On implementing Euro-Bund futures pricing

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Abstract

One of the exciting developments in finance over the last 25 years has been the growth of the derivatives markets. In many situations, both hedgers and speculators find it more attractive to trade a derivative on an asset than to trade the asset itself.

In this thesis, we will take a look at the bond futures contract and derive a method for the pricing of the contracts as well as the dynamics behind it. The focus will be on the implementation part and how to acquire all the essential tools needed to make an efficient and valid pricing of the bond futures contract. Much effort has been put in making the implementation and its presentation as clean and efficient as possible.

The implementation is presented in a stepwise manner, starting with the necessary background discussion in regards to bond futures and the factors that have bearing on the contract followed by the steps to set up the mathematical framework. Further the computational implementation applied is thoroughly discussed and the different computational structures are highlighted. We also take a look at the data utilised, where and how it can be acquired.

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Chapter 1

Introduction

Growth in the derivatives markets has brought with it an ever-increasing volume and range of interest rate dependent derivative products. To allow profitable, efficient trading in these products, accurate and mathematically sound valuation techniques are required to make pricing, hedging and risk management of the resulting positions possible.

The importance of managing interest rate risk cannot be overstated and bond futures are widely used to hedge interest rate risk on long maturities, especially by swap dealers that need to cover their risk against various points of the interest rate curve. Bond futures bear an additional risk often referred to as the basis risk (the price differential between the cash bond and the underlying bonds price for future delivery implied by the futures contract) compared to a swap.

Bond futures traded on an exchange are a very liquid product and hence a key component of the global bond markets. The nominal value of the bonds represented by daily trading, far exceed the actual value of the cash bond market itself and the futures market is arguably as important as the cash market. The futures contract are used as the main hedging and risk management tool by cash bond traders and investors and thus are essential to maintain liquidity and transparency in the market.

The aim of this paper is to derive an implementation method for pricing the bond futures contract. In particular, we will study the German Euro bund contracts that are traded on the Eurexchange. The method for doing this is outlined below. The mathematical recipe we employ is to a large extent inspired by van Straaten [11] in his 2009 masters thesis. It should be noted that all financial theories follows from well established and widely published work in financial mathematics. The main objective of this paper is to present a hands-on computational and implementation aspect to bond futures pricing. Computational structures and formulations are presented and analysed.

1.1 Outline of the Thesis

In the first chapter we give a thorough introduction of the bond futures and the tools that we need to be able to price the futures contract. We introduce the dynamics and instruments needed in a stepwise manner explaining the background and logic behind it. Further, we discuss the choice of a term structure model and argue for the reasoning of our choice; the Ho-Lee model. Finally we give a background to how the Ho-Lee model can be fitted to the current observe term structure.

In following chapter we attempt to explain the mathematical background of the whole pricing process. The bigger part of this chapter will be dedicated to solving and implementing the Ho-Lee model. Navigating the term structure model is for obvious reasons a very central part in the implementation. We will make use of both the analytical and discrete equations when solving for the bond futures price. To create a short rate tree we need to create a construct a zero curve through stripping bonds that are observed on the market. The zero curve is created by the methods bootstrapping and interpolation.

After having established the mathematical framework, we move on to the actual implementation. The implementation is done by using the mathematical software programme Matlab. Efficient computational structures for the ideas presented in the previous mathematical section, are discussed. In particular, we do a deep-dive into how to handle the short rates, construct a zero curve by bootstrapping and interpolating, determine other pricing parameters, and finally how to put it all together to price a futures efficiently.

Finally, we wrap up by discussing the data utilised and the results of the implementation as well as making suggestions for further studies.

Chapter 2

An introduction to bond futures pricing

A futures contract is an agreement between to counterparties that fixes the terms of an exchange of an underlying asset at a predetermined future time point. Futures contract, as opposed to forward contracts, are standardised agreements and the contracts are traded on a recognised futures exchange. Hence they are often also referred to as exchange-traded futures. One of the characteristics of a futures contract is that they are settled on a daily basis meaning that any profits or losses that are gained or suffered during trading is paid out at the end of the day. The majority of the contracts positions are always closed out by netting out the position to 0 before the expiry date. If a position is held until delivery, in theory the short party must deliver the underlying asset to the long party. The settlement will be a physical delivery in case of a commodity futures or cash in the case of a financial futures contract.

There is no counterparty risk associated with trading exchange-traded futures because of the role of the clearing house. A clearing house acts as the buyer or seller to all contracts sold or bought on the exchange. The clearing house has the ability to guarantee all settlement by maintaining a system of a margin deposits. In other words, one can see the margin as a good-faith cash sum required to provide comfort to the exchange that the futures trader is able to meet the obligations of the contract. There are generally two types of margins, maintenance margin and variation margin. The size of both depends on the size of the contracts net open position. Maintenance margin is the minimum level required to have the contract traded through a clearing house whereas variation margin is the additional amount that must be deposited to cover any trading losses.

2.1 Bond futures

A bond future is a futures contract that obliges the holder to buy or sell a bond at maturity. Bond futures contracts are a widely used trading and risk management instrument and an important part of the bond markets. The contracts are mainly used for hedging and speculative purposes by traders and portfolio managers. Most futures contracts on exchanges around the world trade at 3-month maturity intervals. Generally the maturity dates are fixed at March, June, September and December each year. This is also the case for the German Bund futures that are traded on the Eurexchange, the leading clearing house in Europe. Since the maturity dates are fixed, at pre-set times during the year a contract for each of the four months will expire and a final settlement price will be determined for the contract. Usually the most liquid trading takes place only in the front month contracts (the contract closest to maturity) and the further out in maturity one goes the less liquid the trading is in that contract.

The underlying asset of a bond futures contract consists of a basket of bonds. The delivery basket consists of several bonds with different coupons and maturities. Every bond in the delivery basket will have its own conversion factor which is needed to be able to compare the different bonds. The theory behind the conversion factor is to equalise coupon and accrued interest differences of all the delivery bonds provided that the interest rate curve is flat with a given yield.

The design of the bond futures purposely avoids a single underlying security. One reason for this is that if the underlying bond should lose liquidity, perhaps because it has been accumulated over time by buy and hold investors and institutions, then the futures contract would lose its liquidity as well. If we assume that there is only one bond deliverable in the futures contract, a trader may profit by simultaneously purchasing a large fraction of that bond issue and a large number of contracts. As the short party of the contract scrambles to buy that bond to deliver or buy back the contract she has sold, the trader can sell the holding of both bonds and contracts at prices well above their fair values. However, by making shorts hesitant to take positions, the threat of a squeeze can prevent a contract from attracting volume and liquidity. The existence of a basket of securities effectively avoids the problems of a single deliverable only if the cost of delivering the next to CTD is not that much higher than the cost of delivering the actual CTD.

When trading with bonds and derivatives on bonds, it is important to know about all the factors that have bearing on the bond price. Bonds can be priced at premium, discount or par.

2.1.1 Conversion factor

The concept of the conversion factors was developed by CBOT in the 1970s and has since been a standardised tool when dealing with bond futures. The conversion factor gives the price of an individual cash bond such that its yield to maturity on the delivery day of the futures contract is equal to the notional coupon of the contract. The notional coupon in the contract specification has relevance in that it is the basis of the calculation of each bond's conversion factor; otherwise it has no bearing on understanding the price behaviour of the futures contract.

The FGBL contract that we are studying in this thesis has a notional coupon of 6%. Each bond in the delivery basket is given a conversion factor and the factor for a bond will change over time, but remains fixed for one individual contract. For the FGBL contracts it is assumed that the cash flows from the bonds are discounted at six percent and the notional amount equals to one. This means that when a bond has a yield of six percent, the conversion factor is equal to one. Further, if the bond has a yield larger than six percent, the conversion factor is larger than one but due to the pull-to-par effect the shorter the maturity, the closer the conversion factors come to one. Likewise, when the yield of a bond is less than six percent, the conversion factor is smaller than one, but with shorter maturity the conversion factor converges to one. Comparing bonds with different maturity and both with coupons lower than the notional; we will see that the conversion factor is smaller for the bond with longer maturity. The opposite holds for bonds hat carry coupons in excess of the notional coupon rate; the conversion factor is larger for the bond with longer maturity. This effect follows from the fact that bonds with coupons lower than the current market rates will trade at a discount. Since it is a disadvantage to hold a bond paying lower coupon than the market rates for a longer period of time, the discount is larger for the longer maturity bond. Conversely, bonds with coupons above the current market yields trade at a premium which will be greater the longer the maturity. Conversion factors are set by the Exchange at the inception of the contract and stay unchanged for the life of the contract. They are unique to each bond and to each delivery month.

Although the conversion factors equalise the yield on bonds, bonds in the delivery basket will trade a different yields, and for this reason they are not equal at the time of delivery. Certain bonds will be cheaper than others and one bond will be the cheapest-to-deliver bond. The cheapest-to-deliver bond is the one that gives the greatest return from a strategy of buying a bond and simultaneously selling the futures contracts and closing out the position at the futures expiry. This type of strategy is referred to as cash-and-carry trading and is pursued by proprietary traders who actively exploit arbitrage price differentials between the future and the cheapest-to-deliver bond.

