the observed spreads between corporate debt and otherwise identical Treasury securities. These models produce a low spread for reasonable parameter values. The reduced form models are not designed to address this issue, as they take as inputs such market data. Another problem is that financial distress is often accompanied by renegotiations, debt rescheduling, forgiveness and sometimes costly liquidations. Such empirical facts cannot be reconciled in either the conventional structural models or in reduced form models. Moreover these models are unable to generate the levels of default premium that one observes for short-term debt securities. As the maturity shrinks, these models imply that the default premium goes to zero.

But these categories of model still are not designed to address the empirical regularities in the financial distress literature which are germane to the issue of spreads in the corporate debt market.

The third category of papers includes the work of Leland (1994), Leland and Toft (1996), Anderson and Sundaresan (1996) and Mella-Barral and Perraudin (1993). Leland (1994) *endogenizes* the lower reorganization boundary by permitting the payment of promised coupons by selling additional equity until the equity value is driven to zero. Leland also analyzes the effect of a positive net worth covenant on the value of debt. Leland and Toft (1995) consider several extensions of the paper by Leland (1994) including finite maturities and cash payout. Anderson, Sundaresan and Tychon (1996) model the strategic debt servicing possibilities that arise in the presence of liquidation costs. Using a noncooperative game theoretic formulation, they conclude that the possibility of strategic debt service results in deviations from absolute priority and significantly increases the spread between risky and riskless bonds even at moderate volatility and debt levels. Fries, Miller and Perraudin (1997) characterize industry equilibrium in a model of debt

pricing. Fan and Sundarsan (1999) explore alternative bargaining formulations and endogenize dividend policy and the optimal value of the firm under alternate formulations.

**Table VI : Structural Models of Default Risk  
with Negotiations or Strategic Behaviour**

Lower Boundary	Reorganization	Nature of Default	References and Comments
Endogenous		Strategic Default occurs when cash flows are insufficient to meet coupons. Liquidation is costly. Single layer of debt.	Anderson and Sundaresan (1996) - Risk is due to the fluctuations in the issuing firm's value.
Endogenous		Strategic default and optimal closures	Mella-Barrel and Perraudin (1996). Risk is due to the fluctuations in the issuing firm's value.
Endogenous		Strategic default - endogenous dividends, Nash allocation of residual values.	Fan and Sundaresan (1999)
Endogenous		Dynamic Recapitalizations	Mella-Barral (1998)
Endogenous		Industry equilibrium	Miller, Fries and Perraudin (1997)

The default risk literature has had a big impact in the industry. The structural modeling approach initiated by Merton is used by KMV, a leading firm which specializes in measuring default risk, probabilities of default and credit risk management. The reduced form models have been successfully used in the valuation of credit derivatives such as default swaps. Despite their success, much work remains to be done: we need to reconcile more comprehensively the empirical regularities in the corporate distress literature in our models of debt valuation. The key role played by the bankruptcy code in the allocation of residual cash flows upon financial distress is yet to be modeled satisfactorily in the valuation models. In fact, the presence of the code will endogenously determine the valuation of residual cash flows and the timing of renegotiations.. This is also the key in satisfactorily distinguishing sovereign loans, which putatively do not come under a bankruptcy code, from domestic loans which are under an established bankruptcy code.

## VI. Real Options Applications

The real options literature stresses the simple intuition that when reversibility of an investment decision is costly, *and* the payoffs associated with that investment are stochastic, then making that investment entails the sacrifice of the option to delay it. This intuition is most famously seen in finance in the realm of capital budgeting: the conventional net present value criterion is not necessarily the right one when decisions are irreversible. Early papers to formalize this idea were by Brennan and Schwartz (1985) and McDonald and Siegel (1986). This intuition leads to the following economic trade-off: the expected profits foregone by delay in investments must be weighed against the option value that will be relinquished when the option to delay is sacrificed. In a series of influential papers, Dixit (1989a, 1989b, 1991) has applied the real options idea persuasively to many economic applications. In an influential book, Dixit and Pindyck (1986) have



provided this as the theory for studying the general question of investment under uncertainty. Abel and Eberly (1994, 1996, 1997) and Abel, et al. (1996) have explored the question of investment under uncertainty under varying assumptions about the costs of adjustments, reversibility, etc. In a series of papers Grenadier (1995, 1996, 1999) have applied the real options theory to study the valuation of lease contracts, development options and strategic behavior. Kalra and Jain (1997) have development optimal intervention strategies for the Pension Benefits Guarantee Corporation using this approach. Other related papers in this context include the ones by He and Pindyck (1992), Majd and Pindyck (1987), Pindyck (1988), Triantis and Hodder (1990) and Williams (1993).

