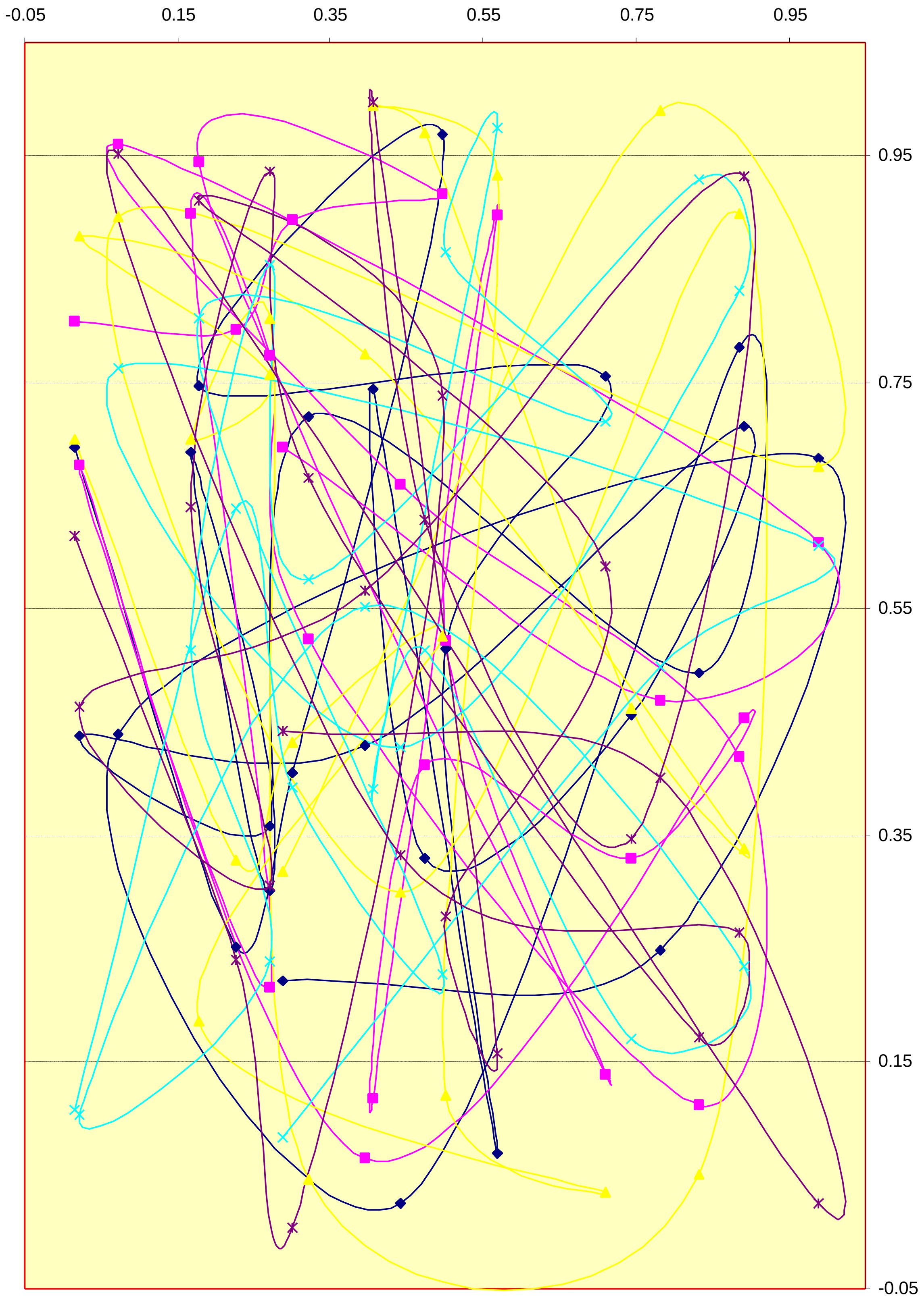
Interest rate bond futures



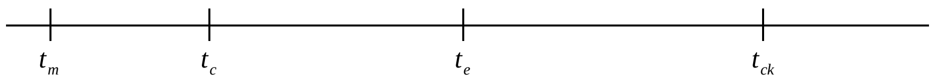
Bond Futures

In this paper we do the distinction that a forward contract is based on a known underlying whereas the future is defined on a collection of underlying bonds. There are of course exceptions from this, but we facilitate the nomenclature in the discussion.

The matter of delivery days and the fact that interest rates and bonds have different delivery periods are not highlighted here. This paper has a theoretical approach and the aim is not to construct numerical examples that are “absolutely true”.

1 *A traditional spot price based forward*

Let us look at a picture of the money flow for a couple of transactions:



t_m : Market date for the future/forward.

t_c : Coupon, if any, before delivery date.

t_e : Delivery date for the future/forward.

t_{ck} : The first coupon after the delivery date.

C : The coupon of the bond in actual money.

C_{me} : This is C if the coupon occurs between t_m and t_e, if not it is 0. (Short notation to avoid having to write down criteria for including the coupon.)

R_{me} : The interest rate for the interval between the time t_m and t_e.

Δt_{me} : The interval between the time t_m and t_e expressed in actual days.

A_m : Accrued interest for the bond at time t_m.

P_m : Spot bond price expressed as the clean price at time t_m.

F_{me} : Forward prices contracted at t_m and valid on t_e.