Summarising the features of the conversion factor

- The conversion factor is used to calculate the invoice price of a bond that is delivered into a futures contract
- Conversion factors remain constant for a bond from the moment they are determined to the expiry of the contract
- How Conversion factors are different for each bond and for each contract and exhibit the pull-to-par effect.
- Bonds with coupons greater than the notional coupon have a conversion factor higher than one, while bonds with coupons lower than the notional coupon have a conversion factor lower than one.

2.1.2 FGBL - Euro-Bund futures contract

In this thesis, we will study the FGBL contract with expiry March 2010 and exact delivery day 10th March 2010. As all Euro-Bund contracts the March 2010 contract has and underlying deliverable basket consisting of three bonds

Bond (ISIN)	Settle	Coupon	First Coupon	Maturity	Coversion Factor
DE0001135374	2008-11-14	3.75	2010-01-04	2019-01-04	0.849118
DE0001135382	2009-05-22	3.50	2010-07-04	2019-07-04	0.825135
DE0001135390	2009-11-13	3.25	2011-01-04	2020-01-04	0.799913

Table 2.1: Deliverable bonds for the March 2010 FGBL contract

For simplicity, we rename the bonds

Following is a graph of the price of Bund 1, 2 and 3 from 25 Janury - 8 March 2010

Bund 1	DE0001135374
Bund 2	DE0001135382
Bund 3	DE0001135390

Table 2.2: Bund 1, 2 and 3

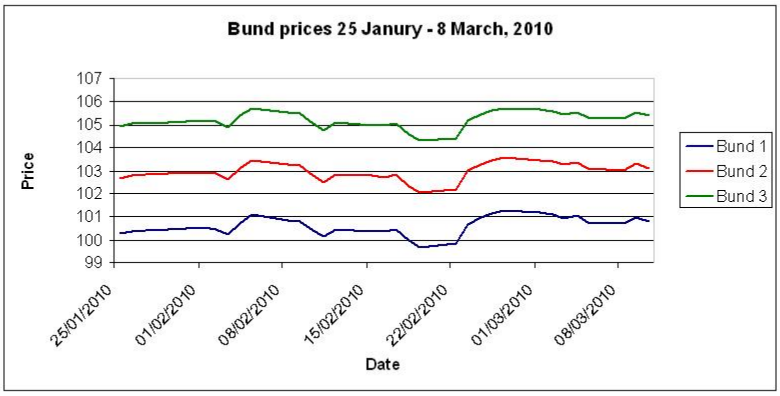

Figure 2.1: Prices of Bund 1, 2 and 3 from today until maturity

The price of the March contract will be determined on the last trading day of the contract, 8th of March, by the bond that is cheapest to deliver on that particular day. There are many factors that have bearing on the bond price. As discussed under the previous section on conversion factors, when yields of the underlying bonds are higher than the notional yield, the conversion factor tends to favour bonds that have low coupons and long maturities. Similarly, when yields are lower, the CTD bonds are often high coupon bonds with short maturities. Further, when the yield curve is upward-sloping, there is a tendency for bonds with a long time to maturity to be favoured whereas when it is downward-sloping, there is a tendency for bonds with short time to maturity to be delivered.

To determine the cheapest-to-deliver bond and hence the price of the future on can think of a trading strategy that would involve taking simultaneous and opposite positions in the cheapest-to-deliver bond and the futures contract. By the law of no arbitrage pricing, the payoff from such a trading strategy should be zero. If we set the profit from such a trading strategy to zero, we can obtain a pricing formula for the fair value of a futures contract. This implies that at time of delivery, the futures price equals to the cheapest-to-deliver bond price divided by the conversion factor.

In order to be able to select which bond is the cheapest-to-deliver we need to be able to calculate the bond prices at delivery. Since the interest rate is stochastic is impossible to know the exact future bond price. One way to make an approximation of the future outcome is to model different scenarios of probable bond prices given the stochastic movement of the interest rate. This is done by selecting and modelling the term structure.

2.2 Term structure models

In order to study interest rate derivatives or other situations where interest rates are assumed to be random, we need to study the dynamics of the short rate of interest. Interest rate derivates' price behaviour is crucially depended ion the term structure and its stochastic movements. The pricing of interest rate contingent claims has two parts. Firstly, a finite number of relevant economic fundamentals are used to price all 'default-free' zero coupon bonds of varying maturities. This gives rise to an interest rate term structure, which describes the relation between the pricing of zero coupon bonds and various maturities. Secondly, taking these zero coupon bond prices as given, all interest rate derivatives may be priced. As with asset prices, the movement of interest rates is assumed to be determined by a finite number of random shocks, which is fed into the model through stochastic processes. Assuming continuous time and hence also continuous interest rates, these sources of randomness are modelled by Brownian motions (Wiener processes). The theory of interest rate dynamics relies on the assumption that the assets are default-free and available in a continuum of maturities.

Term structure models can more or less be divided into two categories; the equilibrium models and the no-arbitrage models. There are a large number of proposals on how to specify the short rate dynamics in the risk-neutral world. The equilibrium models have a deterministic drift and therefore do not automatically fit today's term structure of interest rates. They can only provide an approximate fit to many of the term structures that are encountered in practice. This can be of great disadvantage since the model cannot price the underlying bond correctly which can lead to large errors in the pricing of interest rate derivatives. The no-arbitrage models are designed to be exactly consistent with today's term structure of interest rates. The main difference between an equilibrium and a no-arbitrage model is therefore as follows. In an equilibrium model, today's term structure of interest rates is an output and the drift of the short rate is not usually a function of time. In a no-arbitrage model, today's term structure of interest rates is an input and the drift is in general dependent on time. This is because the shape of the initial zero curve governs the average path taken by the short rate in the future. If the zero curve is steeply upward-sloping for maturities between t1and t2, then r has a positive drift between these times and vice verse. In this thesis we will use the Ho-Lee model for bond pricing.

2.3 The Ho-Lee model

Thomas S. Y. Ho and Sang Bin Lee proposed the first no-arbitrage model of the term structure in 1986. At that time, Ho-Lee's proposal was a new and alternative approach to the existing pricing models. They derived an arbitrage-free interest rate movements model and instead of modelling the short-term interest rate, Ho-Lee developed a discrete time model of the evolution of the whole yield curve. The model was presented in the form of a binomial tree of bond prices with two parameters, namely the short-rate standard deviation and the market price of risk of the short rate.

The Ho-Lee model has many advantages. Bond prices, forward and zero rates are explicitly computable from the model; it is an exogenous term structure model and very well suited for building recombining lattices. The model is relatively simple and can be calibrated so as to fit the current term structure perfectly as the model is a relative pricing model in the sense that contingent claims are priced relative to the observed market term structure. However, there are two features with this model that are a disadvantage. The first is that it does not incorporate mean reversion which means that all shocks to the short rate are permanent and do not wear off with time. On the basis of economic theory, there are compelling arguments for the mean-reversion of interest rates. When rates are high, the economy tends to slow down and investments will decline. This implies that there is less demand for money and rates will tend to decline. Vice verse, when rates are low, it is relatively cheap to invest; hence rates will tend to rise. The second weakness of the Ho-Lee model is that interest rates are assumed to be normally distributed, which implies that the interest rate can become negative with positive probability. And far into the future the short rate will eventually take negative values with a probability that approaches one half.

Next, we take a look at how the Ho-Lee model can be fitted to the initial term structure

2.3.1 Fitting the Ho-Le model

Fitting the Ho-Lee model to the initial term structure can be done by taking a number of bonds from the market from which we can calculate the forward rates that are used to derive the intermediate rates in the Ho-Lee model. The term structure of interest rates is defines as the relationship between the yield-to-maturity on a zero coupon bond and the bond's maturity. There is a vast array of different methods to construct a yield curve. Yield curves are often derived by using methods such as bootstrapping and interpolation. Bootstrapping is a general approach to build a sampling distribution for a statistic by resampling from existing data. In the context of this paper we will apply bootstrapping to calculate zero-coupon yield curve from market observed coupon bearing bonds. One great advantage of the bootstrapping methodology is that it is fairly simple method to utilise. Given a limited data set, one can derive a sampled distribution pretty straight forward by bootstrapping. However, the simplicity of bootstrapping may come at a price of making important simplified assumptions that would be more formally stated in other approaches. Bootstrapping also has a tendency to be overly optimistic. Bootstrapping is utilised in conjunction with an interpolation methodology and the reason for this is because bootstrapping in itself returns insufficient number of values, meaning that we will have a discontinuous data series with gaps and missing information. Those gaps can be filled by applying an interpolation scheme that constructs a continuous function of the data set which means that we can determine yields for zero-coupon bonds with various maturities.

There are a number of criteria that needs to be fulfilled in order to see if the interpolation methodology is valid

- How good does the forward rate look when constructing yield curves? We strive for continuous forward rates however smoothness should not be achieved at the expense of other criteria
- How local is the interpolation method? This can be quantified by studying if the affect of a change in one input affects the whole yield curve or only in a local area.
- How stable are the forward rates? The stability can be sized by measuring changing an input up or down by one point and see the maximum basis point change in any given point in the forward curve.