There are many review papers and books that are currently available on this topic. The ones by Trigeorgis (1996) and Brennan and Trigeorgis (1999) are especially suited for gaining an insight into the wide ranging applications of the real options theory. *Financial Management* (1993) published an issue in which several topics in real options and applications are presented. These topics are wideranging: corporate investment strategies under competition, production flexibility and the role of spawning investment opportunities.

Another branch of the real options theory is directed towards modeling investment and financing decisions simultaneously in the presence of costs. Brennan and Schwartz (1985) and Mello and Parsons (1992) are two papers which study optimal investment and closure decisions. Fisher, Heinkel and Zechner (1989) examine the dynamic capital structure problem when there are costs to recapitalizations. Mauer and Triantis (1994) examine the interactions between investment, financing and operating policies in the presence of adjustment costs and costs of recapitalization. As we have seen in the context of the default risk sub-field, the approach taken by Leland (1994) and Leland and Toft (1996) can also be thought of as an application of the

real option theory where the trigger points for some action are determined in the presence of some costs.

A consequence of the real options approach to investment is that it is generally optimal not to accept the project the first time the net present value of the investment becomes positive. Often, it is beneficial to delay investment. While this is intuitive, it overlooks the fact that the value of the option to delay depends on what the industry market structure is and what are the entry barriers, such as intellectual property rights protection. If a firm is fearful that a competitor may enter sooner by moving in first and the market is perceived to be not deep enough to support more than one firm, then the option value of delaying may not be very high. There may be other instances, where the market is sufficiently deep and it may be beneficial for the firm to delay investments if there are informational externalities associated with a competitor's investment. [examples here might include property development or drilling for oil]. A number of recent papers have contributed to the theory of real options where precisely this type of strategic behavior is explicitly modeled. Grenadier (1996) explores a real estate development problem in which developers strategically choose threshold levels of state variables to make their investments. Trigeorgis (1996) also considers strategic investment decisions by firms in a binomial setting. Fries, Miller and Perraudin (1997) explore in a perfectly competitive equilibrium model real investment decisions. All these papers work with complete information.

More recently Lambrecht and Perraudin (1998) have studied strategic behavior under incomplete information. This is clearly an exciting area of research. Bringing in incomplete information [much like Duffie and Lando (1999) in a debt pricing context] will help to understand the strategic behavior in a much richer setting. Absent informational differences, if firms have similar costs, then the fact that the first-mover gets an advantage will

induce the other firm to move in as well, potentially destroying the option value. But if the costs of investments of different firms are not fully known, then there may be a restoration of the option value to delay the investment and any action by one of the firms may reveal information about that firm's costs.

Equilibrium formulations have recently significantly expanded the scope of real options applications. In a recent paper Kogan (1999) explores a general equilibrium model of asset pricing when the underlying goods technologies display varying costs of irreversibility. An insight that emerges out of his analysis is that firm-specific factors [such as book-to-market ratio] play an important role through their effects on real activities on financial asset prices and their moments. This strand of research has the potential to link in a formal way the real activity, the frictions and the options that they induce with the asset prices in the economy.

## VII. Capital Market Frictions

Frictions in capital markets, such as taxes, transactions costs (bid-offer spreads, brokerage commissions, etc.), informational costs, etc., may have consequences for optimal consumption and portfolio selection and as a consequence on equilibrium asset pricing. This subfield has seen a number of important contributions in the last decade.

An important dimension of market friction is the presence of taxes and transactions costs. The role of transactions costs on the optimal policies pursued by investors was studied by Constantinides (1986). In this model the investor has the choice of investing in a liquid asset as well as in an illiquid asset. The asset price process is specified exogenously. A number of papers in this vein have been written including the ones by Duffie and Sun (1990), Davis and Norman (1990), Grossman and Laroque (1990), Dumas and Luciano (1990), etc. Constantinides (1986) and Constantinides and In-

gersoll (1984) have explored the presence of transactions costs and taxes on allocation decisions and asset prices. Constantinides (1986) examines proportional costs of transaction costs and concludes that while they affect the allocation decisions, they do not affect the asset prices in a significant manner. Davis and Norman (1990) provide the complete extension of Merton (1971) to an economy with transaction costs. In a more recent contribution by Vayanos (1998), the relative price of an illiquid asset is determined in terms of the price of the liquid asset which is given exogenously.

