1. Probability spaces, Random Variables and Stochastic Processes

$(\Omega\ ,\ \mathcal{F}\ ,\ \mathrm{P})$–a probability space
Ω: a given set
$\mathcal{F}$: a family of subsets of Ω with the following properties:

(a) $\phi \in \mathcal{F}$
(b) $A \in \mathcal{F} \Rightarrow A^c = \Omega \setminus A \in \mathcal{F}$
(c) $A_1, A_2, \cdots \in \mathcal{F} \Rightarrow \bigcup\limits_i A_i \in \mathcal{F}$

P: a function $\mathcal{F} \longrightarrow [0, 1]$ such that

(a) $P(\phi) = 0\ ,\ P(\Omega) = 1$
(b) if $A_1, A_2, \cdots \in \mathcal{F}$ and $A_i \cap A_j = \phi$ for $i \neq j$, then $P(\bigcup\limits_{i=1}^{\infty} A_i) = \sum\limits_{i=1}^{\infty} P(A_i)$

$\mathcal{F}$ is an σ-algebra on Ω and $A \in \mathcal{F}$ is an event.
P(A) = the probability that the event A occurs.

$\mathcal{U}$: a family of subsets of Ω

$$
\begin{aligned}
\sigma(\mathcal{U}) &= \bigcap\{\mathcal{V}, \mathcal{V} \text{ is } \sigma - \text{algebra on } \Omega, \mathcal{U} \subseteq \mathcal{V}\} \\
&= \text{ the } \sigma - \text{algebra generated by } \mathcal{U}
\end{aligned}
$$

Example.

$\mathcal{U}$ = the collection of all open subsets of $\mathbf{R^n}$
$\mathcal{B} = \sigma(\mathcal{U})$ = Borel σ-algebra on $\mathbf{R^n}$
The elements $B \in \mathcal{B}$ are called Borel sets.

$Y : \Omega \longrightarrow \mathbf{R^n}$ is $\mathcal{F}$-measurable if

$$Y^{-1}(\mathcal{O}) = \{\ \omega \in \Omega \mid Y(\omega) \in \mathcal{O}\} \in \mathcal{F}$$

for all open sets $\mathcal{O} \subset \mathbf{R^n}$

$X : \Omega \longrightarrow \mathbf{R^n}$: any function.

$$
\begin{aligned}
\sigma(X) &= \text{ the } \sigma - \text{algebra generated by X} \\
&= \{\ X^{-1}(B) \mid B \in \mathcal{B}\ \}
\end{aligned}
$$

A random variable X is an $\mathcal{F}$-measurable function
X : $\Omega \longrightarrow \mathbf{R^n}$.

The distribution of X $\equiv \mu_X(B) \equiv P(X^{-1}(B))$
$\int_\Omega X(\omega)P(d\omega) = \int_{\mathbf{R}} x\mu_X(\mathrm{dx})$ = The expectation of X $\equiv E(X)$

$\{X_t\}_{t\in T}$:a stochastic process
T: an index set
X_t: random variable
$t \longrightarrow X_t(w)$: a sample path of $\{X_t\}$
($w \in \Omega$ = a experiment or a particle)

2. Independence ,Conditional Expectation and Martingales

$\{A_i\}_{i\in I}$ is independent if

$$P\{A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}\} = \prod_{j=1}^{k} P(A_{i_j})$$

for all $i_j \in I$and $i_j \neq i_l$ for $j \neq l$
$\{\mathcal{A}_i\}_{i\in I}$, $\mathcal{A}_i \subseteq F$, is independent if

$$P\{A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}\} = \prod_{j=1}^{k} P(A_{i_j})$$

$\forall A_{i_j} \in \mathcal{A}_{i_j}$, $i_j \neq i_l$ for $j \neq l$
$\{X_i\}_{i\in I}$ is independent if $\{\sigma(X_i)\}_{i\in I}$ is independent.

(Ω , $\mathcal{F}$, P)–a probability space.
X: a random variable , $E|X| < \infty$
$\mathcal{G}$: a sub σ-algebra of $\mathcal{F}$
$Y = E[X|\mathcal{G}] =$ the conditional expectation of X given $\mathcal{G}$ if

(a) Y is $\mathcal{G}$-measuable .

(b) $E(Y1_A) = E(X1_A), \forall A \in \mathcal{G}$

<u>Exercise</u>

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $X : \Omega \to \mathbf{R}$ be a random variable with $E[|X|] < \infty$. If $\mathcal{G} \subset \mathcal{F}$ is a finite σ-algebra , then there exists a partition $\Omega = \bigcup\limits_{i=1}^{n} G_i$ such that $\mathcal{G}$ consists of $\emptyset$ and unions of some (or all) of $G_1, \cdots, G_n$.

(a) Explain why $E[X|\mathcal{G}](\omega)$ is constant on each G_i

(b) Assume that $P(G_i) > 0$. Show that

$$E[X|\mathcal{G}](\omega) = \frac{\int_{G_i} X dP}{P(G_i)} \quad for\ \omega \in G_i$$

(c) Suppose X assumes only finitely many values $a_1, \cdots, a_m$. Then from elementary probability theory we know that

$$E[X|G_i] = \sum_{k=1}^{m} a_k P(X = a_k | G_i)$$

Compare with (b) and verify that

$$E[X|G_i] = E[X|\mathcal{G}](\omega) \quad \omega \in G_i$$

Thus we may regard the conditional expectation as a (substantial) generalization of the conditional expectation in elementary probability theory.

$\{\mathcal{F}_t\}_{t\geq 0}$ = a filtration = a family of sub σ-algebra such that $\mathcal{F}_s \subseteq \mathcal{F}_t$ for $0 \leq s \leq t$

$\{\mathcal{M}_t\}_{t\geq 0}$ is a martingale if

(a) $\mathcal{M}_t \in \mathcal{F}_t$, and $E|\mathcal{M}_t| < \infty \ \ \forall t \geq 0$

(b) $E(\mathcal{M}_t|\mathcal{F}_s) = \mathcal{M}_s \ \forall s \leq t$

$\{X_t\}_{t\geq 0}$: a stochastic process
$\mathcal{F}_t = \sigma\{X_s, 0 \leq s \leq t\}$
$\widetilde{\mathcal{F}}_t$ = the σ-algebra generated by $\mathcal{F}_t$ and $\mathcal{N}$,
where $\mathcal{N} = \{A \in \mathcal{F} \ |P(A) = 0\}$
= the natural filtration of the process $\{X_t\}_{t\geq 0}$

3. Brownian motion

A real-valued stochastic process W_t is called a Brownian motion if

(a) $W_0 = 0$

(b) For P-a.s. ω , $t \longrightarrow W_t(\omega)$ is continuous

(c) W has independent , normally distributed increments

   i. On $0 \le t_0 < t_1 < t_2 < \cdots < t_n$, $W_{t_0}, W_{t_1} - W_{t_0}, W_{t_2} - W_{t_1}, \cdots, W_{t_n} - W_{t_{n-1}}$ are independent

   ii. For $0 \le s < t$, $W_t - W_s \sim N(0, t-s)$ where $N(\mu, \sigma^2)$ denote the normal distribution with mean μ and variance σ^2

Transition probability

$$P_t(x,y) = (2\lambda t)^{-\frac{1}{2}} e^{-\frac{|y-x|^2}{2t}}$$

Finite dimensional distribution

$$P(W_{t_1} \in F_1, W_{t_2} \in F_2 \cdots W_{t_n} \in F_{t_n})$$
$$= \int_{F_1 \times \cdots \times F_n} P_{t_1}(0, x_1)\, P_{t_2 - t_1}(x_1, x_2) \cdots P_{t_n - t_{n-1}}(x_{n-1}, x_n)$$
$$dx_1 dx_2 \cdots dx_n$$

Exercise

Show that

$$E(e^{i\lambda W_t}) = e^{-\frac{1}{2}\lambda^2 t} \quad \forall\ \lambda \in \mathbf{R}$$

and

$$E(W_t^{2k}) = \frac{2k!}{2^k k!} t^k$$

for all positive integer k

4. Quadratic variation of Brownian motion

$$\begin{aligned}
\Delta &= \text{a partition of [S, T]} \\
&= \{S = t_0 < t_1 < t_2 < \cdots < t_n = T\} \\
\Delta t_k &= t_{k+1} - t_k \\
\Delta W_k &= W_{t_{k+1}} - W_{t_k} \\
\|\Delta\| &= \max_k \Delta t_k
\end{aligned}$$

$$
\begin{aligned}
& E\{[\sum_{k=0}^{n-1}(\Delta W_k)^2 - (T-S)]^2\} \\
&= E\{[\sum_{k=0}^{n-1}(\Delta W_k)^2 - \Delta t_k]^2\} \\
&= \sum_{k=0}^{n-1} E\{[(\Delta W_k)^2 - \Delta t_k]^2\} \\
&= \sum_{k=0}^{n-1}[E(\Delta W_k)^4 - (\Delta t_k)^2] \\
&= 2\sum_{k=0}^{n-1}(\Delta t_k)^2
\end{aligned}
$$

As $\|\Delta\| \longrightarrow 0$, $\sum_{k=0}^{n-1}(\Delta W_k)^2 \longrightarrow T-S$ in $L^2(P)$

Theorem

The Brownian paths are a.s. of infinte total variation on any interval.

Exercise

Write down a detail proof of the above theorem.

5. *Itô* Integrals

$(\Omega \ , \mathcal{F} \ , \mathrm{P})$–a probability space
$W = (W_t)_{t\geq 0}$–a Brownian motion
$(\mathcal{F}_t)_{t\geq 0}$: the natural Filtration of W.

<u>*Elementary Process*</u> (or simple process).

$$f(t,\omega) = \sum_{j=0}^{n-1} e_j(\omega) 1_{[t_j,t_{j+1})}(t)$$

where $S = t_0 < t_1 < t_2 \cdots < t_n = T$,
$e_j \in \mathcal{F}_{t_j}$ and $\mathrm{E}(\mathrm{e}_\mathrm{j}^2) < \infty$

$$\begin{aligned} \int_S^T f(t,\omega) dW_t(\omega) &= \int_S^T f dW \\ &\equiv \sum_{j=0}^{n-1} e_j(\omega)[W_{t_{j+1}}(\omega) - W_{t_j}(\omega)] \end{aligned}$$

The Itô integral.