- How local are hedges? The forward rates must be positive to avoid arbitrage

The perfect scenario would be to find an interpolation methodology that satisfies all the above criteria. However, this will be a balance between creating a too complex interpolation method and a more easily manageable method that ticks a sufficient amount of the criteria. In this paper we will use linear interpolation on the log of discount factors.

Once we have derived a complete yield curve, we can calculate the instantaneous forward rates at all nodes in the Ho-Lee binomial lattice, hence we have fitted the model to the current observed term structure.

By using the Ho-Lee model, we only need to model the term structure up to delivery time and the bond prices at delivery can be calculated analytically which we will see in a later section. Knowing the bond prices at delivery, we can determine the futures price at delivery. Finally, to calculate the futures price today we need to take the weighted average of the future outcomes. However, it should be pointed out here that no discounting is needed as the futures prices settle on a daily basis as mentioned before.

In the next section, we turn to the mathematical approach to bond futures pricing.

Chapter 3

Mathematical background

3.1 The fundamental equations of bond futures pricing

When trading with bonds and derivatives on bonds, it is important to know about the factors that have bearing on the bond price. Bonds can be priced a premium, discount or at par. If the bond's price is higher than its par value, it will sell at a premium, which means that its interest rate is lower than current prevailing interest rates. Usually, the required yield on a bond is equal to or greater than the current interest rate in order to offer a decent rate of return to encourage investors. Some bonds trade at a premium because they offer attractive particularities such as that the coupon can be stripped easily from the bond etc. Fundamentally, however, the price of a bond is the sum of the present values of all expected coupon payments plus the present value of the par value at maturity. If $B(c_i, t_i, t)$ denotes the coupon-bearing bond price with coupons c_i paid at time t_i and final payment at maturity L_N we have the following formula for the bond price.

$$\begin{aligned} B(c_i, t_i, t) &= \sum_{i=1}^{N} c_i P(t, t_i) + L_N P(t, t_N) \\ &= \sum_{i=1}^{N} c_i e^{-(t_i - t) z(t, t_i)} + L_N e^{-(t_N - t) z(t, t_N)} \end{aligned}$$

$P(t, t_i)$ denotes discount function between t, t_i, i.e. the price of a bond at time t that pays one at time t_i (a zero coupon bond) The equation of $P(t, t_i)$ is as follows

$$P(t, t_i) = e^{-z(t, t_i)(t_i - t)} \tag{3.1}$$

The discount function must satisfy a number of conditions. To start with all discount function values must be positive as it represents the value of an asset (zero coupon bond). The boundary conditions are

$$P(0, 0) = 1$$

and

$$\lim_{T \to \infty} P(0, T) = 0$$

Which simply means that a zero coupon bon maturing instantly is worth 1 and a zero maturing in a far distant has a negligible value.

A bond's yield y is defined as the interest rate at which the present value of the steam of cash flows equals the bond's market price.

$$B(c_i, t_i, t) = \sum_{i=1}^{N} c_i e^{-(t_i - t)y} + L_N e^{-(t_N - t)y}$$

This equation is usually solved by using an iterative method

Further we introduce the forward rate which is needed to derive the Ho Lee binomial lattice. The forward is denoted as the rate between two future time points. It can be calculated be assuming a no arbitrage scenario; the return of 2 months interest rate must equal the return of 1 month interest rate plus the forward rate between 1 and 2 months. Thus the equation for forward rate is as follows

$$F(0, t_1, t_2) = \frac{P(0, t_2)}{P(0, t_1)} = e^{-z(0,t_2)\cdot t_2} \cdot e^{z(0,t_1)\cdot t_1} \tag{3.2}$$

with the instantaneous forward rate denoted as

$$f(0, t_d) = -\frac{\partial}{\partial t} \ln P(0, t)|_{t=t_d} \tag{3.3}$$

We need to be able to price a bond between coupon payments. In order to this we need to determine an appropriate day-count convention. The day count is a way of measuring the fraction of interest rate between two coupon payment time points. In this thesis, an actual/actual day count will be applied since this is the type of day count used for Euro bund futures.

When an interest is either payable or receivable has been recognised but not yet paid or received that fraction of interest is often referred to as the accrued. In other words, accrued interest is the interest that has accumulated on a bond since the last coupon payment up to, but not including the settlement date. The accrued interest must be taken into account in the bond value and is added to the contract price of a bond transaction. The original bond price plus the accrued interest is called the 'dirty price' of the bond. Typically, bonds are quoted in clean prices, hence investors and other parties trading in bonds must know how to take the accrued interest into account to obtain the actual value of the bond. Accrued interest is calculated as follows

$$AI = \frac{\text{Interest in the reference period} \times \text{Days between settlement and the last coupon}}{\text{Total days in period}} \tag{3.4}$$

$$\textit{Dirty price} = \textit{Quoted clean price} + \textit{Accrued Interest.}$$

Now that we have determined all the factors that have bearing on the bond price; we turn to the calculations of bond futures pricing. As previously mentioned, the futures price is depend on the underlying bond prices as it is solely determined by the cheapest to deliver at the point of maturity of the contract. To find out which bond is the cheapest to deliver we make use of a conversion factor to make the bonds comparable despite different maturities and yields. The conversion factor is calculated as follows

$$\begin{aligned} CF &= \sum_{i=1}^{N} \frac{c_i}{(1+0.06)^{t_i}} + \frac{L_N}{(1+0.06)^{t_N}} \\ &= \sum_{i=1}^{N} c_i e^{-(t_i-t)0.06} + L_N e^{-(t_N-t)0.06} \end{aligned}$$

At maturity of the contract, the difference between the future and cheapest to deliver bond should be zero to avoid arbitrage opportunities. Hence yielding the bond futures equation

$$\textit{Futures price} \ = \ {}^{min}_{i=1,2,3} \ \frac{\textit{(Clean Price of Bond)}_i}{\textit{(Conversion Factor of Bond)}} \tag{3.5}$$

3.2 Bond prices at future delivery time - the Ho-Lee approach

Our goal is to price bond futures. For this we need to know which bond is the cheapest at the futures delivery time td. However, we cannot today know for sure which of the three underlying bonds will be the cheapest. Each bond price at td depends on the prevailing term structure at that time. The term structure is contingent upon the evolution of the short rate from t_0 to t_d. Viewing bonds as a collection of cash flows, we can see that the pricing of a single cash flow is the base for pricing bonds. $1 in the future in a stochastic short rate environment is priced as

$$P(t,T) = E_Q(e^{\int_t^T r(s)ds} \mid F_t) \tag{3.6}$$

for all $t < T$.

From this equation we see that a probabilistic model for the short rate $r(t)$ is necessary. As previously mentioned, we choose here to use the Ho-Lee model.

The Ho-Lee model operates in the standard perfect capital market assumptions in a discrete time frame with main assumptions stated below

Basic assumptions

- The market is frictionless. The market is perfectly liquid and no transaction costs etc
- The market clears at discrete point in times
- The bond market is complete meaning that there exists bonds with maturity for all time point we look at
- At each time n, there are a finite number of states of the world. Pi(n) denotes the discount price at time n state i. Within the model the discount prices completely describers the term structure of rates at that specific state

In the following steps we follow the presentation of Björk [3].

The continuous time Ho-Lee model can be defined on differential form as

$$dr = \theta(t)dt + \sigma dW_t \tag{3.7}$$

Integrating this we obtain

$$r(u) = r(t) + \int_t^u \theta(s)\,ds + \sigma(W_u - W_t)$$

r is normally distributed with mean $E[r(u)|F_t] = r(t)+\int_t^u \theta(s)\,ds$ and variance $Var(r(u)|F_t) = E[\sigma^2(W_u - W_t)^2|F_t] = \sigma^2(u-t)$. Inserting this into equation (3.6) we obtain

$$P(t,T) = E^Q[e^{-\int_t^T r(u)\,du}|F_t] = E^Q[e^{-\int_t^T [r(t)+\int_t^u \theta(s)\,ds+\sigma(W_u-W_t)]\,du}|F_t] \tag{3.8}$$

where

$$e^{-\int_t^T [r(t)+\int_t^u \theta(s)\,ds+\sigma(W_u-W_t)]\,du} = e^{-r(t)(T-t)-\int_t^T\int_t^u \theta(s)\,ds,du+\sigma\int_t^T (W_u-W_t)\,du}$$

The first two terms in the exponent are deterministic and can be taken out of the expectation. For the third term, $z = -\sigma\int_t^T (W_u - W_t)\,du$ is normally distributed with mean zero and variance

$$\begin{aligned}
Var(-\sigma\textstyle\int_t^T (W_u - W_t)\,du|F_t) &= \sigma^2 Var(\textstyle\int_0^{T-t} W_u\,du) \\
&= \sigma^2 Var((T-t)W_{T-t} - \textstyle\int_0^{T-t} u\,dW_u) \\
&= \sigma^2 E[\textstyle\int_0^{T-t}(T-t-u)\,dW_u]^2 \\
&= \sigma^2 \textstyle\int_0^{T-t}(T-t-u)^2\,du \\
&= \tfrac{1}{3}\sigma^2(T-t)^3
\end{aligned}$$

Using the well known result of the moment generating function of a normal distribution,

$$E[e^{tz}] = e^{\mu t+\frac{1}{2}\sigma^2 t^2}$$

where $z \in N(\mu, \sigma^2)$.