A number of papers have explored the problem of consumption and portfolio choice in an incomplete market setting. Duffie, Fleming, Soner and Zariphopoulou (1997) show the existence of optimal policies in an incomplete market setting using the stochastic dynamic programming approach for CRRA utility functions. In a complete market setting, Cox and Huang (1989a) and Karatzas, Lehoczky and Shreve (1986) have shown that the martingale representation theory can be used to characterize the optimal choice variables. The key insight is that the marginal utility of terminal wealth is tied to the density of the Martingale measure. This approach is considerably more intricate in the context of incomplete markets. He and Pearson (1991) have studied this problem. See also the contribution by Karatzas, Lehoczky, Shreve and Xu (1991). The technique employed is to solve a dual variational problem and then use that to determine the solution to the original problem by convex duality. Cvitanic and Karatazas (1992) treat the problem with fairly general constraints. Cuoco (1997) examines the optimal consumption and portfolio choice in the presence of non-traded stochastic labor income and portfolio constraints. He and Peges (1993) explore the effects of labor income and borrowing constraints and asset prices. The presence of leverage constraints has also been studied by Grossman and Vila (1992). Grossman and Laroque (1990) examine the optimal portfolio

choice and asset pricing in the presence of an illiquid durable good such as housing.

Back (1992) in his insightful paper examines insider trading in continuous-time. This is one of the early papers in finance to model market microstructure questions in a continuous-time context. He incorporates explicitly asymmetric information and obtains pricing rules. Back (1993) also considers options pricing with asymmetric information. Merton (1987) presents an asset pricing model with incomplete information, which can form the basis for reconciling the home-bias in asset allocation. The role of imperfect information and its implications for portfolio choice and asset pricing has been explored in Detemple (1986), Dothan and Feldman (1986), Gennotte (1986), Shapiro (1998), Veronesi (1998) and Xia (1999)

## VIII. Estimation of Continuous-time Models

Perhaps the most significant development in the continuous-time field during the last decade has been the innovations in econometric theory and in the estimation techniques for models in continuous-time. In the early 1990s, some pioneering work on the relationship between GARCH processes and diffusion processes was done by Nelson (1989, 1990, 1991). An important insight that Nelson brought to this field was that GARCH processes can be seen as approximations to diffusion processes with stochastic volatility. In Nelson (1990) it was shown that many interesting GARCH processes converge in distribution to diffusion processes. This contribution is valuable as it is easier to perform MLE for GARCH processes with discretely recorded data. The result that they converge to diffusions then allows one to use the GARCH estimates as useful approximations for the underlying diffusion process.

Broadly speaking, the estimation strategies in continuous-time field can

be grouped into the following areas:

1. Maximum likelihood methods.
2. Generalized method of moments or GMM.
3. Simulated method of moments or SMM.
4. Efficient method of moments or EMM.
5. Non-parametric approaches.
6. Method based on empirical characteristic function

We will summarize the major developments in each area as they relate to continuous-time finance.

### Maximum Likelihood Methods

For a continuous-time diffusion process, the conditional density of the process is to be found by solving the Fokker-Plank equation. Consider a diffusion process

$$dY = a(Y, \theta, t) dt + b(Y, \theta, t) dW_t \quad (14)$$

where  $\theta$  is a vector of parameters to be estimated. The conditional density,  $f$ , of this process is implicitly given in the Fokker-Plank equation (or the forward equation) shown below.

$$\frac{\partial f}{\partial t} = -\frac{\partial [af]}{\partial Y} + \frac{1}{2} \frac{\partial^2 [b^2 f]}{\partial Y^2} \quad (15)$$