$f \in \mathcal{V}[S,T] = \{f : [S,T] \times \Omega \longrightarrow \mathbf{R} | f$ adapted and $E[\int_S^T f^2(t,\omega)dt] < \infty\}$

f_n : a sequence of elementary process such that

$$E[\int_S^T |f(t,\omega) - f_n(t,\omega)|^2 dt] \longrightarrow 0$$

$$as\ n \longrightarrow 0$$

$$\int_S^T f(t,\omega) dW_t(\omega) = \lim_{n\to\infty} \int_S^T f_n(t,\omega) dW_t(\omega)$$

(limit in $L_2(P)$)

(By Itô's isometry , we have

$$\begin{aligned} & E(|\int_S^T f_n(t,\omega) dW_t(\omega) - \int_S^T f_m(t,\omega) dW_t(\omega)|^2) \\ = \; & E(|\int_S^T (f_n(t,\omega) - f_m(t,\omega)) dW_t(\omega)|^2) \\ = \; & E(\int_S^T |f_n(t,\omega) - f_m(t,\omega)|^2 dt) \end{aligned}$$

(The Itô isometry)

$$E(\int_S^T f(t,\omega) dW_t(\omega))^2$$

$$= E[\int_S^T f^2(t,\omega) dt]$$

for all f $\in \mathcal{V}[S,T]$

Example

$$\int_0^t W_s dW_s = \frac{1}{2}W_t^2 - \frac{1}{2}t$$

Put

$$f_n(s,\omega) = \sum_j W_{t_j}(\omega) 1_{[t_j,t_{j+1})}(t)$$

Then

$$\begin{aligned} & E(\int_0^t |W_s(\omega) - f_n(s,\omega)|^2 ds) \\ = \; & \sum_j E(\int_{t_j}^{t_{j+1}} |W_s(\omega) - W_{t_j}(\omega)|^2 ds) \\ = \; & \sum_j \int_{t_j}^{t_{j+1}} (s - t_j) ds \\ = \; & \frac{1}{2}\sum_j (t_{j+1} - t_j)^2 \longrightarrow 0 \; as \; \|\Delta\| \longrightarrow 0 \end{aligned}$$

Therefore

$$\int_0^t W dW = \lim_{n\longrightarrow\infty} \int_0^t f_n dW = \lim_{|\Delta|\rightarrow 0} \sum_j W_{t_j}(W_{t_{j+1}} - W_{t_j})$$

Write

$$\Delta W_j = W_{t_{j+1}} - W_{t_j}$$

and

$$\Delta W_j^2 = W_{t_{j+1}}^2 - W_{t_j}^2$$

Then

$$\begin{aligned}
\Delta W_j^2 &= W_{t_{j+1}}^2 - W_{t_j}^2 \\
&= (W_{t_{j+1}} - W_{t_j})^2 + 2W_{t_j}(W_{t_{j+1}} - W_{t_j}) \\
&= (\Delta W_j)^2 + 2W_{t_j}\Delta W_j
\end{aligned}$$

Hence

$$2\sum_j W_{t_j}\Delta W_j = \sum_j \Delta W_j^2 - \sum_j (\Delta W_j)^2 = W_t^2 - \sum_j (\Delta W_j)^2$$

Since

$$\sum_j (\Delta W_j)^2 \longrightarrow t \;\; in \; L^2(P) \;\; as \; \|\Delta\| \longrightarrow 0$$

the result follows

Exercise.

Prove that

(a)

$$\int_0^t s dW_s = tW_t - \int_0^t W_s ds$$

(b)

$$\int_0^t W_s^2 dW_s = \frac{1}{3}W_t^3 - \int_0^t W_s ds$$

Theorem.

Let $f \in \mathcal{V}[0,T]$
Then there exists a t-continuous version of
$\int_0^t f(s,\omega)dW_s(\omega) \;\; 0 \leq t \leq T$.
(i.e. there exists a continuous stochastic process J_t on $(\Omega, \mathcal{F}, P)$ such that $P[J_t = \int_0^t f dW] = 1$ for all $0 \leq t \leq T$)
Moreover the process $\mathcal{M}_t = \int_0^t f(s,\omega)dW_s(\omega)$ is $\mathcal{F}_t$-martingale
(i.e. $E[\mathcal{M}_t | \mathcal{F}_s] = \mathcal{M}_s$ as for all $0 \leq s \leq t \leq T$)

Remark.

We are able to define the stochastic integral $(\int_0^t f_s dW_s)_{0 \leq t \leq T}$ as soon as $\int_0^T (f_s)^2 ds < \infty$ P a.s. It is crucial to notice that in this case $(\int_0^t f_s dW_s)_{0 \leq t \leq T}$ is not necessarily a martingale.

6. It$\hat{o}$'s Formula

(It$\hat{o}$'s Formula - Simplest case)
If $f : \mathbf{R} \longrightarrow \mathbf{R}$ has a continuous second derivative , then $f(W_t) = f(0) + \int_0^t f'(W_s)dW_s + \frac{1}{2}\int_0^t f''(W_s)ds$.

<u>*Example*</u>

Consider the function $f(x) = x^2$
By It$\hat{o}$'s formula , we have

$$\begin{aligned} W_t^2 &= 2\int_0^t W_s dW_s + \frac{1}{2}\int_0^t 2ds \\ &= 2\int_0^t W_s dW_s + t \end{aligned}$$

Assume $F \in C^2(\mathbf{R}) \quad F' = f$, and F$(0) = 0$,
then we have

$$\int_0^t f(W_s)dW_s = F(W_t) - \frac{1}{2}\int_0^t f'(W_s)ds$$

(It$\hat{o}$'s Formula with Space and Time Variable)
For any function $f \in C^{1,2}(\mathbf{R}^+ \times \mathbf{R})$, we have the representation

$$f(t, W_t) = f(0,0) + \int_0^t \tfrac{\partial f}{\partial x}(s, W_s)dW_s + \int_0^t \tfrac{\partial f}{\partial t}(s, W_s)ds + \frac{1}{2}\int_0^t \tfrac{\partial^2 f}{\partial x^2}(s, W_s)ds$$

$\{X_t\}_{0\leq t\leq T}$ is an Itô's process if it can be written as

$$X_t = X_0 + \int_0^t a_s ds + \int_0^t b_s dW_s \quad 0 \leq t \leq T$$

where

- X_0 is $\mathcal{F}_o$ − measurable.
- $\{a_t\}_{0\leq t\leq T}$ and $\{b_t\}_{0\leq t\leq T}$ are adapted process.
- $\int_0^T |a_s|^2 ds < \infty$ P-a.s.
- $\int_0^T |b_s|^2 ds < \infty$ P-a.s.

(We write $dX = a\ dt + b\ dW$ for the Itô's process X)

(The General Itô' formula)

If $f \in C^{1,2}(\mathbf{R}^+ \times \mathbf{R})$, then we have

$$f(t, X_t) = f(0,0) + \int_0^t \frac{\partial f}{\partial t}(s, X_s)ds + \int_0^t \frac{\partial f}{\partial x}(s, X_s)dX_s + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s, X_s)b^2(s,\omega)ds$$

(Itô Formula in differential form)

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}b^2 dt$$

(As before , $dt \cdot dt = dt \cdot dW_t = 0$ and $dW_t \cdot dW_t = dt$)

$\underline{Example}$ $(Black - Scholes$ model)

Find the solutions $\{S_t\}_{t\geq 0}$ of

$$dS_t = \mu S_t dt + a S_t dW_t \quad with\ S_0 = x_0 > 0$$

We try to solve the equation by hunting for a solution of the from $S_t = f(t, W_t)$
By It$\hat{o}$'s formula , we see

$$dS_t = (f_t + \frac{1}{2} f_{xx})dt + f_x dW_t$$

where $f_t = \frac{\partial f}{\partial t}, f_x = \frac{\partial f}{\partial x}$ and $f_{xx} = \frac{\partial^2 f}{\partial x^2}$

Consider the two equation :

$$\begin{aligned} \mu f(t,x) &= f_t(t,x) + \frac{1}{2} f_{xx}(t,x) \\ \sigma f(t,x) &= f_x(t,x) \end{aligned}$$

Solving $\sigma = \frac{f_x}{f}$ gives

$$f(t,x) = e^{\sigma x + g(t)}$$

Plugging into the first one gives

$$f(t,x) = x_0 e^{\sigma x + (\mu - \frac{1}{2}\sigma^2)t}$$

Exercise

(a) Use Itô's formula to check that the process

$$S_t = x_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}$$

satisfies the SDE

$$S_t = x_0 + \int_0^t \mu S_u du + \int_0^t \sigma S_t dW_t$$

(b) Use Itô's formula to prove that

$$\int_0^t W_s^2 dW_s = \frac{1}{3} W_t^3 - \int_0^t W_s ds$$

(Integration by parts formula)

$$\begin{aligned} X_t &= x_0 + \int_0^t a(s,\omega)ds + \int_0^t b(s,\omega)dW_s \\ Y_t &= y_0 + \int_0^t \alpha(s,\omega)ds + \int_0^t \beta(s,\omega)dW_s \end{aligned}$$

$$\Longrightarrow X_t Y_t = x_0 y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s + \langle X, Y \rangle_t$$

with

$$\langle X, Y \rangle_t = \int_0^t b(s,\omega)\beta(s,\omega)ds$$

$$(\; dX = XdY + YdX + \langle X, Y \rangle \;)$$

Proof By Itô's formula

$$(X_t+Y_t)^2 = (X_0+Y_0)^2 + 2\int_0^t (X_s+Y_s)d(X_s+Y_s)$$
$$+\int_0^t (b(s,\omega)+\beta(s,\omega))^2 ds$$

$$\begin{aligned} X_t^2 &= X_0^2 + 2\int_0^t X_s dX_s + \int_0^t b^2(s,\omega)ds \\ Y_t^2 &= Y_0^2 + 2\int_0^t Y_s dY_s + \int_0^t \beta^2(s,\omega)ds \end{aligned}$$

The equality follow by substracting equations 2 and 3 from the first one .

<u>*Example*</u> ($Ornstein-Uhlenbeck$ Process)

$$\begin{aligned} dX_t &= -cX_t dt + \sigma dW_t \\ X_0 &= x_0 \end{aligned}$$

Consider $Z_t = X_t e^{ct}$ Integration by parts yields

$$dZ_t = e^{ct}dX_t + ce^{ct}X_t dt + < X, e^{ct} >_t$$

Since $< X, e^{ct} >_t = 0$, it follows

$$dZ_t = \sigma e^{ct} dW_t$$

Thus

$$e^{ct} X_t = x_0 + \sigma \int_0^t e^{cs} dW_s$$

i.e.

$$X_t = x_0 e^{-ct} + \sigma \int_0^t e^{-c(t-s)} dW_s$$

Exercise

Consider the $Ornstein\!-\!Uhlenbeck$ Process $\{X_t\}_{t \geq 0}$. Prove that

(a) $E(X_t) = x_0 e^{-ct}$

(b) $Var(X_t) = \sigma^2 \frac{1-e^{-2ct}}{2c}$

(c) X_t is a normal random variable

(d) The process $\{X_t\}_{t \geq 0}$ is Gaussian.

Remark

In finance , the $Ornstein - Uhlenbeck$ process was used by $O.A.Vasi\check{c}ek$ in one of the first stochastic models for interest rates.

7. Stochastic Differential Equations

$$\begin{aligned} (\Omega, \mathcal{F}, P) &: \text{ a Probability space} \\ W = \{W_t\}_{t\geq 0} &: \text{ Brownian motion} \\ \{\mathcal{F}_t\}_{t\geq o} &: \text{ the natural filtration of W} \end{aligned}$$

Consider the stochastic differential equation

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s$$

(or in differential form

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_s \quad)$$

We say an $\{\mathcal{F}_t\}$ -adapted process $\{X_t\}_{t\geq 0}$ is a solution of the above SDE if

(a) For any $t \geq 0$, the integrals $\int_0^t b(s, X_s)ds$ and $\int_0^t \sigma(s, X_s)dW_s$ exist.

(b) For any $t \geq 0$,
$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s$ P-a.s.