We see that the third term in equation (3.8) can be written as

$$E^Q[e^Z|F_t] = e^{\frac{1}{6}\sigma^2(T-t)^3}$$

This brings us to

$$P(t,T) = e^{-(T-t)r(t)-\int_t^T\int_t^u \theta(s)\,ds\,du+\frac{1}{6}\sigma^2(T-t)^3} \tag{3.9}$$

The above results can also be obtain through an alternative method of calculation

If we know a priori that the Ho-Lee model possesses an affine term structure we can derive the above equation by taking this into account. Possessing an affine term structure implies that the zero coupon bon can be priced with the following equation

$$P(t,T) = e^{A(t,T)-B(t,T)\cdot r_t} \tag{3.10}$$

where A and B are deterministic functions. To find A and B we apply the Ito formula on equation (3.10)

$$\begin{aligned} dp^T &= (A_t - B_t \cdot r)p^T dt - Bp^T dr + \tfrac{1}{2}B^2 d^T (dr)^2 \\ &= (A_t - B_t \cdot r - \theta B + \tfrac{1}{2}B^2\sigma^2)p^T dt - \sigma B \rho^T dU \\ &= (A_t - \theta B + \tfrac{1}{2}\sigma^2 B^2 - B_t \cdot r)p^T dt - \sigma B \rho^T dU \end{aligned}$$

Under the risk neutral measure Q we know that $p(t,T)/B(t)$ is a martingale, which means that $p(t,T)$ has to have local return equal to the short rate r. Hence the following holds

$$(A_t - \theta B + \frac{1}{2}\rho^2 B^2) - (B_t + 1)\cdot r = 0$$

for $t \geq 0$ and $r \in (-\infty, \infty)$.

The only solution for the above equality is if both terms within the parentheses are equal to zero yielding and ordinary differential equation solved by A and B.

The boundary conditions for the differential equations are

$$\begin{aligned} B_t(t,T) &= -1 \\ B(T,T) &= 0 \end{aligned}$$

and

$$\begin{aligned} A_t(t,T) &= \theta B(t,T) - \tfrac{1}{2}\sigma^2 B^2(t,T) \\ A(T,T) &= 0 \end{aligned}$$

Integrating above gives us

$$B(t,T) = (T-t)$$

and

$$A(t,T) = \int_t^T \theta(s)(s-T)ds \frac{\sigma^2}{2}\frac{(T-t)^3}{3}$$

Inserting A and B back to equation (3.10)

$$\begin{aligned} P(t,T) &= e^{\int_t^T \theta(s)(s-T)ds \frac{\sigma^2}{2}\frac{(T-t)^3}{3} - (T-t)\cdot r_t} \\ &= e^{-(T-t)\cdot r_t - \int_t^T \Theta(s)(s-T)ds \frac{\sigma^2 (T-t)^3}{6}} \end{aligned}$$

3.2.1 Fitting the Ho-Lee model to the initial term structure

By ensuring that equation

$$f(t,T) = -\frac{\partial}{\partial t}\ln P(t,T)$$

is obeyed we can fit the Ho-Lee model to the initial term structure

$$\begin{aligned} f(0,T) &= -\tfrac{\partial}{\partial T}(-Tr(0) - \int_0^T \int_0^u \theta(s)\,ds\,du + \tfrac{1}{6}\sigma^2 T^3) \\ &= r(0) + \int_0^T \theta(s)\,ds - \tfrac{1}{2}\sigma^2 T^2 \end{aligned}$$

Taking logarithms and differentiating twice with respect to T yields

$$\theta(T) = -\frac{\partial}{\partial T} f(0,T) + \sigma^2 T$$

Inserting this into the second term of equation (3.9) yields

$$\begin{aligned} \int_t^T \int_t^u \theta(s)\,ds\,du &= \int_t^T \int_t^u (\tfrac{\partial}{\partial s} f(0,s) + \sigma^2 s)\,ds\,du \\ &= \int_t^T (f(0,u) - f(0,t) + \tfrac{1}{2}\sigma^2(u^2 - t^2))\,du \\ &= -(T-t)f(0,t) - \tfrac{1}{2}\sigma^2(T-t)t^2 + \int_t^T (f(0,u) + \tfrac{1}{2}\sigma^2 u^2)\,du \\ &= reintroducing(f(0,u) = -\tfrac{\partial}{\partial u}\ln P(0,u)) \Rightarrow \\ &= -(T-t)f(0,t) - \tfrac{1}{2}\sigma^2(T-t)t^2 + \int_t^T (-\tfrac{\partial}{\partial u}\ln P(0,u) + \tfrac{1}{2}\sigma^2 u^2)\,du \\ &= -(T-t)f(0,t) - \tfrac{1}{2}\sigma^2(T-t)t^2 - \ln P(0,T) + \ln P(0,t) + \tfrac{1}{6}\sigma^2(T^3 - t^3) \end{aligned}$$

Hence,

$$\begin{aligned} P(t,T) &= e^{-(T-t)r(t) - \int_t^T \int_t^u \theta(s)\,ds\,du + \frac{1}{6}\sigma^2(T-t)^3} \\ &= e^{-(T-t)r(t) + (T-t)f(0,t) + \frac{1}{2}\sigma^2(T-t)t^2 + \ln P(0,T) - \ln P(0,t) - \frac{1}{6}\sigma^2(T^3-t^3) + \frac{1}{6}\sigma^2(T-t)^3} \end{aligned}$$

Finally we arrive at the equation for pricing bonds at future point in time t,

$$P(t,T) = \frac{P(0,T)}{P(0,t)} e^{-(T-t)(r(t) - f(0,t)) - \frac{1}{2}\sigma^2 t(T-t)^2} \tag{3.11}$$

What we need to know to price our bond futures $P(t_d,T)$ for all the coupon bearing bonds at t_d. The above equation becomes

$$P(t_d,t_i) = \frac{P(0,t_i)}{P(0,t_d)} e^{-(t_i-t_d)(\hat{r}(t) - f(0,t_d)) - \frac{1}{2}\sigma_2^2 t(t_i-t_d)^2} \tag{3.12}$$

$$B(c,t_c,t) = \sum_{i=1}^{N} c_{ti} P(t_d,t_i) + P(t_d,t_N) \tag{3.13}$$

This equation is fundamental to bond futures pricing in the Ho-Lee framework. It implies that, if all the other parameters can be derived, only the short rate r up to the time t_d is needed to calculate the price of a bond that matures after t_d. As we shall see in the next section, this greatly simplifies the generation of $r(t_d)$ samples.

The as of yet unknown terms in this equation are: $r(t_d)$, $f(0,t_d)$, $P(0,t_d)$, $P(0,t_i)$, and σ. The term $r(t_d)$ is, as the above derivations make clear, stochastic. For the above results to hold true, $r(t_d)$ must follow the Ho-Lee process 3.7. To price the bond futures, we need several samples of $r(t_d)$ and their associated probabilities, and then an expectation is taken over the cheapest bond to deliver in each scenario. This will be detailed in section 3.5.

3.3 Generating samples and associated probabilities of $r(t_d)$

There are several ways of generating samples of $r(t_d)$* ($*$ = *observed*). Monte Carlo simulating is one way. Another way is by creating a binomial tree. This is detailed extensively in the original Ho-Lee article [6]. To create a Ho-Lee binomial tree we to establish a framework for three main steps

- Binomial lattice
- Perturbation function
- Binomial probability

The evolution of the term structure is illustrated through a binomial lattice. The binomial lattice assumption requires an up-move followed by a down-move to be equal to a down-move followed by and up-move. This is the definition of a recombining tree, see figure (3.1). The discount functions are said to be at any state (node) is solely dependent on the number of up-movements and not in the particular order they occur. The structure of the binomial lattice is such that the state at any vertex in the lattice is defined by (n, i).

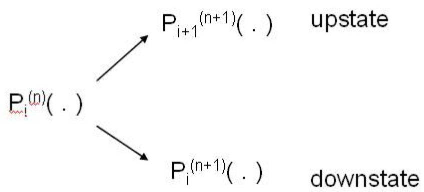

Figure 3.1: On step in a binomial tree

In the future, i denotes time and for each time $i = n$ there are exactly $(n + 1)$ states. The term structure evolves from one time point (vertex) to another by different paths. The longer the time the more alternative routes can be taken, however the discount function at the end of the path is not dependent on the specific route taken. The structure of the Ho-Lee binomial lattice can be seen below

3.3.1 Perturbation function

Ho and Lee describe the differences with an up and a down state by a perturbation function. Without the perturbation function all the nodes at the next state would equal the implied forward rate. Hence the perturbation is the difference between the rates in and up and down state. The perturbation functions are defined as

$$P_{i+1}^{n+1}(T) = \frac{P_i^{(n)}(T+1)}{P_i^{(n)}(1)} h(T)$$

in an upstate, and

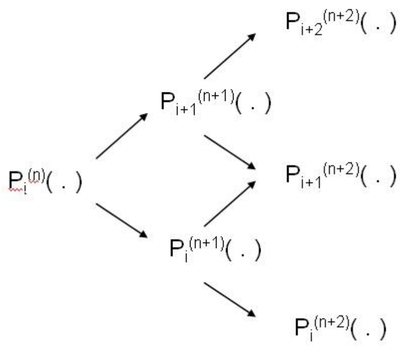

Figure 3.2: Ho-Lee binomial lattice

$$P_i^{n+1}(T) = \frac{P_i^{(n)}(T+1)}{P_i^{(n)}(1)} h^*(T)$$

in a downstate.