We will construct a position at t_m that cost us nothing to enter; this position will also give a **known** cash flow at t_e.

t_m : We want to buy a bond that we can sell in a forward. To be able to buy this bond we must borrow some money. We borrow two amounts, one of size:

$$C_{me}\left(1+\frac{\Delta t_{mc}}{360}R_{mc}\right)^{-1} \tag{1.1}$$

This one should be returned at t_c and another amount:

$$P_m + A_m - C_{me}\left(1+\frac{\Delta t_{mc}}{360}R_{mc}\right)^{-1} \tag{1.2}$$

This one is returned at the forward delivery t_e. The first amount is intended to be covered by the coupon that is given us from the purchased bond. The second one shall cover the forward delivery. The sum of these amounts, $P_m + A_m$, is exactly the amount needed to purchase one bond in the spot market. And at last sell one forward position in this bond at forward price F_{me}.

At this time we have actually not spent any money at all, every transaction we made has been covered by various loans.

t_c : At this point in time we receive a coupon of size C_{me}. But at the same time the first loan is due to pay. The net sum of transaction is zero.

$$C_{me} - C_{me}\left(1 + \frac{\Delta t_{mc}}{360} R_{mc}\right)^{-1}\left(1 + \frac{\Delta t_{mc}}{360} R_{mc}\right) = 0 \tag{1.3}$$

t_e : At this time the remaining loan is due to pay. The amount of money to hand over is:

$$\left[P_m + A_m - C_{me}\left(1 + \frac{\Delta t_{mc}}{360} R_{mc}\right)^{-1}\right]\left(1 + \frac{\Delta t_{me}}{360} R_{me}\right) \tag{1.4}$$

We also have to fulfil our forward contract. In this paper we presume that the forward price is expressed in clean price, the accrued interest is then added when delivered. The forward contract gives us a cash flow of $F_{me} + A_e$. In exchange for this we deliver the previously bought bond.

Since all these figures are known to us at the time we enter the position, and the cost of entry was zero, the value of this position at this time should also be zero (If any part of (1.4) would be unknown at t_m we would be unable to use this argument).

$$F_{me} + A_e - \left[P_m + A_m - C_{me}\left(1 + \frac{\Delta t_{mc}}{360} R_{mc}\right)^{-1}\right]\left(1 + \frac{\Delta t_{me}}{360} R_{me}\right) = 0 \tag{1.5}$$

This gives us a way of calculating the forward price for a specified interest rate bond. The building blocks are, apart from the bond specific, the discount rates to use. One should admit that when we are dealing with long time forwards we have difficulties in pricing these because of the lack of notations for discount rates. (The solution to this is of course to fully use a yield curve based approach but in this paper we do not incorporate techniques for using yield curves.) The forward price can be written as:

$$F_{me} = \left[P_m + A_m - C_{me}\left(1 + \frac{\Delta t_{mc}}{360} R_{mc}\right)^{-1}\right]\left(1 + \frac{\Delta t_{me}}{360} R_{me}\right) - A_e \tag{1.6}$$

This may be seen as somewhat complicated and many people prefer to use forward-forward rates to take care of the possible coupon. First we state the fundamental connections between interest rates:

$$\left(1 + \frac{\Delta t_{mc}}{360} R_{mc}\right)\left(1 + \frac{\Delta t_{ce}}{360} R_{ce}\right) = \left(1 + \frac{\Delta t_{me}}{360} R_{me}\right) \tag{1.7}$$

This gives us equation (1.6) in a new shape

$$F_e = \left[P_m + A_m\right]\left(1 + \frac{\Delta t_{me}}{360} R_{me}\right) - C_{me}\left(1 + \frac{\Delta t_{ce}}{360} R_{ce}\right) - A_e \tag{1.8}$$

The annoying part of this formula is of course that the use forward-forward rate R_{ce} must be calculated from today’s term structure. This is of course a small detail, but nevertheless makes (1.6) a little bit more tempting.

Example 1

The basic market notations:

Market date	1998-01-03
Delivery date	1998-03-18
Rate to delivery	5.55 %
Rate to coupon	5.80 %
Bond expire date	1999-01-21
Days to next coupon	18
Days to delivery date	75
Days between next coupon and delivery	57
Bond coupon	11 %
Bond market rate	6 %
Bond dirty price	115.380

From the above we can calculate the things that we have discussed. Accrued interest at delivery date should also be calculated.

$$A_e = 11 \cdot \frac{57}{360} = 1.742 \tag{1.9}$$

Using equation (1.6) would give us:

$$\begin{aligned} F_{me} &= \left[P_m + A_m - C_{me}\left(1 + \frac{\Delta t_{mc}}{360} R_{mc}\right)^{-1} \right]\left(1 + \frac{\Delta t_{me}}{360} R_{me}\right) - A_e \\ &= \left[115.380 - 11\left(1 + \frac{18}{360} 0.0580\right)^{-1} \right]\left(1 + \frac{75}{360} 0.0555\right) - 1.742 \\ &= 103.877 \end{aligned} \tag{1.10}$$

103.877 would be the clean forward price and expressed in rate it would be 6.08 %.

2 Implied repo rate for forwards

As a complement to this way of calculating forward price we also want to calculate the "implied repo rate" for a forward contract. This means that we already have the forward price and want to know the rate R_{me} in an effort to deduce arbitrage opportunities.

We start this part of the investigation by looking at the case when we have no coupon during the time from market date and delivery date.

$$F_{me} + A_e - \left[P_m + A_m \right]\left(1 + \frac{\Delta t_{me}}{360} R_{me}\right) = 0$$

$$F_{me} + A_e - P_m - A_m = \left[P_m + A_m \right] \frac{\Delta t_{me}}{360} R_{me}$$

$$R_{me} = \frac{F_{me} + A_e - P_m - A_m}{\left[P_m + A_m \right] \frac{\Delta t_{me}}{360}} \tag{2.1}$$

When we have a coupon to deal with there are some different approaches in the literature. The issue at stake is how we should handle the different interest rates that are included in all the equations. One approach is to assume that R_{ce} is equal to the coupon rate. This would be a rather uninteresting case since the coupon rate has nothing to do with market rates and are therefore not further investigated.