Theorem (Existence and Uniqueness)

If b and σ are continuous functions and if there exists a constant K such that

- $|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq K|x - y|$
- $|b(t,x) + |\sigma(t,x)| \leq K(1 + |x|)$

then for any $T \geq 0$, the SDE admits an *unique solution* in the interval $[0,T]$.
Moreover , the solution $\{X_t\}_{0 \leq t \leq T}$ satisfies

$$E(\underset{0 \leq t \leq T}{sup} |X_t|^2) < \infty$$

Exercise (The $Va si\check{c}ek$ model)

Solve the SDE

$$dX_t = (-\alpha X_t + \beta)dt + \sigma dW_t$$

where $X_0 = x_0$ and $\alpha > 0$.
and verify that the solution can be written as

$$X_t = e^{-\alpha t}(x_0 + \frac{\beta}{\alpha}(e^{\alpha t} - 1) + \sigma \int_0^t e^{\alpha s} dW_s)$$

Show that X_t converges in distribution as $t \longrightarrow \infty$, and find the limiting distribution. Find the convariance $Cov(X_s, X_t)$

8. The $Black - Scholes$ model

$$\begin{aligned} \text{Bond model} &: \ d\beta_t = r\beta_t dt \\ \text{Stock model} &: \ dS_t = \mu S_t dt + \sigma S_t dW_t \end{aligned}$$

$h(S_T) =$ the contingent claim at time T

Example

$$\begin{aligned} h(S_T) &= (S_T - K)^+ \quad \text{(European call option)} \\ h(S_T) &= (K - S_T)^+ \quad \text{(European put option)} \end{aligned}$$

Problem :

What is the time 0 value of the contingent claim?

replicating the contingent claim

$a_t =$ the number of units of stock that we hold at time t
$b_t =$ the number of units of the bond at time t
$V_t = a_t S_t + b_t \beta_t$
$=$ the total value of the portfolio at time t

$\underline{Self-financing\ condition}$

$$V_t = V_0 + \int_0^t a_u dS_u + \int_0^t b_u d\beta_u$$

i.e.

$$dV_t = a_t dS_t + b_t d\beta_t$$

terminal replication condition : $V_t = h(S_T)$

$\underline{Assumption}$:

$V_t = f(t, S_t)$ for an appropriately smooth function f

Then

$$\begin{aligned} df = dV_t &= a_t dS_t + b_t d\beta_t \\ &= a_t(\mu S_t dt + \sigma S_t dW_t) + b_t r \beta_t dt \\ &= \{a_t \mu S_t + b_t r \beta_t\} dt + a_t \sigma S_t dW_t \end{aligned}$$

$$\begin{aligned} df &= f_t(t, S_t)dt + f_x(t, S_t)dS_t + \frac{1}{2} f_{xx}(t, S_t) dS_t dS_t \\ &= \{f_t(t, S_t) + \frac{1}{2} f_{xx}(t, S_t)\sigma^2 S_t^2 + f_x(t, S_t)\mu S_t\} dt \\ &\qquad + f_x(t, S_t)\sigma S_t dW_t \end{aligned}$$

Coeffient Matching

$$\begin{aligned} a_t \sigma S_t &= f_x(t, S_t) \sigma S_t \\ a_t \mu S_t + b_t r \beta_t &= f_t(t, S_t) + \frac{1}{2} f_{xx}(t, S_t) \sigma^2 S_t^2 + f_x(t, S_t) \mu S_t \end{aligned}$$

then

$$a_t = f_x(t, S_t)$$

and

$$b_t = \frac{1}{r\beta_t} \{ f_t(t, S_t) + \frac{1}{2} f_{xx}(t, S_t) \sigma^2 S_t^2 \}$$

$$\begin{aligned} f(t, S_t) &= a_t S_t + b_t \beta_t \\ &= f_x(t, S_t) S_t + \frac{1}{r\beta_t} \{ f_t(t, S_t) + \frac{1}{2} f_{xx}(t, S_t) \sigma^2 S_t^2 \} \beta_t \end{aligned}$$

$Black - Scholes$ PDE :

$$f_t(t, x) = -\frac{1}{2} \sigma^2 x^2 f_{xx}(t, x) - r x f_x(t, x) + r f(t, x)$$

with its terminal boundary condition $f(T, x) = h(x)$
for all $x \in \mathbf{R}$

Exercise

Consider the stock and bond model given by

$$dS_t = \mu(t, S_t)dt + \sigma(t, S_t)dW_t$$

and

$$d\beta_t = r(t, S_t)\beta_t dt$$

(a) Show that arbitrage price at time t of a European option with terminal time T and payout $h(S_T)$ is given by $f(t, S_T)$ where f is the solution of the terminal value problem :

$$f_t(t, x) = -\frac{1}{2}\sigma^2(t, x)f_{xx}(t, x) - r(t, x)xf_x(t, x)$$
$$+r(t, x)f(t, x)$$
$$f(T, x) = h(x).$$

(b) Find a_t and b_t for the self - financing portfolio $a_tS_t + b_t\beta_t$ that replicates $h(S_T)$.

9. The $Black - Scholes$ formula

Consider the terminal-value problem

$$\begin{cases} u_t(t,x) = \frac{-1}{2}\sigma^2(t,x)u_{xx}(t,x) - r(t,x)xu_x(t,x) \\ \qquad\qquad +r(t,x)u(t,x) \\ u(T,x) = h(x) \end{cases}$$

<u>$Feynman - Kac\ Formula$</u>

$$u(t,x) = E\left[h(X_T^{t,x})e^{-\int_t^T r(s,X_s^{t,x})ds}\right]$$

where $X_s^{t,x}$ is the solution of the SDE

$$dX_s^{t,x} = r(s,X_s^{t,x})X_s^{t,x}ds + \sigma(s,X_s^{t,x})dW_s\ , \quad \forall s \geq t$$

and

$$X_t^{t,x} = x$$

<u>$The\ Black - Scholes\ formula\ for\ call\ option$</u>

$$\begin{aligned} h(x) &= (x-K)^+ \\ X_s^{t,x} &= xe^{(r-\frac{1}{2}\sigma^2)(s-t)+\sigma(W_s-W_t)} \end{aligned}$$

Hence

$$
\begin{aligned}
u(t,x) &= E\left[e^{-r(T-t)}\{xe^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma(W_T-W_t)}-K\}^+\right] \\
&= E\left[xe^{-\frac{1}{2}\sigma^2\theta+\sigma\sqrt{\theta}g}-Ke^{-r\theta}\right]^+
\end{aligned}
$$

$$\theta = T-t\ ,\ g \sim N(0,1)$$

Set

$$d_1 = \frac{log\frac{x}{K}+(r+\frac{\sigma^2}{2})\theta}{\sigma\sqrt{\theta}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{\theta}$$

Then

$$
\begin{aligned}
u(t,x) &= E\left[(xe^{\sigma\sqrt{\theta}g-\frac{\theta\sigma^2}{2}}-Ke^{-r\theta})\big|_{\{g+d_2\geq 0\}}\right] \\
&= \int_{-d_2}^{\infty}(xe^{\sigma\sqrt{\theta}y-\frac{\theta\sigma^2}{2}}-Ke^{-r\theta})\frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}dy \\
&= \int_{-\infty}^{d_2}(xe^{-\sigma\sqrt{\theta}y-\frac{\theta\sigma^2}{2}}-Ke^{-r\theta})\frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}dy \\
&= xN(d_1)-Ke^{-r\theta}N(d_2)
\end{aligned}
$$

where

$$N(d) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d} e^{-\frac{y^2}{2}}dy$$

Exercise

Using identical notations and through similar calculations , show that the price of the put is

$$u(t,x) = Ke^{-r\theta}N(d_2) - xN(-d_1)$$

Remark

In practice two methods are used to evaluate σ: the historical method , the implied method.

10. Risk - neutral valuation

Theorem (Girsanov)

$\{\theta_t\}_{0\leq t\leq T}$: an adapted process satisfying

$$\int_0^T \theta_s^2 ds < \infty$$

and such that the process $\{Z_t\}_{0\leq t\leq T}$ defined by

$$Z_t = exp\{-\int_0^t \theta_s dW_s - \frac{1}{2}\int_0^t \theta_s^2 ds\}$$

is a martingale.

Then

$$\widetilde{W}_t \equiv W_t + \int_0^t \theta_s ds \ , \ 0 \le t \le T$$

is a standard Brownian motion under the probability $\tilde{P}$ given by

$$\tilde{P}(A) = \int_A Z_T dP \ \ \forall A \in \mathcal{F} \ .$$

Remark

(a) A sufficient condition for $\{Z_t\}_{0 \le t \le T}$ to be a martingale is

$$E\left[exp\{\frac{1}{2}\int_0^T \theta_s^2 ds\}\right] < \infty$$

(b) $Z_T = \frac{d\tilde{P}}{dP}$ = the density of $\tilde{P}$. relative to P
In this case , we say $\tilde{P}$ is absolutely continuous with respect to P and denoted by $\tilde{P} \ll P$. In fact P and $\tilde{P}$ are equivalent.

Exercise

(a) Assume $\{H_t\}_{0 \le t \le T}$ is adapted and $\int_0^T H_s^2 ds < \infty$ P-a.s.
Set

$$X_t = \int_0^t H_s dW_s + \int_0^t H_s \theta_s ds \quad \text{(under P)}$$

and

$$Y_t = \int_0^t H_s d\widetilde{W}_s \quad (\text{under } \tilde{\mathrm{P}})$$

Prove that $X_t = Y_t$ a.s.

(b) If X is $\mathcal{F}_t$ - measurable , show that

$$\tilde{E}(X) = E\left[XZ_t\right]$$

and

$$\tilde{E}\left[X \mid \mathcal{F}_s\right] = \frac{1}{Z_s} E\left[XZ_t \mid \mathcal{F}_s\right]$$

A probability under which $\tilde{S}_t$ is a martingale

Assume

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

and set

$$\tilde{S}_t = e^{-rt} S_t$$

Then

$$\begin{aligned} d\tilde{S}_t &= -re^{-rt} S_t dt + e^{-rt} dS_t \\ &= \tilde{S}_t\left[(\mu - r)dt + \sigma dW_t\right] \\ &= \tilde{S}_t \sigma d\widetilde{W}_t \end{aligned}$$

where

$$\widetilde{W}_t = \frac{\mu - r}{\sigma} t + W_t$$

From Girsanov's Theorem , $\widetilde{W}_t$ is a Brownian motion under $\tilde{P}$ defined by

$$\tilde{P}(A) = \int_A Z_T dP$$

where

$$Z_T = exp\{-\frac{\mu - r}{\sigma} W_T - \frac{1}{2}(\frac{\mu - r}{\sigma})^2 T\}$$

This implies that $\tilde{S}_t$ is $\tilde{P}$ - martingale.

<u>Exercise</u>

Check that $\tilde{E}\left[\int_0^T \tilde{S}_t^{\ 2} dt\right] < \infty$ and show that

$$S_t = S_0 exp\{(r - \frac{1}{2}\sigma^2)t + \sigma \widetilde{W}_t\}$$

$$(i.e.\ \ dS_t = rS_t dt + \sigma S_t d\widetilde{W}_t\ \ under\ \tilde{P})$$

Consider contingent claim $X \in \mathcal{F}_T$ that satisfies

$$X \geq 0\ \ and\ \tilde{E}(X^2) < \infty$$

Set

$$V_t = \beta_t \tilde{E}\left[\frac{X}{\beta_T} \mid \mathcal{F}_t\right]\ \ for\ 0 \leq t \leq T$$

Need to show that

$$V_t = a_t S_t + b_t \beta_t\ \ for\ 0 \leq t \leq T$$

$$dV_t = a_t dS_t + b_t d\beta_t$$

for some a and b.