The conditional density has to be solved by imposing an appropriate initial condition to the Fokker Plank equation. In some circumstances the univariate diffusions may not have a stationary distribution and we need to exercise care to ensure that appropriate regularity conditions are satisfied

before the MLE method is used to estimate the parameters. In some interesting applications, we can extract the conditional density in closed form. Examples include the Cox, Ingersoll, and Ross (1985b) model of the term structure with a univariate square root diffusion, the Black-Scholes model with geometric Brownian motion and the Vasicek (1977) model with the O-U process. In such situations, we can apply the maximum likelihood methods to estimate the parameters of the diffusion process consistent with the restrictions imposed by the underlying model on the data. An example of this approach is the paper by Pearson and Sun (1994) who use the MLE method to estimate the Cox, Ingersoll and Ross (1985b) model. See also Chen and Scott, (1993) for an MLE implementation. In many interesting situations, it may not be possible to solve for the conditional density in closed form. In principle, the likelihood function can be estimated by Monte Carlo simulation methods. This is computationally infeasible in most situations as the simulation has to be performed for every conditioning variable and for every parameter value. Lo (1988) shows how to use numerical methods to perform the MLE procedure. But this procedure can be computationally an expensive and time consuming proposition. This is due to the fact that the partial differential equation has to be solved numerically for each iteration in the estimation process. In two recent papers, Aït-Sahalia (1999a, 1999b) has shown that MLE procedures can be used even under circumstances when the conditional densities are unavailable in closed form. Aït-Sahalia produces approximations in closed form to the unknown but true transition density functions for many univariate diffusion processes. By using a Hermite expansion of the transition density around a normal density up to order  $K$ , he is able to get explicit approximations. In many applications Aït-Sahalia (1999b) shows that values of  $K$  equal to one or two is sufficient to get the required degree of precision. This is a striking development in the field of

the MLE method for diffusion processes. If this procedure can be extended to multi-variate diffusion processes, then it will have a very powerful impact in the estimation of continuous-time models in financial economics.

### Generalized Method of Moments or GMM

For a restricted number of univariate diffusions and some multi-variate diffusions we have an analytical characterization of the conditional density. For many diffusion processes, it is difficult if not impossible to write down explicitly the solution to the Fokker-Plank equation which contains the conditional density of the process. In such situations the application of maximum likelihood methods is not feasible. Hansen and Scheinkman (1995) show in an important paper how to apply the GMM approach in such situations. They derive moment restrictions in continuous-time models with discretely sampled data. The key feature of their paper is the use of the infinitesimal generators to characterize continuous-time Markov processes and show that these generators can be used to construct moment conditions implied by stationary Markov processes. Generalized method of moments estimators and tests can be constructed using these moment conditions. The resulting econometric methods are designed to be applied to discrete-time data obtained by sampling continuous-time Markov processes. Their paper is quite technical and is difficult to apply when there are unobserved state variables, such as stochastic volatility, for example. GMM has been applied in the term structure literature by Chan, et. al (1992) and Gibbons and Ramaswamy (1993).

### Simulated-Based Methods

Recently, simulation-based approaches and indirect inference methods have been developed to estimate continuous-time models. We review the approaches below.

### Simulated Method of Moments or SMM



Duffie and Singleton (1993) in another important paper provide a procedure for obtaining simulated moments estimators (SME) that are consistent and asymptotically normal. They confine attention to time-homogeneous Markov processes in developing their approach. Their approach begins with simulating the forcing variables. Using this information, asset prices (which are functions of the forcing variables) can be simulated. Then the parameter vector  $\theta$  is chosen so as to match moments. Their procedure extends Hansen's GMM approach to situations where asset prices that are observable are not analytic functions of the forcing variables and the unknown parameter vector. This expands significantly the class of economic problems that can be estimated using the SMM. In providing this method, Duffie and Singleton (1993) overcome some important hurdles: simulation requires some initial conditions and they may not be drawn from a stationary distribution. Furthermore, the simulated asset prices (which are functions of the state variables) exhibit dual dependency on the parameter vector both through a structural model and through the generation of the data via simulation. The SMM approach is particularly relevant in many finance applications where closed-form solutions are difficult if not impossible to obtain. Broze, Scaillet, and Zakoian (1998) develop an estimation procedure for continuous-time models based on discretely sampled data. Their method is based on a paper on indirect inference by Gouriou, Monfort, and Renault (1993). Essentially they perform simulations of a discretized model. They study the simulated model and examine the asymptotic properties of this indirect estimator. Brandt and Santa-Clara (1999) apply the simulated likelihood estimation procedures to multivariate diffusion processes.