Another way is to assume that the two rates R_{ce} and R_{me} are equal, this should at least be a rather good approximation.

$$F_{me} + A_e - \left[P_m + A_m\right]\left(1 + \frac{\Delta t_{me}}{360} R_{me}\right) + C_{me}\left(1 + \frac{\Delta t_{ce}}{360} R_{me}\right) = 0$$

$$F_{me} + A_e - P_m - A_m + C_{me} = \left[P_m + A_m\right]\frac{\Delta t_{me}}{360} R_{me} - C_{me}\frac{\Delta t_{ce}}{360} R_{me}$$

$$R_{me} = \frac{F_{me} + A_e - P_m - A_m + C_{me}}{\left[P_m + A_m\right]\frac{\Delta t_{me}}{360} - C_{me}\frac{\Delta t_{ce}}{360}} \tag{2.2}$$

The attractive part is of course that we only have to deal with one interest rate, but keep in mind that this **is** an approximation. The most accurate way to do this calculation should be to use (1.5):

$$F_{me} + A_e - \left\{P_m + A_m - C_{me}\left(1 + \frac{\Delta t_{mc}}{360} R_{mc}\right)^{-1}\right\}\left(1 + \frac{\Delta t_{me}}{360} R_{me}\right) = 0$$

$$\left\{P_m + A_m - C_{me}\left(1 + \frac{\Delta t_{mc}}{360} R_{mc}\right)^{-1}\right\}\frac{\Delta t_{me}}{360} R_{me} = F_{me} + A_e - P_m - A_m + C_{me}\left(1 + \frac{\Delta t_{mc}}{360} R_{mc}\right)^{-1}$$

$$R_{me} = \frac{360}{\Delta t_{me}} \frac{F_{me} + A_e - P_m - A_m + C_{me}\left(1 + \frac{\Delta t_{mc}}{360} R_{mc}\right)^{-1}}{P_m + A_m - C_{me}\left(1 + \frac{\Delta t_{mc}}{360} R_{mc}\right)^{-1}} \tag{2.3}$$

Or (1.8) without any simplifications:

$$F_e - \left[P_m + A_m\right]\left(1 + \frac{\Delta t_{me}}{360} R_{me}\right) + C_{me}\left(1 + \frac{\Delta t_{ce}}{360} R_{ce}\right) + A_e = 0$$

$$\left[P_m + A_m\right]\frac{\Delta t_{me}}{360} R_{me} = F_e + C_{me}\left(1 + \frac{\Delta t_{ce}}{360} R_{ce}\right) + A_e - \left[P_m + A_m\right]$$

$$R_{me} = \frac{360}{\Delta t_{me}} \frac{F_e + A_e - \left[P_m + A_m\right] + C_{me}\left(1 + \frac{\Delta t_{ce}}{360} R_{ce}\right)}{P_m + A_m} \tag{2.4}$$

Do observe that the two equations (2.3) and (2.4) are equivalent, and the choice of which one to use is more a question about how we easiest handle interest rates (a more correct question is if the user has some system to extract forward-forward rates).

Let us compare these different approaches with a little example.

Example 2

The basic market notations:

Market date	1998-01-03
Delivery date	1998-03-18
Rate to delivery	5.55 %
Rate to coupon	5.80 %
Bond expire date	1999-01-21
Days to next coupon	18
Days to delivery date	75
Days between next coupon and delivery	57
Bond coupon	11 %

Bond market rate	6 % (we note bonds in rate)
Forward price	103.877
Clean price	115.380

Accrued interest at market date and delivery date should be calculated.

$$A_e = 1.742 = 11 \cdot \frac{57}{360} \tag{2.5}$$

We start with the first approximation (2.2) and insert the data:

$$\begin{aligned} R_{me} &= \frac{F_{me} + A_e - P_m - A_m + C_{me}}{\left[P_m + A_m\right]\frac{\Delta t_{me}}{360} - C_{me}\frac{\Delta t_{ce}}{360}} \\ &= \frac{103.877 + 1.742 - 115.380 + 11}{115.380\frac{75}{360} - 11\frac{57}{360}} \\ &= 5.5556\ \% \end{aligned} \tag{2.6}$$

And now we take the full expression (2.3) and do the same thing:

$$\begin{aligned} R_{me} &= \frac{360}{\Delta t_{me}} \frac{F_{me} + A_e - P_m - A_m + C_{me}\left(1 + \frac{\Delta t_{mc}}{360} R_{mc}\right)^{-1}}{P_m + A_m - C_{me}\left(1 + \frac{\Delta t_{mc}}{360} R_{mc}\right)^{-1}} \\ &= \frac{360}{75} \frac{103.877 + 1.742 - 115.380 + 11\left(1 + \frac{18}{360} 0.0580\right)^{-1}}{115.380 - 11\left(1 + \frac{18}{360} 0.0580\right)^{-1}} \\ &= 5.5481\ \% \end{aligned} \tag{2.7}$$

As one can see we have some differences between the two equations. The difference is not big but significant, whether this is “good enough” is up to the user. The fact that (2.7) is not exactly 5.55 % is that we round the included parts.

Let us continue with the futures.

3 *Introduction to futures*

3.1 A discussion

For the forward case we have a **known** deliverable bond. In most futures we have a collection of deliverable bonds in which the seller is free to pick anyone. Except for this collection we have the daily mark-to-market that gives us a daily settlement amount that is to be administrated for each trading day. This administrative task makes futures exchange traded instruments. Except for these differences the actual construction is very similar to that of forwards. We follow a case to get the feeling for it. In this set-up we have sold 100 future contract. We position us at expiration of the future.