A strategy $\phi = (a_t, b_t)_{0 \leq t \leq T}$, is admissible if it is self - financing and if the discounted value $\tilde{V}_t \equiv b_t + a_t \tilde{S}_t$ is non-negative and such that $sup_{0 \leq t \leq T} \tilde{V}_t$ is square - integrable under $\tilde{P}$.
A option is said to be replicable if it's payoff at maturity is equal to the final value of an admissible strategy.

$\underline{Theorem}$ (Martingale Representation Theorem)

$\{X_t\}_{0 \leq t \leq T}$: a $\mathcal{F}_t$ - martingale such that $E(X_T^2) < \infty$
$\Longrightarrow$ there exists an unique $\phi \in H^2 [0, T]$ such that

$$X_t = X_0 + \int_0^t \phi(\omega, s) dW_s(\omega) \ for \ all \ 0 \leq t \leq T.$$

$M_t = \tilde{E} \left[e^{-rt} X \mid \mathcal{F}_t \right]$ is a $\tilde{P}$ - square integrable martingale.
By martingale representation theorem , there exist an adapted process $\{K_t\}_{0 \leq t \leq T}$ such that

$$\tilde{E}(\int_0^T K_s^2 ds) < \infty$$

and

$$M_t = M_0 + \int_0^t K_s d\widetilde{W}_s$$

Set

$$a_t = \frac{K_t}{\sigma \tilde{S}_t} \quad and \quad b_t = M_t - a_t \tilde{S}_t$$

Then

$$\begin{aligned} V_t &= a_t S_t + b_t \beta_t \\ &= e^{rt} M_t = \tilde{E}\left[e^{-r(T-t)} X \mid \mathcal{F}_t\right] \end{aligned}$$

Moreover

$$\widetilde{V_t} = \frac{V_t}{e^{rt}} = M_t = V_0 + \int_0^t a_s d\tilde{S}_t$$

Hence (a_t, b_t) is a self - financing replication of X

$\underline{Theorem}$ (Risk - neutral valuation)

In the $Black - Scholes$ model , any option defined by a non-negative , $\mathcal{F}_t$ - measurable variable X , which is square - integrable under the probability $\tilde{P}$, is replicating and the value of any replicating portfolio is given by

$$V_t = \tilde{E}\left[e^{-r(T-t)} X \mid \mathcal{F}_t\right]$$

Thus , the option value at time t can be naturally defined by the expression $\tilde{E}\left[e^{-r(T-t)} X \mid \mathcal{F}_t\right]$

Examples

$$
\begin{aligned}
X &= (S_T - K)^+ && (European\ call\ option) \\
X &= (K - S_T)^+ && (European\ put\ option) \\
X &= (\frac{1}{T}\int_0^T S_t dt - K)^+ && (Asian\ call\ option) \\
X &= \max_{0\le t\le T} S_t && (look\ hack\ option)
\end{aligned}
$$

11. Up and Out European call option

(a) $0 < K < L$

$$X = (S_T - K)^+|_{S_T^* < L} \ \ ,\ where\ S_T^* = \max_{0\le t\le T} S_t$$

$$
\begin{aligned}
S_t &= S_0 exp\{\sigma \left[\widetilde{W}_t + (\frac{r}{\sigma} - \frac{\sigma}{2})t\right]\} \\
S_T^* &= \max_{0\le u\le t} S_u \\
B_t &= \widetilde{W}_t = W_t + \theta t \\
B_t^{'} &= B_t + (\frac{r}{\sigma} - \frac{\sigma}{2})t \\
M_t^{'} &= \sup_{0\le u\le t} \widetilde{B_u}
\end{aligned}
$$

Then

$$S_t = S_0 exp\{\sigma B_t^{'}\}$$

$$S_t^* = S_0 exp\{\sigma M_t^{'}\}$$

Hence

$$\begin{aligned} e^{rT} V_0 &= \tilde{E}\left[(S_T - K)^+ 1_{S_T^* > L}\right] \\ &= \tilde{E}\left[(S_0 exp\{\sigma B_T^{'}\} - K)^+ 1_{S_0 exp\{\sigma M_T^{'}\} > L}\right] \\ &= \tilde{E}\left[(S_0 exp\{\sigma B_T^{'}\} - K) 1_{B_T^{'} > \frac{1}{\sigma} log \frac{K}{S_0} \ , \ M_T^{'} > \frac{1}{\sigma} log \frac{L}{S_0}}\right] \end{aligned}$$

Under $\tilde{P}$, $B_T^{'}$ is a Brownian motion with drift rate $\theta = \frac{r}{\sigma} - \frac{\sigma}{2}$

12. Joint distribution of $(B_T^{'}, M_T^{'})$ under $\tilde{P}$ (without drift)

<u>$reflection\ principle$</u>

Assume $m > 0$, $0 < b < M$

$$\begin{aligned}
\tilde{P}\,[B_T < b\ ,\ M_T > m] &= \tilde{P}\,[B_T > 2m - b] \\
&= \frac{1}{\sqrt{2\pi T}} \int_{2m-b}^{\infty} exp\{-\frac{x^2}{2T}\} dx \\
&= N(\frac{b - 2m}{\sqrt{T}})
\end{aligned}$$

where

$$N(d) = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{d} e^{-\frac{y^2}{2}} dy$$

$$\begin{aligned}
\tilde{P}\,[B_T < b\ ,\ M_T < m] &= F_T(b, m) \\
&= \tilde{P}\,[B_T < b] - \tilde{P}\,[B_T < b\ ,\ M_T > m] \\
&= N(\frac{b}{\sqrt{T}}) - N(\frac{b - 2m}{\sqrt{T}})
\end{aligned}$$

Hence

$$F_T^B(b, m) = \begin{cases} N(\frac{b}{\sqrt{T}}) - N(\frac{b-2m}{\sqrt{T}}) & \text{if} \quad m > 0, b < m \\ N(\frac{b}{\sqrt{T}}) - N(\frac{-b}{\sqrt{T}}) & \text{if} \quad m > 0, m \geq b \end{cases}$$

$\underline{density\ function\ for\ (B_T, M_T)\ under\ \tilde{P}}$

$$\begin{aligned}
f_T^B &= \frac{2(2m - b)}{T\sqrt{T}} \frac{1}{\sqrt{2\pi}} exp\{-\frac{(2m - b)^2}{2T}\} \\
&= \frac{2(2m - b)}{T\sqrt{T}} \phi(\frac{2m - b}{\sqrt{T}})|_{m>0, b<m}
\end{aligned}$$

(with drift)

Write $\theta = \frac{r}{\sigma} - \frac{\sigma}{2}$. and hence $B_t^{'} = B_t + \theta t$. and $M_t^{'} = \max\limits_{0 \le u \le t} B_u^{'}$.

$\tilde{P}\left[B_T^{'} < b, M_T^{'} < m\right]$
$= \tilde{E}\left[1_{\{B_T^{'} < b, M_T^{'} < m\}}\right]$

$\frac{dP^{'}}{d\tilde{P}} = \Lambda_T$, $where\ \Lambda_t = e^{-\theta B_t - \frac{1}{2}\theta^2 t}$, $\frac{d\tilde{P}}{dP'} = \Lambda_T^{-1} = \frac{1}{\Lambda_T}$
$= E^{'}\left[\Lambda_T^{-1} 1_{\{B_T^{'} < b, M_T^{'} < m\}}\right]$
$= E^{'}\left[e^{\theta B_T^{'} - \frac{1}{2}\sigma^2 T} 1_{\{B_T^{'} < b, M_T^{'} < m\}}\right]$
$= \int_0^m \int_{-\infty}^b e^{\theta z - \frac{1}{2}\sigma^2 T} f_T^B(z, y) dz dy \qquad z = b, y = m$
$= \int_{-\infty}^b e^{\theta z - \frac{1}{2}\sigma^2 T} (\int_z^m \frac{2(2y-z)}{T\sqrt{T}} \phi(\frac{2y-z}{\sqrt{T}}) dy) dz$
$= \int_0^m \int_{-\infty}^b \frac{2(2y-z)}{T\sqrt{T}} e^{\theta z - \frac{1}{2}\sigma^2 T} \phi(\frac{2y-z}{\sqrt{T}}) dz dy$

density of $(B_T^{'}, M_T^{'})$ under $\tilde{P}$:

$$f_T^{B^{'}}(b^{'}, m^{'}) = \frac{2(2m^{'} - b^{'})}{T\sqrt{T}} e^{\theta b^{'} - \frac{1}{2}\theta^2 T} \phi(\frac{2m' - b'}{\sqrt{T}})$$
on $0 < m', b' < m'$

Consider the case $S_0 < K < L$.
Set $b' = \frac{1}{\sigma} log \frac{K}{S_0}$ and $m' = \frac{1}{\sigma} log \frac{K}{S_0}$

$$
\begin{aligned}
V_0 &= \tilde{E}\left[(S_0 e^{\sigma B_T'} - K) 1_{\{B_T' > \frac{1}{\sigma} log \frac{K}{S_0} \ , \ M_T' < \frac{1}{\sigma} log \frac{1}{S_0}\}} \right] \\
&= \int \left[S_0 e^{\sigma x} - K \right] 1_{\{m > b' \ , \ y < m'\}} f_T^{B'}(x, y) dx dy \\
&= \int_{b'}^{m'} \int_x^{m'} (S_0 e^{\sigma x} - K) \frac{2(2y-x)}{T\sqrt{2\pi T}} e^{-\frac{(2y-x)^2}{2T} + \theta x - \frac{1}{2}\theta^2 T} dy dx \\
&= - \int_{b'}^{m'} (S_0 e^{\sigma x} - K) \frac{1}{\sqrt{2\pi T}} e^{-\frac{(2y-x)^2}{2T} + \theta x - \frac{1}{2}\theta^2 T} \Big|_{y=x}^{y=m'} dx \\
&= + \int_{b'}^{m'} (S_0 e^{\sigma x} - K) \frac{1}{\sqrt{2\pi T}} \left[e^{-\frac{x^2}{2T} + \theta x - \frac{1}{2}\theta^2 T} - e^{-\frac{(2m'-x)^2}{2T} + \theta x - \frac{1}{2}\theta^2 T} \right] dx \\
&= \frac{1}{\sqrt{2\pi T}} S_0 \int_{b'}^{m'} e^{\sigma x - \frac{x^2}{2T} + \theta x - \frac{1}{2}\theta^2 T} dx - \frac{1}{\sqrt{2\pi T}} K \int_{b'}^{m'} e^{-\frac{x^2}{2T} + \theta x - \frac{1}{2}\theta^2 T} dx \\
&- \frac{1}{\sqrt{2\pi T}} S_0 \int_{b'}^{m'} e^{\sigma x - \frac{(2m'-x)^2}{2T} + \theta x - \frac{1}{2}\theta^2 T} dx \\
&- \frac{1}{\sqrt{2\pi T}} K \int_{b'}^{m'} e^{-\frac{(2m'-x)^2}{2T} + \theta x - \frac{1}{2}\theta^2 T} dx
\end{aligned}
$$

$$
\begin{aligned}
&\sigma x - \frac{x^2}{2T} + \theta x - \frac{1}{2}\theta^2 T \\
&= -\frac{1}{2T}(x^2 - (\theta + \sigma) 2Tx) - \frac{1}{2}\theta^2 T \\
&= -\frac{1}{2T}(x - \theta T - \sigma T)^2 - \frac{1}{2}\theta^2 T + \frac{T}{2}(\theta + \sigma)^2 \\
&= -\frac{1}{2T}(x - \theta T - \sigma T)^2 + \frac{1}{2}\sigma^2 T + \sigma\theta T \\
&= -\frac{1}{2T}(x - \frac{r}{\sigma} T - \frac{\sigma}{2} T)^2 + rT
\end{aligned}
$$

$$\frac{1}{\sqrt{2\pi T}}\int_{b'}^{m'} e^{\sigma x-\frac{x^2}{2T}+\theta x-\frac{1}{2}\theta^2 T}dx$$
$$= \frac{1}{\sqrt{2\pi T}}\int_{b'}^{m'} e^{-\frac{(x-\frac{r}{\sigma}T-\frac{\sigma}{2}T)^2}{2T}+rT}dx$$
$$= \frac{e^{rT}}{\sqrt{2\pi T}}\int_{\frac{b'}{\sqrt{T}}-\frac{r}{\sigma}\sqrt{T}-\frac{\sigma}{2}\sqrt{T}}^{\frac{m'}{\sqrt{T}}-\frac{r}{\sigma}\sqrt{T}-\frac{\sigma}{2}\sqrt{T}} e^{-\frac{y^2}{2}}dy \qquad \frac{x-\frac{r}{\sigma}T-\frac{\sigma}{2}T}{\sqrt{T}} = y$$
$$= e^{rT}\left[N(\frac{m'}{\sqrt{T}}-\frac{r}{\sigma}\sqrt{T}-\frac{\sigma}{2}\sqrt{T}) - N(\frac{b'}{\sqrt{T}}-\frac{r}{\sigma}\sqrt{T}-\frac{\sigma}{2}\sqrt{T})\right]$$