The perturbation function must satisfy the following boundary conditions'

$$h(0) = h^*(0) = 1$$

as well as always have positive value. The boundary conditions follow from that at state zero we only have one state and that is the observed short rate today, hence we do not need to perturb the function in any way.

3.3.2 The binomial probability π

The binomial probability can be calculated as

$$\pi = \frac{r-d}{u-d}$$

where d and u denotes the bond returns for an upstate and down state respectively

$$P_i^{(n)}(T) \cdot u = P_{i+1}^{n+1}(T-1) \Rightarrow$$

$$u = \frac{P_{i+1}^{n+1}(T-1)}{P_i^{(n)}(T)}$$

and

$$P_i^{(n)}(T) \cdot d = P_i^{n+1}(T-1) \Rightarrow$$

$$d = \frac{P_i^{n+1}(T-1)}{P_i^{(n)}(T)}$$

In this thesis we use the probability of 1/2, meaning we assume equal probability for an up-move and a down-move. The probability can be chosen arbitrarily but must fit obey the above equation in a no arbitrage setting.

An alternative representation of deriving a binomial tree in the Ho-Lee frame work is shown by Luenberger [9]

Here the short rate at each node is defined as

$$\hat{r}(k,s) = a(k) + b(k) \cdot s \tag{3.14}$$

Where $a(k)$ is a drift parameter and $b(k)$ is a volatility parameter. As previously mentioned, at a given time point k there are $k+1$ states, hence $s = 0, \ldots, k+1$. Therefore the tree structure becomes as follows

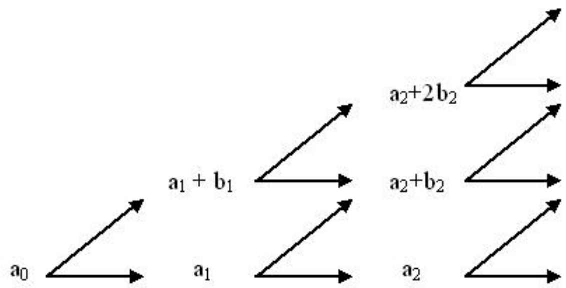

Figure 3.3: Ho-Lee short rate tree

To determine the parameters $a(k)$ and $b(k)$ we match the first and second moments of the continuous and discrete Ho-Lee models.

The conditional expectation and variation in the discrete time frame equal

$$\begin{aligned} E\left[\bar{r}(k+1) - \bar{r}(k) \mid \bar{r}(k) = \hat{r}(k,s)\right] &= \tfrac{1}{2}[a(k+1) - a(k) + (b(k+1) - b(k))s] + \\ &+ \tfrac{1}{2}[a(k+1) - a(k) + (b(k+1) - b(k))s + b(k+1)] \\ &= a(k+1) - a(k) + (b(k+1) - b(k))s + \tfrac{b(k+1)}{2} \end{aligned}$$

$$Var\left(\bar{r}(k+1) - \bar{r}(k) \mid \bar{r}(k) = \hat{r}(k,s)\right) = \frac{(b(k+1))^2}{4}$$

The expectation and variation in the continuous time frame equal

$$E\left[r(t+\Delta t) - r(t) \mid F_t\right] = \theta(t)\Delta t + o(\Delta t)$$

$$Var\left(r(t+\Delta t) - r(t) \mid F_t\right) = \sigma^2 \Delta t + o(\Delta t)$$

We know that the discrete model converges to the continuous model when the first and second moment of the discrete model equals the first and second moment of the continuous model. Which means that the following must hold

$$a(k+1) - a(k) + (b(k+1) - b(k))s + \frac{b(k+1)}{2} = \theta(t)\Delta t \tag{3.15}$$

and

$$\frac{(b(k+1))^2}{4} = \sigma^2 \Delta t \quad \Rightarrow \quad \frac{b(k+1)}{2} = \sigma\sqrt{\Delta t} \tag{3.16}$$

The Ho-Lee model assumes a constant volatility which implies that $\frac{b(k+1)}{2}$ is independent of k and equal to $\sigma\sqrt{\Delta t}$. Therefore $b(k+1) - b(k) = 0$. Hence Equation (3.15) equals

$$a(k+1) - a(k) = \theta(t)\Delta t - \sigma\sqrt{\Delta t} \tag{3.17}$$

Having constructed the short rate tree according to the dynamics of the Ho-Lee model, we have a method for generating scenarios for r at t_d, and its associated probabilities, in a consistent way. We are still missing the parameters $P(0, t_d)$, $P(0, t_i^j)$, and $f(0, t_d)$. Following equations (3.1) and (3.2). These all depend on the zero curve which will be presented in section 3.4

3.3.3 Discount rate tree, Probability of CTD tree

From the short rate it's possible to calculate the discount factor that each short rate corresponds to. This discount factor is defined as

$$p(k, s) = e^{-r(k,s)\cdot\Delta t}$$

The discount rate tree can be used to find the price of some payout at the end of the tree, by discounting these payouts using the discount factors to an earlier time in the tree. This will be done in chapter 5 in order to calculate the price of bonds before delivery of the futures.

A final tree that is useful is a tree indicating the probability that an underlying bond is the cheapest to deliver at a certain tree node. This is done by first setting a '1' at each of the final t_d nodes where a given bond is the cheapest to deliver. Then, by working backwards through the tree and recombining nodes with probability $\frac{1}{2}$, a tree of the probability that that bond is the cheapest to deliver, is obtained. This will also be analysed further in chapter 5.

3.4 Zero curve construction

3.4.1 Bootstrapping

To construct a zero curve we use a method called Bootstrapping. The principle of bootstrapping is that we invert equation (3.1) for zero coupon bonds to get

$$z = -\frac{\ln(P)}{t}$$

From this we can calculate z given market prices for zero coupon bonds P and their respective time to maturity t.

In general, however, zero coupon bonds are not available on a wide variety of maturities, and we must use coupon bearing bonds for deriving the zero curve. In order to do this, we see a coupon bearing bond as consisting of several zero coupon bonds with scaled notional (scaled by the size of the coupon c_i). Each previous coupon can be priced by using the zero curve known up to that point (as computed by earlier iterations). Thus we must construct the zero curve by starting at the short time to maturities, and adding longer dated bonds in subsequent iterations iteration. Mathematically, a coupon bearing bond is priced as

$$B(\vec{c}, t_{\vec{c}}, t) = \sum_{i=1}^{N} c_{t_i} e^{-(t_i - t)z(t,t_i)} + e^{-(t_N - t)z(t,t_N)}$$

Inverting this formula leads to the formula for extraction of a zero rate from a coupon bearing bond

$$z(t, t_N) = \frac{-1}{t_N - t} \ln \frac{B(c, t_{\vec{c}}, t)) - \sum_{i=1}^{N-1} c_{t_i} e^{-(t_i - t)z(t,t_i)}}{(1 + c_{t_n})}$$

As a starting point in constructing our zero curve, we can use short rates. Various short rates are available for various markets and uses. For example, in our case the so-called EONIA swap rates are appropriate, and the zero curve points are immediately available. This will be discussed further in the data section, chapter 5.

3.4.2 Interpolation

The discrete zero curve points that we calculate can then be interpolated in a variety of ways to form a continuous zero 'curve'. For an extensive presentation on various ways of interpolating zero curves, see Hagan et. al. [5]. Following the same paper, we will present and use 'raw' interpolation. Raw interpolation is simply linear interpolation on the zero curve points. Raw interpolation is formulated mathematically as

$$z(t) = \frac{t - t_i}{t_{i+1} - t_i} \frac{t_{i+1}}{t} z(t_{i+1}) + \frac{t_{i+1} - t}{t_{i+1} - t_i} \frac{t_i}{t} z(t_i)$$

Raw interpolation has several advantages. These include simplicity, a guarantee that forward rates are positive, and a guarantee that forward rates are continuous. Positive forward rates are necessary to avoid arbitrage opportunities.

To carry out a bootstrap it is necessary to interpolate to price coupon payments falling on dates between previously calculated zero curve points. Thus a full bootstrap requires an interpolation algorithm to work. As stated by Hagan et. al [5], "What needs to be stressed is that in the case of bootstrapping yield curves, the interpolation method is intimately connected to the bootstrap, as the bootstrap proceeds with incomplete information.".

If there is more than one coupon lying beyond the zero curve available in that iteration, it may be necessary to use a guess-and-check approach for computing the next zero curve point. The intermediate coupons are priced by interpolating between the last zero curve point and the guessed zero curve point. This will be discussed further in the computational implementation section, section 4.2.