Market date	:	1998-03-13
Delivery date	:	1998-03-18
Contract price	:	97.454
Last fix	:	97.465
Nominal amount	:	1 000 000 SEK

Deliverable bond A	Bond expire date	:	2006-10-25
1038	Bond coupon	:	6.500 %
	Days to next coupon	:	217
	Bond market rate	:	6.833 %
	Clean price	:	100.416
	Accrued interest	:	2.582
Deliverable bond B	Bond expire date	:	2008-05-05
1040	Bond coupon	:	6.500 %
	Days to next coupon	:	47
	Bond market rate	:	6.352 %
	Clean price	:	101.060
	Accrued interest	:	5.651
Deliverable bond C	Bond expire date	:	2009-04-20
1034	Bond coupon	:	9.000 %
	Days to next coupon	:	32
	Bond market rate	:	6.386 %
	Clean price	:	120.302
	Accrued interest	:	8.200

First of all the future should be marked-to-market as usual.

$$MM = \left(97.465 - 97.454\right) \cdot 1\,000 \cdot 100 = 1100 \tag{3.1}$$

This amount should be exchanged before delivery procedure takes over. After this point the future price for all positions is 97.465.

We start this discussion with minimum of details and assume that the future price is the whole price that we receive for bonds, the accrued interest is not included. Assume now also that we have to deliver a bond among the deliverable and hand over without adjusting for accrued interest. We now want to buy a bond at the spot market and deliver. The actual profit from receiving the futures fix and deliver the newly bought bond is:

$$\left.\begin{aligned} A&: \quad \left(97.465 - \left(100.416 + 2.582\right)\right) \cdot 1\,000 \cdot 100 = -553\ \ 300 \\ B&: \quad \left(97.465 - \left(101.065 + 5.651\right)\right) \cdot 1\,000 \cdot 100 = -924\ \ 600 \\ C&: \quad \left(97.465 - \left(120.302 + 8.200\right)\right) \cdot 1\,000 \cdot 100 = -3\ \ 103\ \ 700 \end{aligned}\right\} \tag{3.2}$$

Ops! We buy bond A and deliver. But wait a moment, if I am a very fast customer I could have bought that future at the fix price and since we have a net loss for all the deliverable bonds in (3.2), actually made an arbitrage. This should be impossible in a developed market!

What does that mean in practical terms? Well the future price in (3.2) should clearly converge to the dirty spot price for the cheapest deliverable bond. We also see that since different bonds have their coupon at different times the accrued interest is a problem. If we were to use the dirty price directly this would be taken care of, but since the spot market in most countries is denoted in clean price we chose to explicitly include the accrued interest. Let us add the accrued interest to the futures fix for each bond.

$$\left.\begin{aligned} A&: \quad \left(97.465 + 2.582 - \left(100.416 + 2.582\right)\right) \cdot 1\,000 \cdot 100 = -295\ \ 100 \\ B&: \quad \left(97.465 + 5.651 - \left(101.065 + 5.651\right)\right) \cdot 1\,000 \cdot 100 = -359\,500 \\ C&: \quad \left(97.465 + 8.200 - \left(120.302 + 8.200\right)\right) \cdot 1\,000 \cdot 100 = -2\,283\ \ 700 \end{aligned}\right\} \tag{3.3}$$

Why would we want to have this construction? Well one answer is that the deliverable bonds have about 10 years to expiry. This means that the clean price is more of a measure of the cash flow's to be expected and the accrued interest is almost an annoying part of the back office treatment of the bond.

If you have a lot of sold futures positions you must buy deliverable bonds to fulfil your obligations. The other market participants could then rise the price. This is commonly named squeezing. In (3.3) we see that the clean price of bond A could rise up to 101.065 before we would change bond to deliver.

This squeezing risk must be dealt with.

3.2 Introducing price factors

If we where to have only one deliverable bond there could easily be a situation where market actors where **forced** to buy this bond, this could lead to a situation with increasing prices. One way to deal with this is to increase the underlying amount of bonds or with a repo agreement from the government. The drawback of both these solutions is that the future would single out this bond and the bond market would probably treat this as "special". If one were to introduce futures for bonds with private issuers this repo solution would be hard to maintain. This would lead to different contract constructions, which also could be a disadvantage.

A simple solution is to permit delivery of several bonds. But the squeezing part still exists if the cost of delivery differs very much between them. This leads us to try to find a method of narrowing the distance between them, and this is why we introduce price factors. We want a factor that we can multiply with the futures fix price, in (3.3), to narrow the profit/loss from delivering different bonds.

These factors are calculated prior to the start of trading of the future. We can see in (3.3) that if we could multiply the fix with something that is near the bond price with a nominal amount of one, we would be quite satisfied. Therefore we **guess** the market interest rate for a bond with 10 (the same discussion can be used for each future length) years to expiry and a price of 100. The reason for choosing one here is that we prefer the futures price to be around 100. Do observe that we want this at the date of delivery for the future. We guess that the market rate should be around 6%.

This 6% is the market rate as well as the coupon for the bond that **theoretically** defines the future.

We assume that the reader is familiar with the rate-to-price formula for bonds:

$$P_{fi} = \frac{\frac{C_i}{r}\left(\left(1+\frac{r}{100}\right)^{n_i+1}-1\right)+1}{\left(1+\frac{r}{100}\right)^{n_i+m_i/12}} - 1\cdot C_i\left(1-\frac{m_i}{12}\right) \qquad (3.4)$$

n_i : Numbers of **whole** years to expiration of the bond, measured from the next receivable coupon standing at the futures date of delivery for bond *i*.

m_i : Number of **whole** month to next receivable coupon, measured from the futures delivery date for bond *i*.