Pricing formula for up and out European call option

$$V_0(S_0) = S_0\left[N(\frac{m'}{\sqrt{T}}-\frac{r\sqrt{T}}{2}-\frac{\sigma\sqrt{T}}{2}) - N(\frac{b'}{\sqrt{T}}-\frac{r\sqrt{T}}{\sigma}-\frac{\sigma\sqrt{T}}{2})\right]$$
$$-e^{rT}K\left[N(\frac{m'}{\sqrt{T}}-\frac{r\sqrt{T}}{\sigma}+\frac{\sigma\sqrt{T}}{2}) - N(\frac{b'}{\sqrt{T}}-\frac{r\sqrt{T}}{\sigma}+\frac{\sigma\sqrt{T}}{2})\right]$$
$$+e^{-rT+2m'(\frac{r}{\sigma}-\frac{\sigma}{2})}\left[N(\frac{m'}{\sqrt{T}}+\frac{r\sqrt{T}}{\sigma}-\frac{\sigma\sqrt{T}}{2}) - N(\frac{2m'-b'}{\sqrt{T}}+\frac{r\sqrt{T}}{\sigma}-\frac{\sigma\sqrt{T}}{2})\right]$$

where $b' = \frac{1}{\sigma}log\frac{K}{S_0}$ and $m' = \frac{1}{\sigma}log\frac{L}{S_0}$

Remark

If we let $L \longrightarrow \infty$ we obtain the classical $Black-Scholes$ formula.
If we replace T by $T-t$ and replace S_0 by x in the formula $V_0(S_0)$, we obtain a formula for $V_t(x)$, the value of the option at time t if $S_t = x$.

Exercise

(a) A down and in call option give the holder the right to buy a share of the stock for a strike price K at time T provided that at some time $t \leq T$ the price S_T of the stock fell below L, otherwise the option does not yet exist.
Compute the arbitrage-free price of this option.

(b) A look-back call option correspond to a payoff function

$$X = S_T - m_T$$

where

$$m_T = \min_{0 \leq t \leq T} S_t$$

Compute the time $t = 0$ value of this option

13. *American options*

An American option is naturally defined by on adapted non-negative process $\{h_t\}_{0 \leq t \leq T}$. We study processes of the form $h_t = \psi(S_t)$ where ψ is a continuous function from $\mathbf{R}^+$ to $\mathbf{R}^+$, satisfying : $\psi(x) \leq A + Bx$ for some non-negative constants A and B.

Example

$$h_t = (S_t - K)^+ \qquad (\textit{American call option})$$
$$h_t = (K - S_t)^+ \qquad (\textit{American put option})$$
$$h_t = |S_t - K| \qquad (\textit{American straddle})$$

$\phi = (a_t, b_t, c_t)_{0 \le t \le T}$ - a trading strategy with assumption if

(a) $\int_0^T |a_t| dt + \int_0^T |b_t|^2 dt < \infty$ a.s.

(b) $a_t S_t + b_t \beta_t = a_0 S_0 + b_0 \beta_0 + \int_0^t a_u dS_u + \int_0^t b_u d\beta_u - c_t \qquad \forall 0 \le t \le T$

(c) $\{c_t\}_{0 \le t \le T}$ is adapted, non-decreasing process with $c_0 = 0$

The trading strategy with consumption

$$\phi = (a_t, b_t, c_t)_{0 \le t \le T}$$

is said to hedge the American option defined by $h_t = \psi(S_t)$ if, setting $V_t(\phi) = a_t + b_t \beta_t$, we have $v_t(\phi) \ge \psi(S_t)$ a.s.

$\Phi^\psi =$ the set of trading strategies with consumption hedging the American option defined by $b_t = \psi(S_t)$

$\tau : \Omega \longrightarrow [0, \infty]$ is a stopping time if

$$\forall \{\tau \leq t\} \in \mathcal{F}^t, \forall t \geq \text{o}$$

$\mathcal{T}_{t,T}$ = the set of all stopping time taking values in $[t, T]$

$$u(t,x) = \sup_{\tau \in \mathcal{T}_{t,T}} \tilde{E}\left[e^{-r(\tau - t)}\psi(xe^{(r-\frac{\sigma^2}{2})(\tau - t)+\sigma(\widetilde{W_\tau}-\widetilde{W_t})})\right]$$

<u>*Theorem*</u>

There exist a strategy $\bar{\phi} \in \Phi^\psi$ such that

$$V_t(\bar{\phi}) = u(t, S_t) \qquad \forall 0 \leq t \leq T$$

Moreover

$$V_t(\phi) \geq V_t(\bar{\phi}) \qquad for\ all\ 0 \leq t \leq T\ and\ \phi \in \Phi^\psi$$

It is nature to consider $u(t, S_t)$ as a price for the American option at time t, since it is the minimal value of a strategy hedging the option.

<u>*Theorem*</u>

Consider the American call option. Then we have

$$u(t,x) = V(t,x) = \tilde{E}\left[e^{-r(T-t)}(S_T^{t,x} - K)^+\right]$$

where V is the function corresponding to the European call price.
<u>*proof*</u>

we assume that $t = 0$ (the proof is the some for $t > 0$). It suffices to show that

$\tilde{E}\left[e^{-r\tau}(S_\tau - K)^+\right] \leq \tilde{E}\left[e^{-rT}(S_T - K)^+\right]$, $\forall \tau \in \mathcal{T}_{0,T}$

Note that

$$\begin{aligned} \tilde{E}\left[(\tilde{S}_T - e^{-rT}K)^+ \mid \mathcal{F}_\tau\right] &\geq \tilde{E}\left[(\tilde{S}_T - e^{-rT}K) \mid \mathcal{F}_\tau\right] \\ &= \tilde{S}_\tau - e^{-rT}K \\ &\geq \tilde{S}_\tau - e^{-r\tau}K \end{aligned}$$

Hence

$$\tilde{E}\left[(\tilde{S}_T - e^{-rT}K)^+ \mid \mathcal{F}_\tau\right] \geq (\tilde{S}_\tau - e^{-r\tau}K)^+$$

We obtain the derived inequality by computing the expectation of both sides.

14. First passage times for Brownian motion

$(\Omega, \mathcal{F}, P)$ - a probability space
$\{W_t\}_{t\geq 0}$ - the standard Brownian motion

$$M_t = \sup_{0\leq s\leq t} W_s$$

$$T_x = min\{t \geq 0, W_t = x\} \quad \text{(first passage time to x)}$$

Consider the case $x > 0$

Note that

$$\{T_x \leq t\} = \{M_t \geq x\}$$

and recall

$$P(M_t \in d_m, W_t \in db) = \tfrac{2(2m-b)}{t\sqrt{2\pi t}} e^{\frac{-(2m-b)^2}{2t}} dmdb$$

for $m > 0, b < m$

Hence

$$\begin{aligned}
P\left[T_x \leq t\right] &= P\left[M_t \geq x\right] \\
&= \textstyle\int_x^\infty \int_{-\infty}^m \frac{2(2m-b)}{t\sqrt{2\pi t}} e^{\frac{(2m-b)^2}{2t}} dbdm \\
&= \textstyle\int_x^\infty \frac{2}{\sqrt{2\pi t}} e^{-\frac{(2m-b)^2}{2t}} |_{b=\infty}^{b=m} dm \\
&= \textstyle\int_x^\infty \frac{2}{\sqrt{2\pi t}} e^{-\frac{m^2}{2t}} dm \quad (\frac{m}{\sqrt{t}} = y) \\
&= \textstyle 2\int_{\frac{x}{\sqrt{t}}}^\infty \frac{1}{\sqrt{2\pi}} e^{-y^2} dy = N(-\frac{x}{\sqrt{t}})
\end{aligned}$$

Hence

$$\begin{aligned}
P(T_x \in dt) &= \frac{d}{dt} P(T_x \leq t) dt \\
&= \frac{d}{dt}(2N(-\frac{x}{\sqrt{t}}))dt \\
&= \textstyle 2 \cdot \frac{x}{2} t^{-\frac{3}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2t}} dt \\
&= \textstyle \frac{x}{t\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dt
\end{aligned}$$

and

$$
\begin{aligned}
E\left[e^{-\lambda T_x}\right] &= \int e^{-\lambda t}\frac{x}{t\sqrt{2\pi t}}\cdot e^{-\frac{x^2}{2t}}dt \\
&= e^{-\frac{x}{\sqrt{2\lambda}}} \qquad \lambda > 0
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{W}_t &= \theta t + W_t \qquad \text{(Brownian motion with drift)} \\
T_x^\theta &= inf\{t \geq 0 | \widetilde{W}_t = x\}
\end{aligned}
$$

Fix $t > 0$ and choose $T > t$,

$$
\begin{aligned}
P\left[T_x^\theta \leq t\right] &= E\left[1_{T_x^\theta \leq t}\right] \\
&= \tilde{E}\left[1_{T_x^\theta \leq t}\frac{dP}{d\tilde{P}}\right] \\
&= \tilde{E}\left[1_{T_x^\theta \leq t}e^{\theta\widetilde{W}_T - \frac{1}{2}\theta^2 T}\right] \\
&= \tilde{E}\left[1_{T_x^\theta \leq t}\tilde{E}\left[e^{\theta\widetilde{W}_T - \frac{1}{2}\theta^2 T}|\mathcal{F}_{T_x^\theta \wedge t}\right]\right] \\
&= \tilde{E}\left[1_{T_x^\theta \leq t}e^{\theta\widetilde{W}_{T_x^\theta \wedge t} - \frac{1}{2}\theta^2 T_x^\theta \wedge t}\right] \\
&= \tilde{E}\left[1_{T_x^\theta \leq t}e^{\theta x - \frac{1}{2}\theta^2 T_x^\theta}\right] \\
&= \int_0^t e^{\theta x - \frac{1}{2}\theta^2 s}\frac{x}{s\sqrt{2\pi s}}\cdot e^{-\frac{x^2}{2s}}ds \\
&= \int_0^t \frac{x}{s\sqrt{2\pi s}}e^{-\frac{(x-\theta s)^2}{2s}}ds
\end{aligned}
$$

therefore

$$
P(T_x^\theta \in dt) = \frac{x}{t\sqrt{2\pi t}}e^{-\frac{(x-\theta t)^2}{2t}}dt
$$