Knowing the zero curve, it is trivial to calculate $P(0, t)$, $F(0, t_1, t_2)$, and $f(0, t)$ using the equations (3.1), (3.2) and (3.3).

3.5 Futures price from bond price scenarios

With the tree and the zero curve construction algorithms, we now have all the pieces necessary to compute bond prices at t_d using equation (3.12) . The tree algorithm provides scenarios for the short rate at futures delivery, $r_{t_d}(s)$. The zero curve provides $P(0, t_d)$, $P(0, t_i)$ and $f(0, t_d)$. As stated previously, the futures price at td is simply the cheapest bond at that point. We therefore calculate the bond prices for the three underlying bonds under all the scenarios for $r_{t_d}(s)$. The futures price in each scenario is the cheapest of the three bonds. In order to calculate the futures price at t_0 instead of t_d we need to sum the futures prices in each scenario, weighted by each scenario's probability. This can be done by working backwards through the tree and recombining nodes. Each node gets recombined with probability $\frac{1}{2}$.

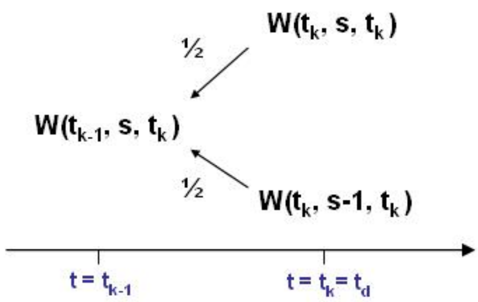

Figure 3.4: Recombination of the futures price

The reason that the prices can be recombined in this way is that no discounting is required. The daily settlement of futures means that no party is holding excess value for any period of time. Having recombined the tree to t_0, the end goal of calculating the bond futures price at t_0 has been reached.

Chapter 4

Computational implementation

We have so far seen the mathematics of how to price a bond futures contract. The central argument hinges on the Ho-Lee result that the bond prices need only be known under different interest rate scenarios at the futures delivery time t_d. Formula (3.12) shows us that the nontrivial parameters are: $r(t_d, s)$, $P(0,t)$, and $f(0,t)$. We will look at each component individually, and determine how to efficiently calculate each from a computational point of view.

4.1 The short rate

Mathematically we have seen how the short rate can be computed using a binomial tree as

$$r(k,s) = a(k) + b(k) \cdot s \tag{4.1}$$

The binomial parameters $a(k)$ and $b(k)$ were matched with the continuous time Ho-Lee results and resulted in equations (3.16) and (3.17). $b(k)$ was thus seen to have a fixed value, and $a(k)$ could be computed as a result of taking the expectation over the nodes at each time step and matching it with the discretised instantaneous forward rate over the next time step. Formalizing this mathematically we see that

$$f(0,t_k) \approx F(0,t_k,t_{k+1}) = \sum_i^k (a(k) + i \cdot b(k)) \cdot p(X_k = i)$$

where $X_k \sim Bin(k, \frac{1}{2})$

$$= b(k) = const. = b = \sigma\sqrt(t)$$

$$= a(k) + b \cdot \sum_{i=1}^k (i \cdot \frac{k!}{i! \cdot (k-i)!} \cdot \frac{1}{2}^k)$$

The fact that X is binomially distributed is due to that we are working with a binomial tree with probabilities $p = \frac{1}{2}$ of stepping up, and $1 - p = \frac{1}{2}$ of stepping down.

Rearranging this we get

$$a(k) = F(0, t_k, t_{k+1}) - b \cdot \sum_{i}^{k} (i \cdot \frac{k!}{i! \cdot (k-i)!} \cdot \frac{1}{2}^k) \tag{4.2}$$

where $F(0, t_k, t_{k+1})$ is the forward rate (non-instantaneous) from t_k to t_{k+1}. We therefore need to be able to compute the forward rate, and add it to our list of parameters.

Reinserting equation (4.2) into equation (4.1) we can determine the short rates at all nodes in the binomial tree from $t_0 = 0$ to $t_n = t_d$. Using the above representations it is easy to implement these in a programming environment. The only values of r that we are really interested in, however, are $r(n, s)$, and these can be determined directly from the results above. So even though we use a binomial tree to simulate the Ho Lee interest rate model, the computation of the relevant short rates does not actually require a tree to be computed. This can only be seen when the calculations are carried all the way to equation (4.2). A highly computationally expensive iterative problem - calculating a short rate tree - is in this way transformed into a relatively computationally cheap problem - calculating r at the end nodes analytically.

4.2 Zero curve computation to determine // $P(0, t)$, $F(0, t_1, t_2)$, and $f(0, t)$

The parameters we are after are

$$P(0, t) = e^{-z(0,t) \cdot t}$$

$$F(0, t_1, t_2) = \frac{P(0, t_2)}{P(0, t_1)}$$

$$f(0, t) = -\frac{\partial}{\partial t} \ln P(0, t)$$

As can be seen, these all require $z(0, t)$, i.e. the value at time t of the current zero curve (term structure). To compute the zero curve we use a method called bootstrapping to determine today's term structure based on market prices for bonds, and then interpolate these discreet values to get the values in between. The mathematical formalism for bootstrapping was derived in section **??**.

4.2.1 Bootstrapping

Computationally, bootstrapping takes the form of iterative calculations of successive zero curve points. However, this only works analytically when there is only one payout at a time is calculated. When several payouts occur in the region beyond which we have no zero curve yet, it is necessary to use a guess-and-check approach to solve both zero rates at the same time. A guess is taken as to the zero rate at the last of the new payouts, and the zero rate for the payouts in between follow from interpolation. The zero rate at the final payout has to be adjusted until the rates for all the points are such that the discounted payouts are equal to the current bond price. For example, the situation where both a coupon and a notional pay out after the known zero curve region, the following formula must hold

$$f(z(0,t_{k+2})) = B - \sum_{i=1}^{k} c_i e^{-z(0,t_i)t_i} - c_{k+1} \cdot e^{-z^*_{interp}(0,t_{k+1})\cdot t_{k+1}} - e^{-z^*(0,t_{k+2})\cdot t_{k+2}} = 0 \tag{4.3}$$

where z^* is the guessed value of zero curve at the last payout, z^*_{interp} is the linearly interpolated value based on that guess, and B is the market price of the bond with coupon payments up to t_{k+2}. To find the point where the above formula holds, an iterative root finding algorithm such as the bisection method, or the secant method, can be used.

4.2.2 The secant method of finding roots

The secant method is an iterative search algorithm akin to the famous Newton-Raphson method. However, the Newton-Raphson method requires a first derivative to be calculated. This is a bit problematic for equation (4.3) because of the interpolation required. A slightly slower, but also very efficient and stable, method is the secant method. The secant method discretises the first derivative using function evaluations already known. Mathematically the secant method is expressed as

$$x_n = x_{n-1} - f(x_{n-1}) \cdot \frac{x_{n-1} - x_{n-2}}{f(x_{n-1}) - f(x_{n-2})}$$

4.2.3 Computational interpolation on functional form

An effective computational structure for calculating an interpolated zero curve value is to use a function that, given inputs of the zero curve values and their times, returns a linearly interpolated value at time t that falls within the zero curve as the mathematics were set out in section 3.4.1. Thus the functional form is

$$z_{interp} = zInterp([z_1, z_2, \ldots, z_k], [t_1, t_2, \ldots, t_k], t) = zInterp(\vec{z}, \vec{t}, t)$$

And in the case of calculating $z^*_{interp}(0, t_{k+1})$ the functional call would be

$$z^*_{interp}(0, t_{k+1}) = zInterp([z_1, z_2, \ldots, z_k, z^*_{t_{k+2}}], [t_1, t_2, \ldots, t_k, t_{k+2}], t_{k+1})$$

This sort of modular implementation for calculating interpolated zero curve values is useful, as interpolated zero curve values are vital to bonds and bonds futures pricing. We will use this implementation heavily in the next steps.

4.2.4 Determining functional forms for the parameters

With an efficient way of bootstrapping, and a functional way of determining interpolated zero curve values, we are now ready to define the remaining parameters on functional form.