C_i : The coupon for bond *i*.

r : The guessed theoretical rate

All the parameters in (3.4) is individual for each bond except the interest rate which we earlier assumed to be 6%.

As one can see is that price factors are just estimations of the clean price for individual bonds. When we use the approximation to round time to whole years and month, we have a good reason for this. To not being as accurate as possible when using (3.4) would lead to a situation at the start of the future when all the deliverable bonds where equally profitable to deliver (provided that 6% would be the market rate at that time). We would then have a situation where CTD could change very rapidly. Since this could inflict negatively at the trading we use some approximations to widen the distance between them.

3.3 The discussion continuos

Using this price factor method described in (3.4) and the individual bond data as the ongoing example give us:

$$\left.\begin{aligned} P_{fA} &= 1.032337 \\ P_{fB} &= 1.036880 \\ P_{fC} &= 1.237680 \end{aligned}\right\} \tag{3.5}$$

With these price factors we are prepared to take a second look at delivery. (Standard for these factors is to round them to six decimals.) Let us be a little bit stricter about what we receive from the futures contract.

$$\left(F_{Fix} P_{fi} + U_i\right) \cdot N \cdot n \tag{3.6}$$

F_{Fix} : The futures fix price

P_{fi} : The individual price factor for each deliverable bond.

N : Nominal amount

U_i : Accrued interest for each deliverable bond.

n : Number of contracts

P_i : Clean price for each deliverable bond.

The actual profit we get from delivering a special bond is given by:

$$\left(F_{Fix} P_{fi} + U_i - \left(P_i + U_i\right)\right) \cdot N \cdot n \tag{3.7}$$

Let us see what the equivalent to the expression in (3.3) would be with this approach.

$$\left.\begin{aligned} A&: \left(97.465 \cdot 1.032337 + 2.582 - \left(100.416 + 2.582\right)\right) \cdot 1\,000 \cdot 100 = 20\ \ 073 \\ B&: \left(97.465 \cdot 1.036880 + 5.651 - \left(101.065 + 5.651\right)\right) \cdot 1\,000 \cdot 100 = -49 \\ C&: \left(97.465 \cdot 1.237680 + 8.200 - \left(120.302 + 8.200\right)\right) \cdot 1\,000 \cdot 100 = 32\ \ 848 \end{aligned}\right\} \tag{3.8}$$

We see that the differences between the deliverable bonds are much less now with the price factors than without.

4 Cheapest to deliver

Now when we know the basic construction of the future we are standing with all the deliverable bonds and wondering which one of them I should buy at the same time as I sell the future. The reason for doing this investigation could be to determine if I should deliver the bond that already is in my possession or do I benefit from buying new ones and deliver. The process of selecting the optimal bond is called determining which one is “cheapest to deliver (CTD)”.

One thing that we do not discuss in the following sections is the fact that futures are mark-to-market on a daily basis. This means that we can not perform true arbitrages with bonds and futures due to the daily cash flows. Because of this we can not, without explanation, use today’s futures price in equations describing future events. If however the future price and the forward price equals we can make positions that **momentarily** can be regarded as an “arbitrage”. With a more strict formulation we have that the future price equals the forward price when we add the expected value of all the mark-to-markets for the period.

$$F_{fut} = \mathrm{E}\left[\sum M_i\right] + F_{for} \tag{4.1}$$

Do observe that the different M_i is the **discounted** mark-to-markets. The important point here is that if the futures price is correlated with the short rate we use for discounting, we can not say that the expected value in (4.1) equals zero. But the size of this part is, if measurable, very small and for this paper we consider equality[1].

$$\mathrm{E}\left[\sum M_i\right] = 0 \tag{4.2}$$

For the rest of this paper we assume equality and do not discuss this further.

4.1 Using the forward price

When we calculate the forward prices for these bonds we get different $F_{me,i}$. If these bonds where to be delivered, the profit expressed in (3.7) would be:

$$\left(F_{fut} P_{fi} - F_{me,i}\right) \cdot N \cdot n \tag{4.3}$$

Deciding which bond that maximises profit gives us the most favourable one. Since the nominal amount and the number of contracts are equal for all bonds we can exclude them in the equation.

$$Max\left[\left(F_{fut} P_{fi} - F_{me,i}\right) \cdot N \cdot n\right] \Rightarrow Max\left[F_{fut} P_{fi} - F_{me,i}\right] \tag{4.4}$$

So maximising profit is the same as finding the bond that maximises (4.4). Let us create an example.

Example 3

Market date		:	1998-01-03
Delivery date		:	1998-03-18
Future price		:	98.000
Nominal amount		:	1 000 000 SEK
Interest rate to delivery		:	4.5 %
Deliverable bond A 1038	Bond expire date	:	2006-10-25
	Bond coupon	:	6.500 %
	Bond market rate	:	6.750 %
	Clean price	:	98.347
	Market date accrued interest	:	1.2278
	Delivery date accrued interest	:	2.5819
	Price factor	:	1.032337
	Forward price	:	97.926
Deliverable bond B 1040	Bond expire date	:	2008-05-05
	Bond coupon	:	6.500 %
	Bond market rate	:	6.697 %
	Clean price	:	98.516
	Market date accrued interest	:	4.2972
	Delivery date accrued interest	:	5.6514

[1] Se for example "Derivatives" by F. D. Arditti page 161

	Price factor	:	1.036880
	Forward price	:	98.126
Deliverable bond C 1034	Bond expire date	:	2009-04-20
	Bond coupon	:	9.000 %
	Bond market rate	:	6.632 %
	Clean price	:	118.359
	Market date accrued interest	:	6.3250
	Delivery date accrued interest	:	8.200
	Price factor	:	1.237680
	Forward price	:	117.653

Deliverable bond A	$F_{fut} P_{fi} - F_{me,i}$	:	3.243026
Deliverable bond B	$F_{fut} P_{fi} - F_{me,i}$	:	3.488240
Deliverable bond C	$F_{fut} P_{fi} - F_{me,i}$	:	3.639640

From this table we se that the bond C is CTD.