Exercise

Show that

$$E\left[e^{-\lambda T_x^\theta}\right] = e^{\theta x - |x|\sqrt{\theta^2+2\lambda}}$$

Recall that $x > 0$ and notice

$$\begin{aligned} P\left[T_x^\theta < \infty\right] &= \lim_{\lambda\downarrow 0} E\left[e^{-\lambda T_x^\theta}\right] \\ &= e^{\theta x - |x|\sqrt{\theta^2}} \\ &= e^{\theta x - |\theta| x} \end{aligned}$$

If $\theta \geq 0$, then $P\left[T_x^\theta < \infty\right] = 1$
If $\theta < 0$, then $P\left[T_x^\theta < \infty\right] = e^{2\theta x} < 1$

15. Perpetual American put

$$\begin{aligned} u(0,x) &= \sup_{\tau\in\mathcal{T}_{0,T}} \tilde{E}(Ke^{-r\tau} - xe^{\sigma\widetilde{W_\tau} - \frac{\sigma^2}{2}\tau})^+ \\ &\leq \sup_{\tau\in\mathcal{T}_{0,\infty}} \tilde{E}(Ke^{-r\tau} - xe^{\sigma\widetilde{W_\tau} - \frac{\sigma^2}{2}\tau})^+ \end{aligned}$$

(The right-hand term can be interpreted as the value of a "perpetual" put)
Write

$$u^\infty(x) = \sup_{\tau\in\mathcal{T}_{0,T}} \tilde{E}(Ke^{-r\tau} - xe^{\sigma\widetilde{W_\tau} - \frac{\sigma^2}{2}\tau})^+$$

Note that $u^\infty(x) \geq (K-x)^+$, $u^\infty(x) > 0$ $\forall x \geq 0$ and u^∞ is decreasing and convex.
Set

$$x^* = sup\{x \geq 0 | u^\infty(x) = K - x\}$$

Then

$$0 \leq x^* \leq K \ , \ u^\infty(x) = K - x \ \forall x \leq x^*$$

and

$$u^\infty(x) > (K-x)^+ \ for \ x > x^*$$

Fix $x \in$ and then the Snell enuelope theory enables us to show

$$u^\infty(x) = \tilde{E}\left[(Ke^{-r\tau_x} - xe^{\sigma\widetilde{W}_{\tau_x} - \frac{\sigma^2}{2}\tau_x})^+ 1_{\tau_x < \infty}\right]$$

where

$$\tau_x = inf\{t \geq 0 \mid e^{-rt}u^\infty(S_t^x) = e^{-rt}(K - S_t^x)t\}$$

and

$$S_t^x = xe^{(r-\frac{\sigma^2}{2})t + \sigma\widetilde{W}_\tau}$$

Note that

$$\begin{aligned}
\tau_\infty &= inf\{t \geq 0 \mid S_t^* \leq x^*\} \\
&= inf\{t \geq 0 \mid (r - \frac{\sigma^2}{2})t + \sigma\widetilde{W}_t \geq log\frac{x^*}{x}\}
\end{aligned}$$

For any $z \in \mathbf{R}^+$, consider

$$\tau_{x,z} = inf\{t \geq 0 \mid S_t^x \leq z\}$$

and set

$$\phi(z) = E\left[e^{-r\tau_{x,z}} 1_{\tau_{x,z}<\infty}(K - S_{\tau_{x,z}}^x)^+\right]$$

Then $\phi(z)$ attains its maximum at $z = x^*$. We are going to calculate ϕ explicitly , then we will maximuize it to determine x^* and $u^\infty(x) = \phi(x^*)$.
Clearly if $z > x$, then $\tau_{x,z} = 0$ and $\phi(z) = (K-x)^+$.
If $z \leq x$, then $\tau_{x,z} = inf\{t \geq 0 \mid S_t^* = z\}$ and $\phi(z) = (K - z)^+ E(e^{-r\tau_{x,z}})$.

Note that

$$\begin{aligned}
\tau_{x,z} &= inf\{t \geq 0 \mid (r - \frac{\sigma^2}{2})t + \sigma\widetilde{W}_t = log\frac{z}{x}\} \\
&= inf\{t \geq 0 \mid \theta t + \widetilde{W}_t = \frac{1}{\sigma}log\frac{z}{x}\}
\end{aligned}$$

where

$$\theta = \frac{r}{\sigma} - \frac{\sigma}{2} .$$

White

$$T_b^\theta = inf\{t \geq 0 \mid \theta t + \widetilde{W}_t = b\}$$

Therefore for $z \le x \wedge K$, we observe

$$\begin{aligned}
\phi(z) &= (K-z)\tilde{E}\left[e^{-rT^{\theta}_{\frac{1}{\sigma}log\frac{z}{x}}}\right] \quad \lambda = r, b = \frac{1}{\sigma}log\frac{z}{x} \\
&= (K-z)e^{\theta b + b\sqrt{\theta^2+2r}} \\
&= (K-z)e^{b(\frac{r}{\sigma}-\frac{\sigma}{2}+\frac{r}{\sigma}+\frac{\sigma}{2})} \\
&= (K-z)e^{\frac{2r}{\sigma^2}log\frac{z}{x}} \\
&= (K-z)(\frac{z}{x})^{\gamma} \qquad\qquad \gamma = \frac{2r}{\sigma^2}
\end{aligned}$$

On $[0,x] \wedge [0,K]$, we get

$$\phi'(z) = \frac{z^{\gamma-1}}{x^{\gamma}}\left[K\gamma - (\gamma+1)z\right] \ .$$

If

$$x \le \frac{\gamma}{\gamma+1}K$$

then

$$\max_z \phi(t) = \phi(x) = K - x \ .$$

If

$$x > \frac{\gamma}{\gamma+1}K$$

then

$$\max_z \phi(t) = \phi(\frac{K\gamma}{\gamma+1}) = (K-x^*)(\frac{x}{x^*})^{-\gamma} \ .$$

where

$$x^* = \frac{\gamma}{\gamma+1}K = K\frac{2r}{\sigma^2+2r}$$

Therefore

$$u^\infty(x) = \begin{cases} K - x & \text{if} \quad x \le x^* \\ (K-x^*)(\frac{x}{x^*})^{-\gamma} & \text{if} \quad x > x^* \end{cases}$$

16. Basic interest rate instruments and terminslugy

A deposit (fixed term) is an agreement between two parties in which one pays the other a cash amount and in return receive this money hack a pre-agrees additional payment of interest.

$$\text{actual fador or day count fraction} = \frac{\text{actual}}{365} \text{ or } \frac{\text{actual}}{360}$$

LIBOR = London Interbank Offer Rate
H (for Hong kong)
S (for Singapole)

Forward Rate Agreements

A forward rate agreement , or FRA , is an agreement between two counter parties to exchange cash payments at some specified date in the future .

T : reset date
S : payment date
A : notional amount
α : accural factor
K : fixed rate
$L_T\,[T,S]$ = LIBOR for the period $[T,S]$ that sets on date T .
$L_t\,[T,S]$ = forward LIBOR rate = the value of K for which the FRA is t-value zero.
at time T : invert his unit capical at the spot LOBOR $L_T\,[T,S]$
at time S : $A\alpha L_T\,[T,S]+[A\alpha L_t\,[T,S]-A\alpha L_T\,[T,S]]$
$= A\alpha L_t\,[T,S]$

Interest Rate Swaps

An interest rate swap , which we will abbreviate to swap , an agreement between two counterparties to exchange a series of cashflow on pre-agreed dates in the future.

T = the start date (the start of the first accural period)
S = maturity date (the date of the last cashflow)

parment frequeney (each leg can in general have a difference payment frequeney)

$$\begin{aligned} K &= \text{par swap rate} \\ &= \text{swap rate (t = T) , forward swap rate (t < T)} \end{aligned}$$

zero coupon bonds (pure discount)

Zero coupon bonds are assets which entitle the holder to receive a cashflow at some future date T.

Discount factors and valuation

D_{tT} = the value, at time t, of a ZCB paying a unit amount at time T, $t \leq T$.
$D_{tt} = 1$

- We will usually assume that the initial discount curve, $\{D_{0T} : T \geq 0\}$ if today is time zero, is known.
- We will see how discount factor can be used to express the value of the basic interest rate instruments.

Deposit valuation

$$
\begin{aligned}
D_{TT} = 1 = D_{TS}(1 + L_T\,[T,S]\,\alpha) \\
D_{TS} = (1 + \alpha L_T\,[T,S])^{-1} \\
L_T\,[T,S] = \frac{D_{TT} - D_{TS}}{\alpha D_{TS}}
\end{aligned}
$$

FRA valuation

$$
\begin{aligned}
\text{payment under an FRA} &= \alpha(L_T\,[T,S] - K) \\
&= \alpha(\frac{D_{TT} - D_{TS}}{\alpha D_{TS}} - K) \\
&\quad \text{(a derivative of ZCBs)}
\end{aligned}
$$

at time t

$$
\text{buy one unit of the T} - \text{bond}
$$

$$
\text{sell } (1 + \alpha \text{K}) \text{ units of the S} - \text{bond.}
$$

at time T

receive a unit payment from the maturing ZCB

deposite this unit payment until time S

at time S

the deposit and ZCB mature

$$(1+\alpha L_T\left[T,S\right])-(1+\alpha K)=\alpha(L_T\left[S,T\right]-K)$$

No Arbitrage

$$D_{tT}-(1+\alpha K)D_{tS}=V_t$$

$$\begin{aligned} L_t\left[T,S\right] &= \text{forward LIBOR rate} \\ &= \text{the value if L for which } \mathrm{V_t} \text{ is zero} \\ &= \frac{D_{tT}-D_{tS}}{\alpha D_{tS}} \end{aligned}$$

Swap valuation

$$\begin{aligned} V_t^{F\times D} &= K\sum_{j=1}^{n}\alpha_j D_{tS_j} \\ &= KP_t\left[T,S\right], S=(S_1,S_2,\cdots,S_n) \end{aligned}$$

$$P_t\left[T,S\right]=\sum_{j=1}^{n}\alpha_j D_{tS_j}$$

(the present value of a basic point)

at time t

$$\text{buy one unit of the T} - \text{bond}$$

$$\text{sell one unit of the S}_\text{n} - \text{bond.}$$

at time T

$$\text{receive one unit from the T} - \text{bond}$$

$$\text{deposit it at LIBOR until time S}_1$$

at time S_1

$$\text{receive one } 1 + \alpha_1 \text{L}_\text{T} \left[\text{T}, \text{S}_1\right]$$

$\alpha_1 L_T \left[T, S_1\right]$ - replicate the swap extra unit of principle deposit at LIBOR until time S_2.

$$\vdots$$

at time S_n

$$\text{receive one } 1 + \alpha_\text{n} \text{L}_{\text{T}_\text{n}} \left[\text{T}_\text{n}, \text{S}_\text{n}\right]$$

$$V_t^{FLT} = D_{tT} - D_{tS_n}$$

$$\begin{aligned} V_t &= V_t^{FLT} - V_t^{F \times D} \\ &= \text{the value of a payer swap at time t} \\ &= D_{tT} - D_{tS_n} - K P_t \left[T, S\right] \end{aligned}$$

$$
\begin{aligned}
y_t[T,S] &= \text{the value of K which set } \mathrm{V_t} = 0 \\
&= \text{forward swap rate} \\
&= \frac{D_{tT} - D_{tS_n}}{P_t[T,S]}
\end{aligned}
$$

$$V_t = P_t[T,S](y_t[T,S] - K)$$

The value of a receiver swap $= -V_t$.