$P(0, t)$ is trivial to calculate given the functional form of determining z_{interp} introduced above, given that t is on the inner region of the zero curve. It is determined as

$$P(0, t) = e^{-zInterp(\vec{z}, \vec{t}, t)\cdot t}$$

$F(0, t_1, t_2)$ is similarly trivial using the interpolation function, and is determined as

$$F(0, t_1, t_2) = \frac{P(0, t_2)}{P(0, t_1)} = \frac{e^{-zInterp(\vec{z}, \vec{t}, t_2) \cdot t_2)}}{e^{-zInterp(\vec{z}, \vec{t}, t_1) \cdot t_1)}}$$

Calculating $f(0, t)$, however, is complicated by the presence of a time derivative. Taking equation (3.3) one step further gives us

$$f(0, t) = -\frac{\partial}{\partial t} \ln(P(0, t)) = f(0, t) = -\frac{\partial}{\partial t}(-z(0, t) \cdot t) = z(0, t) + t \cdot \frac{\partial}{\partial t}(z(0, t))$$

The remaining derivative on the zero curve must be calculated numerically. Since we choose to interpolate the zero rates linearly, a first order numerical approximation will suffice. Furthermore, since the derivative of a piecewise linear function is piecewise constant (a discontinuous function), a choice must be made as to the direction of the numerical derivative approximation. Because the aim is to determine an instantaneous forward rate, which is, essentially, a forward looking rate, it makes the most sense to use a forward derivative. Thus $f(0, t)$ becomes

$$f(0, t) = z(0, t) + t \cdot \frac{zInterp(\vec{z}, \vec{t}, t) - zInterp(\vec{z}, \vec{t}, t + \Delta t)}{\Delta t}$$

Using the same derivative arguments, however, it is simpler to use equation (3.2) with $t_1 = t$ and $t_2 = t + \Delta t$,

$$f(0, t) = F(0, t, t + \Delta t) = \frac{e^{-zInterp(\vec{z}, \vec{t}, t) \cdot t}}{e^{-zInterp(\vec{z}, \vec{t}, t + \Delta t) \cdot (t + \Delta t)}}$$

4.3 Futures price computation

We now know how to determine all the necessary components of equation (3.12), and can therefore calculate the bond prices at each end node. As detailed in section 3.5, the futures price at each end node is simply the cheapest bond at that node since that bond will be the cheapest to deliver. The futures price at time t=0 can be determined by working backwards and recombining node prices. As explained in section 3.5 we do not need to discount the futures prices.

This recombination is a computationally expensive iterative problem, and can be simplified by noting that what is achieved by recombination is simply a weighted sum of the futures prices at the end nodes. The weights are the probabilities of reaching the end nodes. In a binomial tree, the probabilities are binomial distributed. Thus the futures price at $t = 0$ is computed as

$$W(0, t_d) = \sum_{i=1}^{n} W(t_d, i, t_d) \cdot p(X_n = i)$$

where $X_n \sim Bin(n, \frac{1}{2})$. As was the case when computing the short rates $r(n, s)$, we have transformed the computationally expensive problem of recombining a binomial tree, into a much simpler analytical problem at one time period only.

With this formula we have reached our end goal of computing the bond futures price. Having gone from building a basic understanding of bond futures markets, to a mathematical formalism for pricing bond futures, to computationally efficient methods and implementations necessary in doing the calculations on a large scale, we can now continue with applying the model to pricing real bond futures on actual market data.

Chapter 5

Data Discussion

To price the futures selected for this thesis, the FGBL March 2010 contract, the most critical component is the construction of the zero curve. The zero curve must be based on bonds of similar character as the bonds underlying the Euro-Bund futures. As we have seen, it is necessary to construct a full zero curve up to the maturity of the last bond in order to be able to price the regular coupons that occur at times before maturity of the bonds. For this we will use German government issued bonds of varying maturity. The table below summarises the bonds used.

Bond	Settle	Next Coupon	Maturity	Coupon Rate
Bond 2	2002-07-05	2010-07-04	2012-07-04	5.00
Bond 3	2003-07-04	2010-07-04	2013-07-04	3.75
Bond 4	2004-05-28	2010-07-04	2014-07-04	4.25
Bond 5	2005-05-20	2010-07-04	2015-07-04	3.25
Bond 6	2006-11-17	2011-01-04	2017-01-04	3.75
Bond 7	2007-05-25	2010-07-04	2017-07-04	4.25
Bond 8	2008-05-30	2010-07-04	2018-07-04	4.25
Bond 8.5	2008-11-14	2011-01-04	2019-01-04	3.75
Bond 9	2009-05-22	2010-07-04	2019-07-04	3.50
Bond 9.5	2009-11-13	2011-01-04	2020-01-04	3.25
Bond 14	1994-01-04	2011-01-04	2024-01-04	6.25
Bond 17	1997-07-04	2010-07-04	2027-07-04	6.50

Table 5.1: Data of the bonds used for bootstrapping the zero curve

The keen observer will notice that the three bonds that underlie the FGBL contract are included in this table. This is natural since the purpose is to price the futures on them, and including the zero curve information contained in these bonds therefore makes sense.

The first bond included above has a maturity of one year. For earlier maturities, the government bonds traded are illiquid, and prone to poor price discovery. For the short end of the zero curve, we will therefore use the European Overnight Index Average (EONIA) rates. Despite their name, EONIA rates exist from for maturities up to one year. EONIA rates are appropriate because they carry low credit risk, on par with the German government bonds.

Chapter 6

Numerical results and analysis

6.1 Result of the zero curve construction

With the algorithm for constructing a zero curve defined mathematically in chapter 3, and computationally in chapter 4, we can construct a zero curve given the coupon bearing bonds as summarized in table (5), and the EONIA short rate data. The result, including linear interpolation between the discrete points, is the following plot

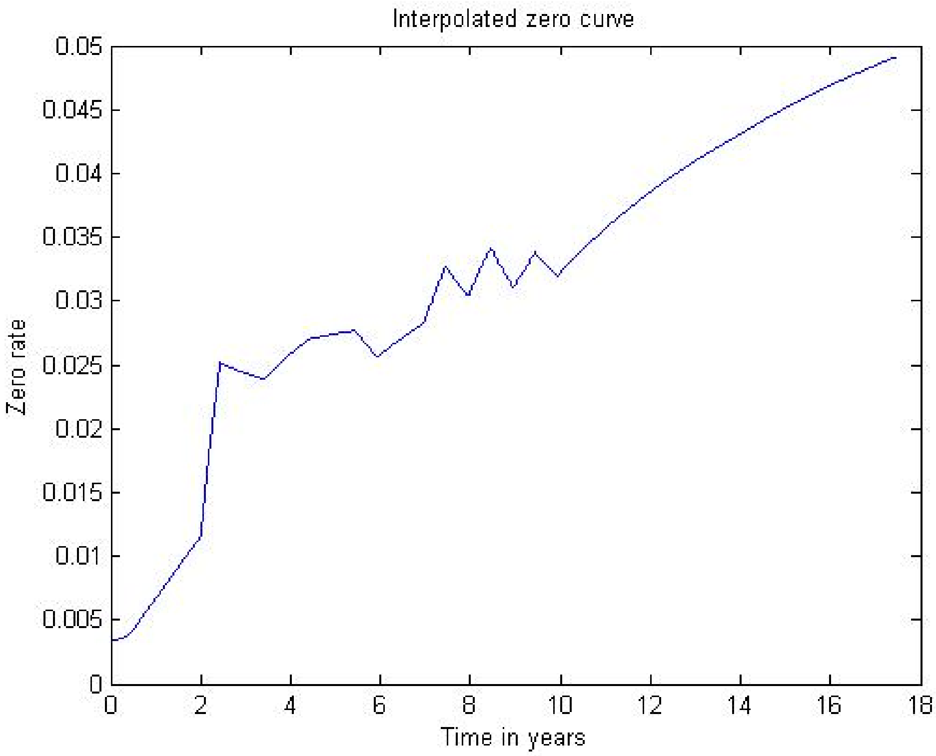

Figure 6.1: Interpolated zero curve

The result appears to be a jagged line, but rising as is often the case for market rates. This plot represents all the information known on January 25th 2010 regarding the future value of capital. As we have seen, it also makes up a cornerstone in bond futures pricing. Although we have shown how an implementation of zero curve construction can be structured efficiently, it makes little sense to analyse the computational time required to construct a curve in our case. The reason is that the number of points is relatively small, and performance gains have no considerable impact on the overall bonds futures pricing algorithm.

6.2 Computational performance analysis in calculating the short rate tree

Having gone from defining and fitting the short rate dynamics using the Ho-Lee method in section 3.2, we arrived at the critical result in equation (3.12). This equation allows us to calculate bond prices under various short rate scenarios at t_d, given that we know the other parameters. It was mentioned that a central feature of the equation was that only calculating rates scenarios at t_d implied huge computational advantages. We are now at a stage where these advantages can be analysed and appreciated.

Following our choice of a binomial tree to generate short rate 'scenarios', the r_{t_d} that go into equation (**??**) were parametrised as the end nodes of the tree, $r_{t_d}(s)$. From the plot below it is easy to see that the work required to calculate the tree up to t_d must be shorter than that required until the maturity of the longest underlying bond, t_M.