One common way (apart from implied repo rate, which we will investigate later) is to calculate Implied futures price (IFP) for the deliverable bonds

Definition 1 : Implied futures price for a bond is the price that provides zero profit on the purchase, carry and delivery of a specified bond.

$$IFP_i \cdot P_{fi} - F_{me,i} = 0 \tag{4.5}$$

$$IFP_i = \frac{F_{me,i}}{P_{fi}} \tag{4.6}$$

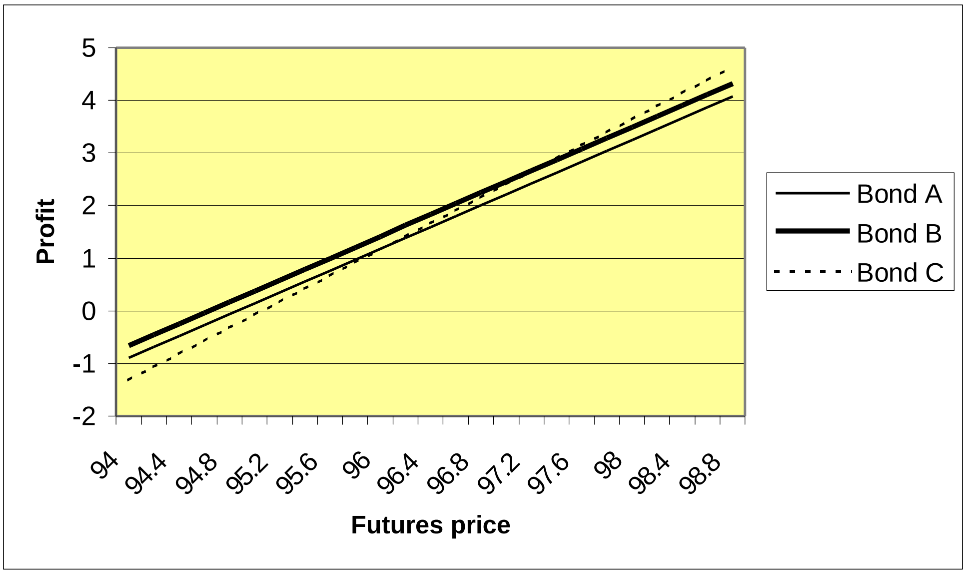
Here one says that the bond with the lowest IFP will be CTD, for at that futures price, any other bond will, upon delivery, provide a negative profit.

Let us discuss what we have calculated from a more theoretical point of view. Well (4.4) is really an equation which has two unknown, $F_{me,i}$ and F_{fut}. If we assume that the forward price can be constructed without much difficulty we are left with the futures price. Taken this as a variable we have (4.4) to be an ordinary linear equation. Two straight lines will intersect, as long as the price factors are not identical, somewhere in a two dimensional space. At that point the two bonds change places when talking about profit. IFP describes when the lines intersect the zero profit line. This means that depending on what future price we have the bond with the lowest IFP is not necessarily the CTD. An example will show the differences.

Use the same table as previous and calculate IFP: s.

Deliverable bond A	IFP_A	:	94.85856
Deliverable bond B	IFP_B	:	94.63583
Deliverable bond C	IFP_C	:	95.05930

This means that bond B should be CTD! Let us plot the individual profit that these bonds produce against futures price, under the assumption that forward prices are constant.



We see that Bond C crosses the other two bonds in the plot interval. This means that we have three intervals, which we must inspect separately:

$F_{fut} \leq 96.069$	CTD = Bond B
$96.069 < F_{fut} \leq 97.246$	CTD = Bond B
$97.246 \leq F_{fut}$	CTD = Bond C

We get problems with the last approach if the different IFP: s is situated in different intervals. If we are unlucky the calculated CTD could be the wrong one.

4.2 Using implied repo rate

According to most books on this topic the standard method for deducing "cheapest to deliver" is to look at the implied repo rate. Assume that you construct the forward and sell the future exactly as before. Now we ask ourselves the question what other part except for the future price can we consider to be a variable? Let us take the interest rate, the one we must pay to borrow the amount of money that we buy the bond for, to be a variable and all other part as explicitly given. We now inspect the profit function (4.3) as in the previous section:

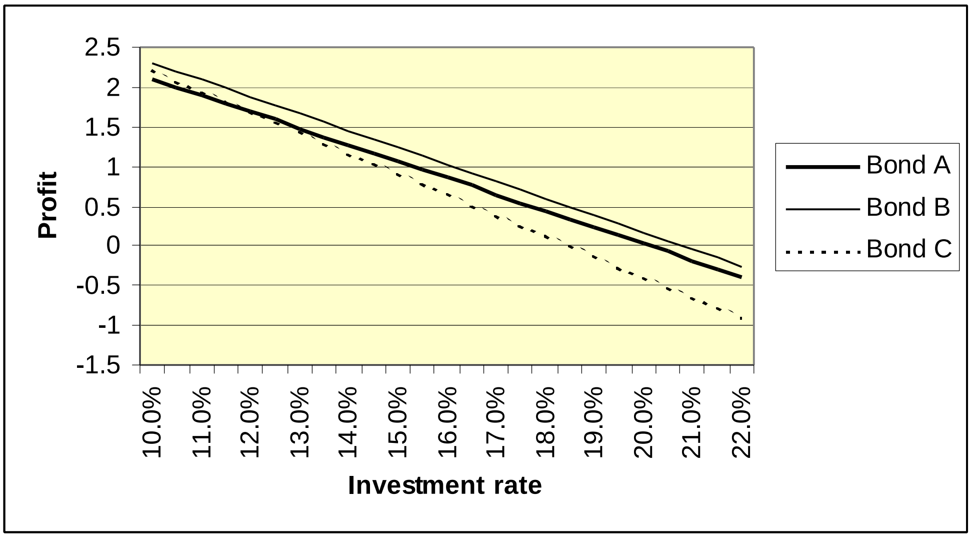
$$F_{fut} P_{fi} - \left[P_{m,i} + A_{m,i}\right]\left(1 + \frac{\Delta t_{me}}{360} R_{me}\right) - C_{me,i}\left(1 + \frac{\Delta t_{ce,i}}{360} R_{ce}\right) + A_{e,i} \qquad (4.7)$$