17. Some standard interest rate derivatives

<u>*Caps and Floors*</u>

Caps and floors are similar to swaps in that they are made of a series of payments on regularly spaced times , $S_j, j = 1, 2, \cdots, n$. On dates S_j the holder of a cap receiver a payment of amount

$$\alpha_j max\{K - L_{T_j}[T_j, S_j] - K, 0\}, T_j = S_{j-1}$$

while the holder of a floor receives a payment of amount

$$\alpha_j max\{K - L_{T_j}[T_j, S_j], 0\}$$

K : the strike of the sption

caplet / floorlet = an option on an FRA.

swaption : an option on a swap.

Future

T : a settlement date
$\{\Phi_s, 0 \leq s \leq T\}$: future price process
$\Phi_t - \Phi_T =$ the net amount a counteopatty who buys the future contract at time t , when the future price is ϕ_t , agrees to pay to the exchange over the time interval $[t, T]$.
payment rules : the rules which determine precisely how the net amount $\Phi_t - \Phi_T$ is paid.

initial margin / maintence margin

18. Term-Structure models

$W = \{W_t\}_{0 \leq t \leq T}$: a Brownian motion on some probability space $(\Omega, \mathcal{F}, P)$
$\{\mathcal{F}_t\}_{0 \leq t \leq T^*}$: the natural filtration generated by W
$\{r(t); 0 \leq t \leq T^*\}$: an adapted interest rate process satisfying $\int_0^T |r(t)|dt < \infty$

$B(t, T)$ = price at time t of the zero-coupon bond payment \$1 at time T , $0 \le t \le T^*$

Fundamental Theorem of Asset Pricing

A term structure model is free of arbitrage if and only if there is a probability measure $\tilde{P}$ equivalent to P , under which for each $T \le T^*$

$$\tilde{B}(t, T) = \frac{B(t, T)}{\beta(t)} \ , \ 0 \le t \le T$$

is a martingale.

Arbitrage – free bond prices

$$\begin{aligned} \tilde{B}(t, u) &= \tilde{E}[\tilde{B}(u, u)|\mathcal{F}_t] \qquad 0 \le t \le u \\ &= \tilde{E}[e^{-\int_0^u r(s)ds}|\mathcal{F}_t] \\ \Longrightarrow B(t, u) &= \tilde{E}[e^{-\int_t^u r(s)ds}|\mathcal{F}_t] \end{aligned}$$

Term-Structure model : Any mathematical model which determines , at least theoretically , the stochastic process $B(t, T)$, $0 \le t \le T$, for all $T \in (0, T^*)$

$\underline{Forward\ rate\ agreement}$: $L_t[T, S]$

at time t : buy a unit of T-bond short $\frac{B(t,T)}{B(t,T+\epsilon)}$ units of $(T+\epsilon)$ - bond . The value of this portfolio at time t is

$$B(t,T) - \frac{B(t,T)}{B(t,T+\epsilon)} B(t,T+\epsilon) = 0$$

at time T : receive \$1 from the T-matuning zero bond.

at time $T+\epsilon$: puy $\$\frac{B(t,T)}{B(t,T+\epsilon)}$

$$\frac{B(t,T)}{B(t,T+\epsilon)} = e^{\epsilon L_t(T,T+\epsilon)}$$

i.e.

$$L_t(T,T+\epsilon) = -\frac{\log B(t,T+\epsilon) - \log B(t,T)}{\epsilon}$$

The forward rate is

$$\begin{aligned} f(t,T) &= \lim_{\epsilon \longrightarrow 0} L_t(T,T+\epsilon) \\ &= -\frac{\partial}{\partial T} \log B(t,T) \end{aligned}$$

= the instantaneous interest rate , agree at time t , for money borrowed at time T.

$$\begin{aligned} \int_t^T f(t,u)du &= -\int_t^T \frac{\partial}{\partial u} \log B(t,u)du \\ &= -\log B(t,u)|_{u=t}^{u=T} \\ &= -\log B(t,T) \end{aligned}$$

i.e.

$$B(t,T) = e^{-\int_t^T f(t,u)du}$$

Remark

$$\begin{aligned} B(t,T) &= \tilde{E}[e^{-\int_t^T r(u)du}|\mathcal{F}_t] \\ \frac{\partial}{\partial T}B(t,T) &= \tilde{E}[-r(T)e^{-\int_t^T r(t)du}|\mathcal{F}_t] \end{aligned}$$

$$\frac{\partial}{\partial T}B(t,T)|_{T=t} = -r(t)$$

i.e.

$$r(t) = f(t,t)$$

19. Bond Options

$W = (W_t)_{0\leq t\leq T^*}$ a BM on a probability space $(\Omega, \mathcal{F}, P)$
$(\mathcal{F}_t)_{0\leq\tau\leq T^*}$: the information generates by W
$(r_t)_{0\leq\tau\leq T^*}$: adapted process with $\int_0^T |r(s)|ds < \infty$ a.s.
$B_t = \exp(\int_o^t r(s)ds)$: accumulation factor

Assume that there exist a probability measure $\tilde{P}$, equivalent to P, such that for every $0 \leq \tau \leq T^*$,

$$\frac{B(t,T)}{B_t}, 0 \leq \tau \leq T,$$

is a $\tilde{P}$-martingale

$B(t,T)$: T-bond price at time $t \leq T$

$B(t,T) = \tilde{E}[e^{-\int_t^T r(s)ds}|\mathcal{F}_t]$

Write $L_T = \frac{d\tilde{p}}{dp}$

(a)For any non-negative random variable X,we have $\tilde{E}[X] = E[XL_T]$

(b)If $X$is $\mathcal{F}_t$ measurable,then $\tilde{E}[X] = E[XL_T]$,where $L_t = E[L_T|\mathcal{F}_t]$ (Hence $\frac{d\tilde{p}}{dp}|_{\mathcal{F}_t} = L_t$)

(c)For any non-negative random variable X ,

$$\tilde{E}[X|\mathcal{F}_t] = \frac{1}{L_t}E[XL_T|\mathcal{F}_t]$$

Indeed if $Y$is $\mathcal{F}_t$-measurable , then

$$\begin{aligned}
&\tilde{E}[Y\frac{1}{L_t}E[XL_T|\mathcal{F}_t]] \\
&= E[YE[XL_T|\mathcal{F}_t]] \\
&= E[E[XYL_T|\mathcal{F}_t]] \\
&= E[XYL_T] \\
&= \tilde{E}[XY]
\end{aligned}$$

Hence

$$\begin{aligned} B(t,T) &= \tilde{E}[e^{-\int_t^T r(s)ds}|\mathcal{F}_t] \\ &= E[e^{-\int_t^T r(s)ds}\frac{L_T}{L_t}|\mathcal{F}_t] \end{aligned}$$

<u>*Proposition*</u>

There is an adapted process $(q(t))_{0\leq t\leq T^*}$ such that, for all $t \in [0,T]$,

$$L_t = exp(\int_0^t q(s)dw_s - \frac{1}{2}\int_0^t q(s)^2 ds) \ a.s.$$

$$B(t,u) = E[exp(-\int_t^u r(s)ds + \int_t^u q(s)dw_s - \frac{1}{2}\int_t^u q^2(s)ds)|\mathcal{F}_s]$$

<u>*Proposition*</u>

For each maturity u, there is an adapted process $(\sigma_t^u)_{0\leq t\leq \ u}$ such that,on [0,u],

$$\frac{dB(t,u)}{B(t,u)} = (r(t) - \sigma_t^u q(t))dt + \sigma_t^u dw_t$$

Note that $-\sigma_t^u q(t)$is the difference between the average yield of the bond and the riskless rate here the interpretation Recall that under $\tilde{p}$,

$\widetilde{W}_t = W_t - \int_0^t q(s)ds$ is a BB and hence of -q(t) as a "risk premium"

$$\frac{dB(t,u)}{B(t,u)} = r(t)dt + \sigma_t^u d\tilde{w}_t$$

For this reason the probability $\tilde{P}$ is often call the risk neutral probability.

Bond options

θ : maturity date
K : strike price
H_t^o : the quantity of riskless asset
H_t : the number of bond with maturity T

$$\begin{aligned} V(t) &= \text{the value of the portfolio at time t} \\ &= H_t^0 B_t + H_t dB(t,T) \end{aligned}$$

Self-financing :
$dV_t = H_t^0 dB_t + H_t dB(t,T)$
A strategy $\phi = (H_t^0, H_t)_{0 \le t \le T}$ is admissible if it is self-financing and if the discounted value $\tilde{V}_t(\phi)$ is non-negative and if $\sup_{t \in [0,T]} \tilde{V_t(\phi)}$ is square-integrable under $\tilde{P}$

Theorem

We assume $\sup_{0\leq t\leq T}|r(t)| < \infty$a.s. and $\sigma_t^T \neq 0$ a.s. for all $t \in [0,\theta]$.
Let $\theta < T$ and h be a $\mathcal{F}_\theta$-measurable random variable such that $he^{-\int_o^\theta r(s)ds}$ is square-integrable under $\tilde{P}$.
Then there exist an admissible strategy whose value at time θ is equal to h. The value at time $t \leq \theta$ of such a strategy is given by

$$V_t = \widetilde{E}(e^{-\int_t^\theta r(s)dsh}|\mathcal{F}_t)$$

Proof

(1) $\phi = (H_t^0, H_t)$:an admissible strategy $d\widetilde{V}_t(\theta) = H_t\tilde{B}(t,T)\sigma_t^T d\widetilde{W}_t$ Hence $\widetilde{V}_t$ is a $\tilde{P}$-measurable i.e
$\widetilde{V}_t(\phi) = \widetilde{E}[\widetilde{V}_\theta(\phi)|\mathcal{F}_t]$ IF $V_\theta(\phi) = h$, then we get $V_t = e^{\int_0^t r(s)ds}\widetilde{E}[e^{-\int_0^\theta r(s)ds}h|\mathcal{F}_t]$
(2)$\exists(J_t)_{0\leq t\leq\theta}$ such that $\int_0^\theta J_t^2 dt < \infty$ a.s. and $he^{-\int_0^\theta r(s)ds} = \widetilde{E}(he^{-\int_0^\theta r(s)ds}) + \int_0^\theta J_s d\widetilde{W}_s$ Set $H_t = \frac{J_t}{\tilde{B}(t,T)\sigma_t^T}$ and $H_t^0 = \widetilde{E}[he^{-\int_0^\theta r(s)ds}|\mathcal{F}_t] - \frac{J_t}{\sigma_t^T}$
Then $\phi = (H_t, H_t^0)$ is a self-financing strategy whose value at time θ is indeed equal to h.