#### Efficient Method of Moments or EMM

Gallant and Tauchen (1996, 1997a, 1997b, 1998) offer a different approach to the estimation problem. They set up an auxiliary model and

compute the “score” which is the derivative of the log density of the auxiliary model with respect to its parameters. The advantage is that the score has an analytical expression. They then use the expectations under the structural model of the score to develop moment conditions. Naturally they now depend on *both* the parameters of the auxiliary model and of the structural model. The parameters of the auxiliary model are replaced by their quasi-MLEs and the estimates of the structural model are then obtained by minimizing the usual GMM criterion function. Two applications of this approach in finance are by Anderson and Lund (1997) who use the EMM approach to estimate continuous-time stochastic volatility models of the short-term interest rate and by Benzoni (1998) who applies the EMM to estimate an options pricing problem with stochastic volatility.

### Non-Parametric Approaches

The non-parametric approach is particularly relevant when we have no structural models that are rich enough to provide a definitive guidance as to the relationship that must prevail between the endogenous variables and exogenous variables and parameters. In addition, non-parametric procedures are typically data-intensive and the availability of a large sample of high quality data is also a pre-requisite for this approach to have any chance of success. Fortunately the data requirement is not necessarily a big issue in many empirical questions in finance. The lack of a clear structural model that provides a definitive guidance is a problem, though, in some key applications in finance. For example, what is the most appropriate short rate process for developing the default-free bond price functions? The answer to this question is ambiguous at best. There are many univariate diffusion specifications in the default-free term structure theory. Vasicek (1977), CIR (1985), Duffie and Kan (1993) and Brennan and Schwartz (1979) are just a few of the competing alternatives. The chief advantage of the non-

parametric approach is that it can accommodate fairly general non-linear functional forms. In recent contributions, Aït-Sahalia and Lo (1998, 2000) and Aït-Sahalia (1996a, 1996b) have applied non-parametric estimation procedures in the context of estimating the state price density and the pricing of interest rate derivatives, respectively. Pritsker (1998) and Stanton (1998) examine how well non-parametric estimation procedures perform when they are applied to U.S. interest rates which display persistence. Pritsker (1998) uses the Vasicek (1977) model of interest rates to characterize the performance of a kernel density estimator in finite samples and contrasts it with asymptotic theory. One conclusion of his paper is that the persistence of the interest rate process matters in the selection of optimal bandwidth. Pritsker also examines the bias in finite samples and compares them with the asymptotic theory. Nonparametric approach has been used by Jiang (1998) and Jiang and Knight (1997) in the context of estimating term structure of interest rates and derivative securities.

### Methods Based On Empirical Characteristic Function

Singleton (1998) considers a pricing equation of the following form

$$p_t = E_t \left( e^{\left(-\int_t^T R(X_s,s)ds\right)} (v_0 + v_1 \cdot X_T) e^{u \cdot X_T} \right), \quad (16)$$

where the expectation is taken conditional on the history of the vector of state variables  $X$  up to  $t$ . This specification represents a class of models known as the affine jump diffusion (AJD) models. The conditional characteristic function of the state variable can be found by setting  $R = 0$ ,  $v_0 = 1$ , and  $v_1 = 0$ . The characteristic function is

$$\Phi(u, X_t, t, T) = E \left( e^{u \cdot X_T} | X_t \right) \quad (17)$$

Knowing the function  $\Phi(u, X_t, t, T)$  is equivalent to knowing the joint conditional density function of  $X_T$ . Singleton exploits this to derive maximum like-

likelihood estimators for AJDs by deriving the conditional density of  $X_{t+1}$  given  $X_t$  by Fourier inversion of the conditional characteristic function. Singleton also provides method of moments estimators of the parameters of the AJD processes by exploiting the fact that the functional form of  $\Phi(u, X_t, t, T)$  is known and that the function satisfies the orthogonality condition:

$$E\left(e^{iu \cdot X_{t+1}} - \Phi(u, X_t, t, t+1)\right) = 0 \quad (18)$$

This approach works for the AJD class of models which appear to span many interesting specifications that have been used in the literature. In a recent paper Chacko and Viceira (1999) have also developed spectral GMM methods for the affine family of diffusions. They have independently reached many of the results that have been obtained by Singleton (1998). Jiang and Knight (1999) apply this technique to estimate models of stochastic volatility.