One very important point here is that the interest rate R_{ce} is taken to be constant even if we alter R_{me}. We continue with the same example as before and calculate this profit equation for Bond A.

$$\begin{aligned}
&F_{fut} P_{fi} - \left[P_{m,i} + A_{m,i}\right]\left(1 + \frac{\Delta t_{me}}{360} R_{me}\right) - C_{me,i}\left(1 + \frac{\Delta t_{ce,i}}{360} R_{ce}\right) + A_{e,i} \\
&F_{fut} P_{fi} - \left[P_{m,i} + A_{m,i}\right] - \left[P_{m,i} + A_{m,i}\right]\frac{\Delta t_{me}}{360} R_{me} - C_{me,i}\left(1 + \frac{\Delta t_{ce,i}}{360} R_{ce}\right) + A_{e,i} \\
&98.000 \cdot 1.032337 - \left[98.347 + 1.2278\right]\left(1 + \frac{75}{360} R_{me}\right) + 2.5819 \\
&4.176126 - 20.74475 R_{me}
\end{aligned} \qquad (4.8)$$

We can see that the derivative of this profit function with respect to R_{me} is strictly negative, which gives us a downward sloping curve. The steepnes of this graph depends on the dirty price of the bond.

To visualise this equation we directly make a plot of the profit versus R_{me} for each deliverable bond. We use same bonds as the previous example.



What can we say about this plot? If the interest rates we pay on the amount of money we borrowed for the bonds are very small, our profit increases. This is nothing extraordinary. One question is at what rate, for each deliverable bond, we get break even? As we see in the plot this is when the curve intersects the zero axes for profit. But these points are the definition of implied repo rate (IRR) for a future. This gives us these well known functions (the equivalent formulas for forwards has already been calculated in the beginning of this paper):

$$IRR_i = \frac{F_{fut} \cdot P_{fi} + A_{e,i} - P_{m,i} - A_{m,i} + C_{me,i}}{\left[P_{m,i} + A_{m,i}\right]\frac{\Delta t_{me}}{360} - C_{me,i}\frac{\Delta t_{ce,i}}{360}} \tag{4.9}$$

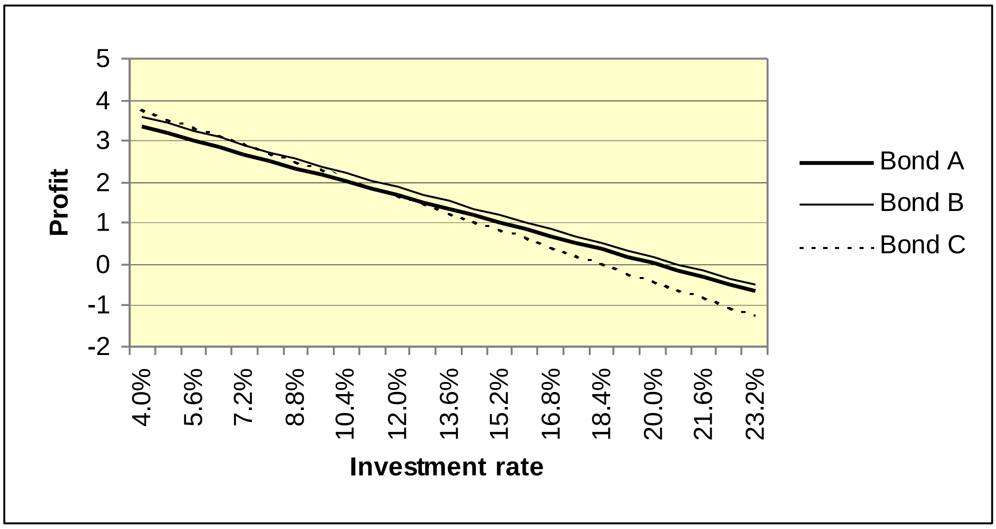
$$IRR_i = \frac{360}{\Delta t_{me}} \frac{F_{me,i} \cdot P_{fi} + A_{e,i} - P_{m,i} - A_{m,i} + C_{me,i}\left(1 + \frac{\Delta t_{mc,i}}{360} R_{mc,i}\right)^{-1}}{P_{m,i} + A_{m,i} - C_{me,i}\left(1 + \frac{\Delta t_{mc,i}}{360} R_{mc,i}\right)^{-1}} \tag{4.10}$$

$$IRR_i = \frac{360}{\Delta t_{me}} \frac{F_{e,i} \cdot P_{fi} + A_{e,i} - \left[P_{m,i} + A_{m,i}\right] + C_{me,i}\left(1 + \frac{\Delta t_{ce,i}}{360} R_{ce,i}\right)}{P_{m,i} + A_{m,i}} \tag{4.11}$$

In the first equation we have taken rates R_{ce} and R_{me} to be equal, this should at least be a rather good approximation. In the second equations we take R_{mc} to be explicitly given and in the third we take R_{ce} to be given.