20. The Vasicek Model

$$dr(t) = a(b - r(t))dt + \sigma dW_t$$

where a,b,σ are non-negative constants.
Set

$$X_t = r(t) - b$$

Then

$$dX_t = -aX_t dt + \sigma dW_t$$

which means $\{X_t\}$ is an $Ornstein-Uhlenbeck$ process.
We deduce that $r(t)$ can be written as

$$r(t) = r(0)e^{-at} + b(1 - e^{-at}) + \sigma e^{-at} \int_o^t e^{as} dW_s$$

- $r(t)$ follows a normal law
- $Er(t) = r(0)e^{-at} + b(1 - e^{-at}) = b + (r(0) - b)e^{-at}$
- $Var(r(t)) = \frac{\sigma^2}{2a}(1 - e^{-2at})$
- $r(t)$ converges in law to a Gaussian random variable with mean b and variable $\frac{\sigma^2}{2a}$.

We also assume that $q(t)$ is a constant $q(t) = -\lambda$, with $\lambda \in \mathbf{R}$.

Then

$$\begin{aligned} B(t,T) &= \tilde{E}[e^{-\int_t^T r(s)ds}|\mathcal{F}_t] \\ &= e^{-\tilde{b}(T-t)}\tilde{E}[e^{-\int_0^T (r(s)-\tilde{b})ds}|\mathcal{F}_t] \\ &= e^{-\tilde{b}(T-t)}\tilde{E}[e^{-\int_t^T \tilde{X}_s ds}|\mathcal{F}_t] \end{aligned}$$

where $\tilde{b} = b - \frac{\lambda\sigma}{a}$ and $\tilde{X}_t = r(t) - \tilde{b}$.
Note that

$$d\tilde{X}_t = -a\tilde{X}_t dt + \sigma d\widetilde{W}_t.....*$$

with

$$d\widetilde{W}_t = \lambda dt + dW_t$$

We can write

$$\widetilde{E}[e^{-\int_t^T \tilde{X}_s ds}|\mathcal{F}_t] = F(T-t, r(t) - \tilde{b})$$

where F is the function defined by

$$F(\theta, x) = \tilde{E}[e^{-\int_0^\theta \tilde{X}_s^x ds}]$$

$\{\tilde{X}_t^x\}$ being the unique solution of (*) which satisfies $\tilde{X}_0^x = x_0$.
Since

$$\tilde{X}_t^x = xe^{-at} + \sigma e^{-at}\int_o^t e^{as} d\widetilde{W}_s$$

we have

$$\begin{aligned}\tilde{E}[\int_0^\theta \tilde{X}_s^x ds] &= \int_0^\theta \tilde{E}[\tilde{X}_s^x] ds \\ &= x\int_0^\theta e^{-as} ds \\ &= \frac{x}{a}(1-e^{-a\theta})\end{aligned}$$

Note that

$$\begin{aligned}Var(\int_0^\theta \tilde{X}_s^x ds) &= Cov(\int_0^\theta \tilde{X}_s^x ds, \int_0^\theta \tilde{X}_s^x ds) \\ &= \int_0^\theta \int_0^\theta Cov(\tilde{X}_s^x, \tilde{X}_t^x) ds dt\end{aligned}$$

and

$$\begin{aligned}Cov(\tilde{X}_u^x, \tilde{X}_t^x) &= \sigma^2 e^{-a(t+u)} \tilde{E}[\int_0^t e^{as} d\widetilde{W}_s \int_0^u e^{as} d\widetilde{W}_s] \\ &= \sigma^2 e^{-a(t+u)} \int_0^{t\wedge u} e^{2as} ds \\ &= \sigma^2 e^{-a(t+u)} \frac{e^{2at\wedge u}-1}{2a}\end{aligned}$$

Therefore

$$Var(\int_0^\theta \tilde{X}_s^x ds) = \frac{\sigma^2\theta}{a^2} - \frac{\sigma^2}{a^3}(1-e^{-a\theta}) - \frac{\sigma^2}{2a^3}(1-e^{-a\theta})^2$$

Hence

$$\tilde{E}[e^{-\int_0^\theta \tilde{X}_s^x ds}] = e^{-\tilde{E}[\int_0^\theta \tilde{X}_s^x ds] + \frac{1}{2}Var(\int_0^\theta \tilde{X}_s^x ds)}$$

and

$$B(t,T) = e^{-(T-t)R(T-t, r(t))}$$

$$\widetilde{E}e^{-\lambda X} = e^{(-\lambda)EX+\frac{1}{2}(-\lambda)^2 varX}$$

where

$$R(\theta, r) = R_\infty - \frac{1}{a\theta}\{(R_\infty - r)(1-e^{-a\theta}) - \frac{\sigma^2}{4a^2}(1-e^{-a\theta})^2\}$$

with

$$R_\infty = \lim_{\theta \longrightarrow \infty} R(\theta, r) = \tilde{b} - \frac{\sigma^2}{2a^2}$$

- $R(T-t, r(t))$can be seem as the average interest rate on the period [t,T]
- $R_i nfty$ can be interpreted as a long-term rate

It is immediately clear that the Vasicek Model does not have enough free parameters so that it can be calibrated to correctly price all pure discount bonds, i.e. we cannot in general choose a, σ, b and λ to simultaneously solve

$$B(0, T) = \widetilde{E}[exp\{-\int_o^T r(s)ds\}]$$

for all maturities $T > 0$.

This led Hull and White (1990) to extend to the Vasicek Model by replacing these constants a,b,λ, and σ whit deterministic function,

$$dr(t) = (\theta(t) - a(t)r(t))dt + \sigma(t)dW_t$$

Hull and white Model. Assume ,under the risk-neutral measure P,we have

$$dr(t) = (\alpha(t) - \beta(t)\gamma(t))dt + \sigma(t)dW(t)$$

where $\alpha(t), \beta(t)$and $\sigma(t)$ are deterministic function. Set

$$k(t) = \int_0^t \beta(s)ds,$$

Then

$$\begin{aligned} d(e^{k(t)r(t)}) &= e^{k(t)}(\beta(t)r(t)dt + dr(t)) \\ &= e^{k(t)}(\alpha(t)dt + \sigma(t)dW(t)) \end{aligned}$$

Integrating,we get

$$r(t) = e^{-k(t)}[r(0) + \int_0^t e^{k(s)}\alpha(s)ds + \int_0^t e^{k(s)}\sigma(s)dW_s]$$

We see that $r(t)$ is a Gaussian process with mean function

$$m(t) = e^{-k(t)}[r(0) + \int_0^t e^{k(s)}\alpha(s)ds]$$

and covariance function

$$\rho(s,t) = e^{-k(t)-k(s)} \int_0^{t\wedge s} e^{2k(u)}\sigma^2(u)du$$

Moreover we have

$$\begin{aligned} B(0,T)b &= E[exp\{-\int_0^T r(e)dt\}] \\ &= exp\{(-1)E(\int_0^T r(t)dt) + \frac{1}{2}(-1)^2 var(\int_0^T r(t)dt)\} \\ &= exp\{-A(0,T) - C(0,T)r(0)\} \end{aligned}$$

Where

$$A(0,T) = \int_0^T \int_0^t e^{-k(t)+k(u)} \alpha(u) du dt$$

$$-\frac{1}{2}\int_o^T e^{2k(v)} \sigma^2(v) (\int_v^T e^{-k(y)} dy)^2 dv$$

and

$$C(0,T) = \int_0^T e^{-k(t)} dt$$

<u>Exercise</u>

Show that

$$var(\int_0^T r(t)dt) = \int_o^T e^{2k(v)} \sigma^2(v) (\int_v^T e^{-k(y)} dy)^2 dv$$

and

$$A(0,T) = \int_0^T [e^{k(v)} \alpha(v) (\int_v^T e^{-k(y)} dy)$$

$$-\frac{1}{2} e^{2k(v)} \sigma^2(v) (\int_v^T e^{-k(y)} dy)^2] dv$$

<u>Exercise</u>

Show that

$$B(t,T) = exp\{-A(t,T) - C(t,T) r(t)\}$$

Where

$$A(t,T) = \int_t^T [e^{k(v)} \alpha(v) (\int_v^T e^{-k(y)} dy)$$

$$-\frac{1}{2} e^{2k(v)} \sigma^2(v) (\int_v^T e^{-k(y)} dy)^2 dv]$$

and

$$C(t,T) = e^{k(t)} \int_t^T e^{-k(y)} dy$$

By Itô's lemma, we get

$$\frac{dB(t,T)}{B(t,T)} = \{-C(t,T)(\alpha(t)-\beta(t)r(t))-\frac{1}{2}C^2(t,T)\sigma^2(t)$$

$$-r(t)C_t(t,T) - A_t(t,T)\}dt - C(t,T)\sigma(t)dW(t)$$

$$= r(t)dt + \sigma(t)C(t,T)dW(t)$$

In particular,the volatility of the bond price is $\sigma(t)C(t,T)$

21. Calibration of the Hull-White Model

$$\begin{aligned}
dr(t) &= (\alpha(t) - \beta(t)r(t)dt + \sigma(t)dW(t)) \\
k(t) &= \int_0^t \beta(u)du \\
A(t,T) &= \int_t^T [e^{k(v)}\alpha(v)(\int_v^T e^{-k(y)}dy) - \frac{1}{2}e^{2k(v)}\sigma^2(v)(\int_v^T e^{-k(y)}
\end{aligned}$$

$$dy)^2]dv$$

$$\begin{aligned}
C(t,T) &= e^{k(t)} \int_t^T e^{-k(y)}dy \\
B(t,T) &= exp\{-r(t)C(t,T) - A(t,T)\}
\end{aligned}$$

Suppose we obtain $B(0,T)$ for all $T \in [0,T^*]$ from market data (with some interpolation).
Can we determine the function $\alpha(t), \beta(t), and \sigma(t)$

for all $t \in [0, T^*]$
We take the following input data foe the calibration:
1.$B(0,T), 0 \leq T \leq T^*$;
2.$r(0)$;
3.$\alpha(t)$;
4.$\sigma(t), 0 \leq t \leq T^*$(usually assumed to be constant)
5.$\sigma(0)C(0,T), 0 \leq T \leq T^*$(the volatility at time zero of bond of all maturities)
<u>Step 1</u>

$$C(0,T) = \int_0^T e^{-k(y)}dy \Longrightarrow -log\frac{\partial}{\partial T}C(0,T) = k(T)$$

$k(T) = \int_0^T \beta(y)dy$
$\Longrightarrow \beta(T) = \frac{\partial}{\partial T}k(T) = -\frac{\partial}{\partial T}log\frac{\partial}{\partial T}C(0,T)$
<u>Step 2</u>
From the formula

$$B(0,T) = exp\{-r(0)C(0,T) - A(0,T)\}$$

We can solve for $A(0,T)$ for all $0 \leq T \leq T^*$.
<u>Sept 3</u>

$\frac{\partial}{\partial T}[e^{k(T)}\frac{\partial}{\partial T}[e^{k(T)}\frac{\partial}{\partial T}A(0,T)]] = \alpha'(T)e^{2k(T)}+2\alpha(T)\beta(T)e^{2k(T)} - e^{2k(T)}\sigma^2(T), 0 \leq T \leq T^*$
This give us an ordinary equation for α

$$\alpha'(T)e^{2k(T)} + 2\alpha(T)\beta(T)e^{2k(T)} - e^{2k(T)}\sigma^2(T)$$

=know function of t.
We can solve the equation numerically to determine the function $\alpha(t), 0 \leq T \leq T^*$.