### Bayesian Methods

Recently, Markov Chain Monte Carlo (MCMC) methods have been used in the estimation of continuous time models. Jacquier, Polson and Rossi (1994) developed this approach to analyze stochastic volatility models. Papers by Eraker (1997) and Jones (1997) are examples of this approach that have recently used this approach in finance. The chief advantages of this approach are: (a) stationarity assumption need not be imposed [unlike the GMM approach], (b) finite sample inferences are possible and latent variables can be accommodated in the estimation procedure.

Econometric theory is thus responding to the impressive challenge posed by the continuous time theory. To reconcile the implications of the theory with stylized facts in the data, scholars are increasingly investigating models with more than one state variable. For example, in order to address the options pricing problem in the context of the stylized fact that the

underlying stock returns are predictable, we need to specify a stock price process with time-varying moments (see Lo and Wang (1995) for a model of options pricing when the underlying returns are predictable). Likewise, in order to reconcile the presence of volatility smiles and skews in the data, many papers have modeled volatility as a stochastic process in addition to modeling the underlying asset as a stochastic process. The challenge to the econometricians is to present a framework for estimating such multivariate diffusion processes which are becoming more and more common in financial economics in recent times. Recent developments in econometric theory give us considerable hope that more realistic multi-factor continuous-time models can be estimated so that their practical implementation will be feasible. The development of estimation procedures for multi-variate AJD processes is certainly a very important step towards realizing this hope.

## IX. International Markets & Exchange Rate Dynamics

Research in this area is too vast to be surveyed in full here. Early applications of continuous-time methods in finance were predictable extensions of Merton's model with relabelling of state variables to capture the international flavor of the problem on hand. Adler and Dumas (1983) provide an excellent survey of the field until the early 80s, and we will not attempt to cover this period. Most papers tended to specify process for prices, interest rates and exchange rates exogenously without enforcing any equilibrium restrictions. The last decade has seen several interesting applications in the international markets area. Much of the progress that we have seen in the literature has come from modeling the heterogeneity and transactions costs. The paper by Dumas (1992) is among the first to provide a truly satisfactory dynamic model of international markets that are spatially separated. Dumas characterizes the real exchange rate dynamics in the context of his

equilibrium model. Uppal (1993) considers a general equilibrium model very much in the spirit of Dumas (1992) to explore whether a bias in domestic goods consumption will necessarily lead to a “home-bias” portfolio selection. The driving force is the costliness in the transfer of capital good from one country to another which leads to hysteresis. A number of authors have applied “real options” theory to problems in international economics and exchange rate dynamics. Dixit (1989) explores the pricing problem facing a firm in the presence of entry and exit costs.

## X. Challenges

There are several challenges that face researchers who use continuous-time methods in finance. I will try to sketch a few of them in this section. As Ross (1989) has noted, an important challenge to the theorists in finance is to explain the level and the pattern of volume of trading in financial markets. While this is a challenge irrespective of whether one uses continuous-time methods or not, this nonetheless represents an interesting challenge to scholars working in continuous-time finance. Models in continuous-time begin by specifying *continuous trading opportunities*. Markets are not open all the time. Received empirical wisdom suggests that the estimated volatility of prices depends on whether one uses closing prices or transaction prices. The activity of trading in itself may generate volatility. There are interesting variations within a trading day in the pattern of volatility. How can one reconcile these facts in the context of a paradigm such as continuous-time trading? In a continuous-time model with transactions costs and taxes, trading intervals are endogenous and trading will only occur when the state variables cross certain trigger levels. Similarly, other sources of frictions such as asymmetric information, liquidation costs, etc., may also produce endogenous trading intervals. Fundamental to explaining the volume of trading is the information structure of the economy, and the manner in which news

gets generated and transmitted via trading. While there is a potential to translate this to models which can be consistent with the stylized facts on volume of trading, the paradigm is a long way from realizing this goal.<sup>12</sup>