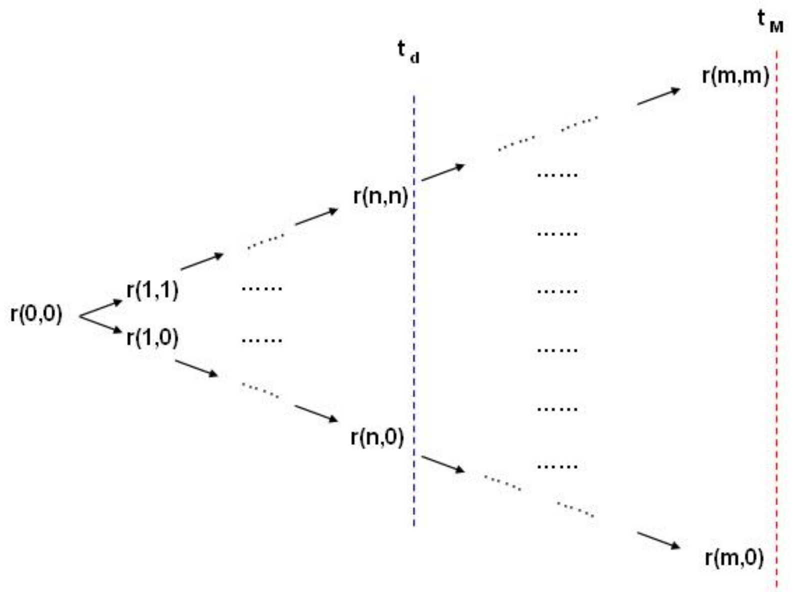

Figure 6.2: Short rate tree to t_d and t_M

Specifically, the time saved is on the order of

$$saving = \frac{M^2}{2} - \frac{N^2}{2} \sim O(M^2 - N^2)$$

However, as we have further seen in the computation section, we can use equation (4.2) to generate the $r_{t_d}(s)$ end nodes immediately. All that is required is the computation of the discrete forward rate $F(0, t_k, t_{t+1})$, and the binomial probabilities of reaching each node. This puts the time required to $\sim O(N)$. To visualize the difference in computational time required, the time taken for Matlab to generate a tree to N, and the binomial results for N end nodes is plotted below.

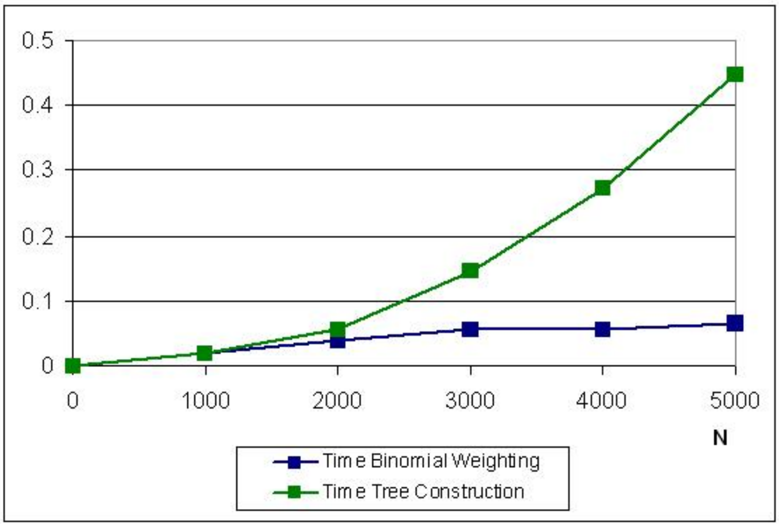

Figure 6.3: Times taken to generate tree vs. create final nodes

The interesting feature to note is that the time taken to construct the tree indeed grows quadratically with N, whereas the binomial result grows linearly with N.

6.3 Accuracy of short rate tree by pricing bonds in reverse

In section 3.3.3, we noted that having created a short rate tree, it's possible to construct a discount rate tree for discount factor between k and $k+1$ under either an up move or a down move in the short rate. We also noted that, although superfluous when pricing bond futures, the discount factor tree could be used to price bonds in 'reverse'. Since the market prices of bonds at t_0 are known, and in fact all the exogenous information is essentially made up of these various bonds, it is also entirely superfluous to price bonds backwards to t_0. It may, however, be of value to know what a bond's price would be at a time between t_0 and t_d, whereby the use of the discount factor tree arises.

We will, however, use the tree for different purposes. Our purposes are twofold: to verify that the Ho-Lee theory and the tree construction theory in section 3.3, are consistent. This is done by calculating and plotting the relative error in bond prices at t_0 for various number of steps in the Ho-Lee short rate tree up to t_d.

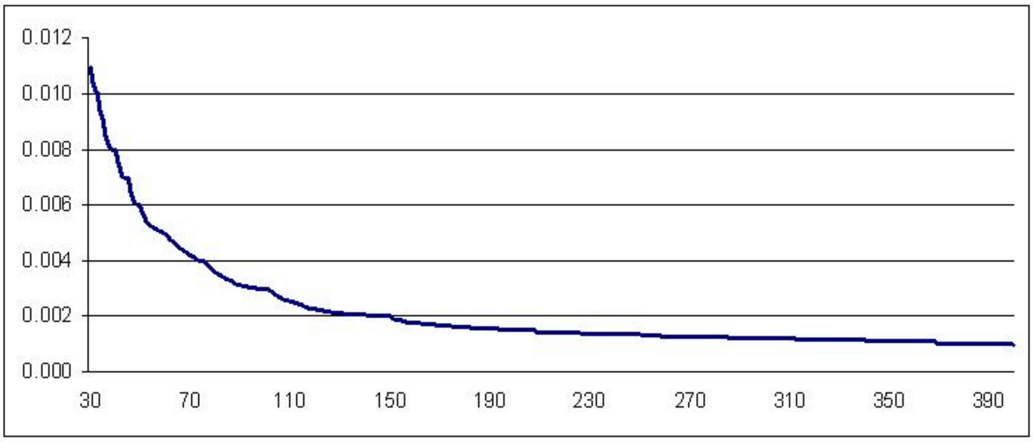

Figure 6.4: Error when comparing theoretical with market prices

From this we see that firstly, the error goes to 0, and secondly the error reduces as the number of steps increases. The first fact means that the discrete time Ho-Lee theory, and tree construction algorithms used here are consistent. The second fact means that the discrete time binomial tree parameterisation of the Ho-Lee model indeed converges to the continuous time version as the number of steps increases (step size decreases).

6.4 Probability of a bond being cheapest to deliver

As a final analysis, we examine the fact that probabilities of a certain bond being cheapest to deliver can be recombined backwards. The theory behind this was formalised at the end of section 3.3. We can thus determine how likely it is at each point in time, and at each short rate state, that a given underlying bond will be the cheapest to deliver. Since there are a large number of points in the short rate tree, the most effective visualisation is by plotting the likelihood of each bond being the cheapest to deliver, in a different colour. The colours are overlaid on one plot, and is seen below. It must be noted that this visualisation should be credited to van Straaten [11], and the reason we show it here for our calculations and numbers is that it is a highly effective way of showing the probabilities the underlying bonds being cheapest to deliver.

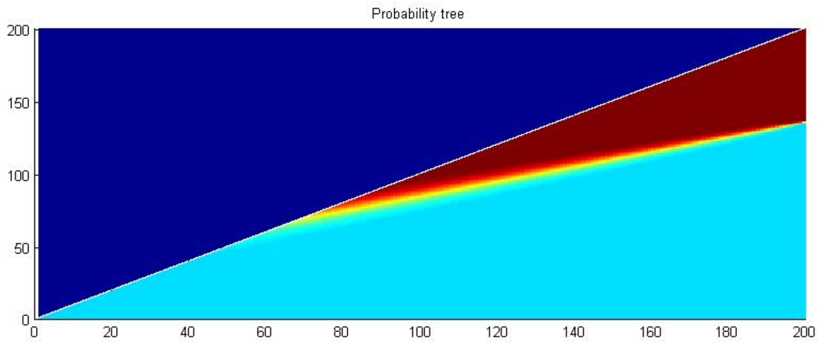

Figure 6.5: Lower right triangle; probability tree

The conclusions that can be drawn from the plot is that for the larger part of potential short rate scenarios, Bund 1 will be the cheapest to deliver. For a smaller part, Bund 3 will be the bond worth delivering. Bund 2 will never be worth delivering. For a very slim portion of short rate scenarios it is uncertain whether Bund 1 or 3 will be the cheapest to deliver.

Chapter 7

Summary and conclusions

Having chosen to take the Ho-Lee approach to modelling bond prices, we came to equation (3.12). The equation stated that we needed various parameters based on the zero curve, and scenarios for the short rate at the delivery of the maturity, t_d. The theory behind each of these required components was treated first mathematically, and then computationally. Finally we could calculate and analyse the properties of the methods we derived. It could be seen that there is a wide disparity in the efficiency of the different ways of implementing the pricing methods.

From the results we have seen from the previous section we can conclude that Ho-Lee is a robust method of simulating short rate movements yielding bond prices that are in line with the market. In particular we see that increasing the number of steps in the Ho-Lee tree yields better results as in accordance with theory, this means that the discrete model converges to the continuous case.

Creating a zero curve based on observed bond prices by bootstrapping and interpolating does not yield a smooth zero curve. Figure 6.1 clearly shows a graph with a lot of discontinuity. One can argue for different methods of interpolation as well as other resampling methodologies to obtain a smoother curve. However, working with market data will never yield smooth results, as the market is govern by supply and demand conditions.

Finally, we see that it is possible to create a set of future scenarios with their associated probabilities for the bond prices and thus, which one will be the cheapest to deliver. This means that we can make an appropriate prediction of the bond futures price.

7.1 Suggestions for further studies

Some ideas for future investigations

- studying the bond and futures prices by another term structure, i.e. the Hull-White model
- utilise methods such as the Monte Carlo simulation to create a set of possible future outcomes rather than constructing a binomial lattice
- derive the zero curve through another interpolation method.

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