Much of the continuous-time theory in the area of term structure of interest rates and in the area of default risk is now fastened on delivering models that are *calibrated* to market conditions. There is a bit of reverse engineering here that is worrisome. Application of such calibrated models in moderate doses can help the researchers to get some insights about the empirical realities that can be a valuable input into developing more satisfactory models. An example should illustrate this point: The fact that there is an implied volatility smile or skew when one uses the Black-Scholes model is in itself very valuable knowledge. This insight came out of a reverse engineering exercise in which we let the market prices tell us what the model volatility should be to reconcile the model value with the market price. This has led to development of models with jump risks, stochastic volatilities, etc., which is a very positive way in which the theory is responding to empirical facts or stylized facts that one has arrived at using reverse engineering in moderation. It appears that one could do too much of this. If the markets are liquid and the pricing is efficient, then there is some merit in calibrating models to such market prices. The calibrated default-free term structure models may be an example of a good application because we have reasons to believe that the Treasury market is reasonably liquid. On the other hand, calibration is likely to be misleading in markets where the liquidity is suspect. Examples would be emerging markets, corporate debt markets, and so on. In such markets some guidance is needed from scholars as to how the underlying assets are to be priced in the first place. Models of credit derivatives that begin by calibrating their models to a curve of zeroes in the credit rating category of (say) BB are essentially starting on a shaky foundation because

what constitutes a satisfactory zero curve in the BB-rated category is yet to be addressed in a theoretically satisfactory manner.

An endemic weakness of the continuous-time methods is that the contractual features are almost always specified exogenously. The optimality of the contracts or endogenizing the contractual provisions has not been the strength of this framework. The derivatives literature is elegant in its intertemporal formulation, richness of the specification of state variables, and the solution procedures, but is typically silent on why a certain contract that is being valued is optimal. In this context the distinction between private optimality and social optimality has to be clearly articulated in models that seek to endogenize contracts. Questions pertaining to efficiency turn on the the notion of social optimality, but the design of contracts for incentive purposes must be consistent with private optimality. Attaining one does not necessarily lead to the attainment of the other. The design and optimality of the contracts is starting to receive more attention in the literature recently. This also turns on the question of welfare issues that are traditionally ignored in the continuous-time finance. The optimality has to be pinned down relative to a measure of welfare. This is an issue that is addressed by Duffie and Huang (1985). They show that the full efficiency can be achieved by trading continuously in a few long-lived securities. Although their model is based on some restrictive assumptions it should set the direction for research in this area. Of course, a rigorous treatment of asymmetric information in continuous-time models is clearly an important prerequisite to achieve this objective.

The principal-agent paradigm has been most helpful in endogenizing contractual arrangements and explain why certain types of contracts that we observe in real-life may be in fact optimal. Recently a number of papers have attempted to incorporate this paradigm in a continuous-time setting. An



influential paper in this context is by Holmstrom and Milgrom (1987). They consider a life-time utility maximizing agent (who has a negative exponential utility) who can expend effort to affect the output from a technology modeled by a diffusion process. In this context they show that the second-best sharing rule is a linear function of the aggregate output. Their paper has been generalized by a number of authors recently. Schattler and Sung (1993) find conditions for optimality in the agent's problem that leads to a semi-martingale representation of the agent's salary. Sung (1995) has extended the Holmstrom and Milgrom (1987) model where the agent can also control the diffusion process. He still shows that the optimal contract is linear. Muller (1998) shows that the first-best sharing rule is also linear in aggregate output by primarily exploiting the fact that both the principal and the agent have CARA utility. Govindaraj and Ramakrishnan (1999) extend the analysis by allowing discounting of future cash flows and permitting mean reversion in the earnings process. Detemple and Govindaraj (1999) extend the analysis to a richer set of stochastic processes and utility functions.

Among the elusive issues taunting the theorists is the problem of liquidity. Many observers have noted that from time to time, markets display lack of liquidity. The hedge funds crisis in the summer of 1998 was attributed by some to the lack of liquidity in the market. Liquidity also is closely related to the probability that there may be a major default or a crisis. Russian default in 1998 was attributed to the lack of liquidity and the flight to quality, which led to the demise and reorganizations of well-known hedge funds. A modeling challenge is the possibility that defaults may be correlated in the economy in an equilibrium, leading to a contagion. We know precious little from a theoretical perspective as to how such episodes occur in the markets. The literature is yet to formulate an interesting framework for studying "contagion in financial markets." The recent work by Kyle

and Xiong (1999) explores this question although they do not model default which one suspects is at the root of any “contagion.”<sup>13</sup>

The home-bias issue is not yet settled in a satisfactory way. The putative advantages of diversification are being ignored by many investors who appear to hold portfolios that are sharply “home-biased.” Despite some success on this front, we are yet to resolve this puzzle. The role of government policies and their consequences for consumption and portfolio decisions, as well as for asset pricing, are yet to be dealt with in a satisfactory manner in the literature. Macro policy variables such as money supply, personal taxes, corporate taxes, etc., affect asset prices and interest rates. Government policies (such as taxes) have important consequences for the nature and volume of trading. Yet, continuous time models have not satisfactorily modeled the government and its actions. The government and its objective functions are seldom explicitly modeled. These are some of the challenges that face researchers in continuous-time methods.

A class of models in economics result in an optimal stopping time problem in which the stopping time has a feedback effect on controls such as consumption and portfolio allocations. An example should illustrate the nature of the optimization problem. Consider an investor who is choosing optimal consumption and portfolio decisions along with the optimal retirement strategy [induced by a labor-leisure trade-off and the inaccessibility of pension wealth until retirement]. The optimal retirement date will influence the asset allocation decisions, and the equilibrium asset prices. A similar problem arises when the investor may optimally choose the time of default. Currently we have computational procedures for solving the free-boundary problems without a feedback effect on controls or the optimal control problems without a free boundary problem. Many economic problems fall in this category [especially with two or more state variables] and as of now we do

not have the tools to solve such problems.

## Conclusion

Over the last three decades, continuous-time methods have become an integral part of research in financial economics. This field has left an indelible mark on several core areas of finance such as asset pricing theory, consumption-portfolio selection, and derivatives valuation. The popularity of this field is also attested by the fact that in every major university, doctoral students in finance are expected to take courses in this discipline and review the important papers in this area irrespective of their ultimate research interest. The availability of several excellent texts and the proliferation of journals in which research in this area is published also point to the growing popularity of this field in finance. Many universities now offer master's degrees in disciplines such as computational finance or financial engineering. The core of the intellectual material in such programs is drawn from the continuous-time methods in finance. This field has made a substantial impact in the financial services industry, proving that sophisticated finance theory can be of practical assistance in the industry.

I hope that this survey has provided the reader with a perspective on this important field in finance and some of the open research questions. In the bibliography of the review I have included an extensive (but not exhaustive) collection of papers and texts in this area.

# References

The references are organized as follows. The papers cited in the body of the review are grouped in each subsection. Papers that have been cited in more than one subsection are put in the subgroup where they are cited first in the review.

## Introduction and Overview

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## Notes

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<sup>2</sup>See Long (1976) for a lucid treatment of this issue in a discrete time setting.

<sup>3</sup>A number of scholars have attacked the equity options pricing formulation and produced formulae that bear varying resemblances to the Black-Scholes model. The earliest formula is by Bachelier (1900) who used arithmetic Brownian motion for the stock price. Other contributions include the paper of Samuelson (1965). These formulations depended on subjective discount rates or risk aversion parameters and were not fully supported by an arbitrage-free argument.

<sup>4</sup>Models with stochastic opportunity sets have made a successful reentry into the literature on dynamic consumption-portfolio behavior now after the recent empirical evidence that equity returns show long-term memory. This is dealt with in a later section of the survey.

<sup>5</sup>Anderson and Sundaresan (1996) provide a game-theoretic analysis in a discrete-time binomial framework to study the design and valuation of debt contracts.

<sup>6</sup>Cox, Ross and Rubinstein (1979) developed the Binomial options pricing

model later which further accentuated the growth of options pricing theory.

<sup>7</sup>Harrison and Pliska (1981, 1983) laid rigorous foundations for the theory of trading in continuous-time models.

<sup>8</sup>Presumably, not all shapes can be accommodated by adding free parameters.

<sup>9</sup>Epstein and Zin (1989, 1990, 1991) study in a series of papers the asset pricing and consumption implications for utility functions in which they explore the substitution effects and risk aversion effects. More recently, Epstein (1998) has developed a framework for ambiguity aversion.

<sup>10</sup>Duffie and Singleton (1995) permit more general point processes. Lando (1998) considers a time of default  $\tau$ , which is represented by the jump of a process  $N_t = 1_{\{\tau \leq t\}}$  with an intensity governed by the following relation:

$$\lim_{h \rightarrow 0} P[N_{t+h} - N_t = 1 | N_t = 0] = \lambda_t.$$

<sup>11</sup>They assume that the recovery rate after default is a fraction of the market value before default.

<sup>12</sup>See Lo, Mamaysky, and Wang (1999) for an initial attack on this question.

<sup>13</sup>The paper by Kodres and Pritsker (1999) explores this issue in a discrete time setting.