In most books this is the way one decides which bond is CTD. The bond with the **highest** IRR is CTD. But even here I will argue against this! Let us discuss what the IRR: s really is.

If I buy a bond and sell a future I will earn a return on this position. The position which gives me the highest return (highest IRR) is clearly the most profitable for me. On the other hand I can sell the future and buy the bond in the forward. This position does not require any cash at the moment and return on the position is therefor undefined. So what is right and what is wrong and can the different approaches give us different result? Lets start by discuss the difference between the two approaches. We follow our example and look at the plot again.



We see that bond B clearly has the largest IRR.

Deliverable bond A	IRR_A	:	20.131 %
Deliverable bond B	IRR_B	:	20.787 %
Deliverable bond C	IRR_C	:	18.512 %

This means that buying B and selling the future gives us the largest return for the period from now to delivery. This does not in any way contradicts the profit function approach, since the profit function does not state **anything** for this period. The profit function says that bond C should be CTD.

Deliverable bond A	$F_{fut} P_{fi} - F_{me,i}$	:	3.243026
Deliverable bond B	$F_{fut} P_{fi} - F_{me,i}$	:	3.488240
Deliverable bond C	$F_{fut} P_{fi} - F_{me,i}$	:	3.639640

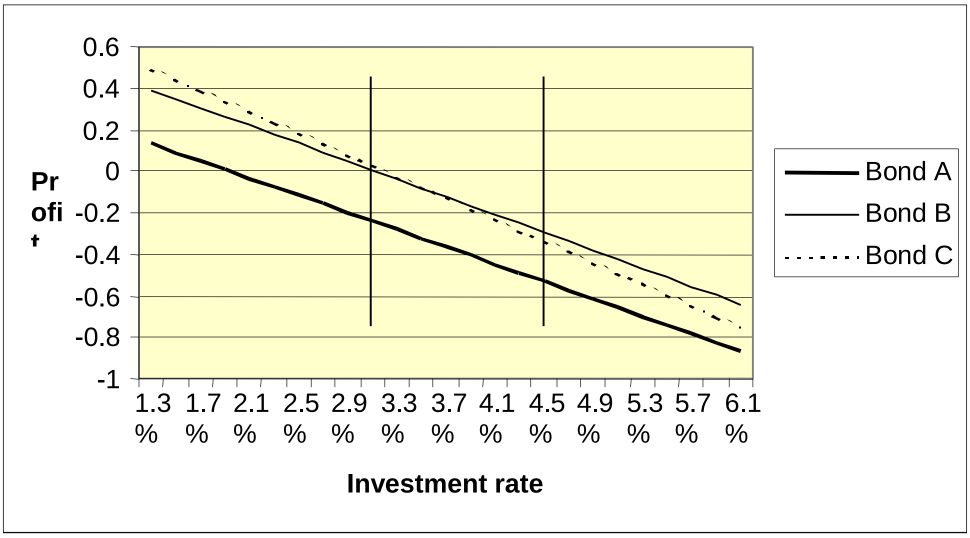
Let us assume that the futures price remains the same until expiry and that we invested money in buying bond B and selling the future. At delivery we find that we have earned the best return of this position compared with all other bonds. The interesting question is now how we will act when we are supposed to deliver a bond. Well the normal use of IRR tells us to deliver B and make a profit of 3.488240. But the profit is larger if we were to sell the bond B in the market (selling something at market price generates a zero profit) buying bond C and deliver. This would give us a profit of 3.639640!

This shows us that IRR does not necessarily pick the CTD. The profit function on the other hand always gives us the CTD. The remaining question is why and when those approaches differ.

First of all we can not have an IRR that is bigger then the actual investment rate! This is because if it was, “everybody” would do this basis trade and have (not a genuine arbitrage) a momentarily “arbitrage” and the prices would change. On the other hand the IRR is free to be as low as can bee since you can not make a reversed position and be certain about which bond will be delivered. (More about delivery option later) This means that the situation is like this:

$$\forall i \quad : \quad IRR_i \le R_{me} \tag{4.12}$$

In a plot we would have something like this:



The first vertical line is the highest IRR and the second line is the actual market rate. If we have an intersection of the lines, including the line with the highest IRR, we have the situation where IRR and the profit function give us different answers. So what conditions attribute to this? The steepness of the lines is of minor interest since all straight lines intersect (as long as they are not parallel). If the lines have approximately the same values around market rates we have a higher probability for intersection! What does this mean? If the profit function should be around the same value for two bonds we have:

$$F_{fut} P_{fi} - F_{me,i} \approx F_{fut} P_{fk} - F_{me,k} \tag{4.13}$$

This implies that the spacing between different bonds when it comes to CTD should be as narrow as possible. This is something that can happen for two reasons:

- Market rates are the same as the decided market rates for constructing price factors. Here we have per definition small differences between the bonds.
- The feature of the bonds is similar. Coupon rate and time to maturity. The closer they are, the smaller the difference to deliver.

Well, do we have any conclusions regarding this? Well IRR can be used in most circumstances but the profit function is the correct one to use for deciding CTD. One very interesting question arises when IRR and profit function points to different bonds. Which one should the future follow most? One usually says that the CTD is very closely correlated to the future when it comes to hedging and likewise! But basis trades should possibly be conducted to the one pointed out by the IRR! One answer to this could be that the closer we come to the expiry of the bond the more interested we are in CTD and less of basis trades!?

5 Hedging

5.1 Analytical approach to hedging CTD bond

To deduce a hedge factor for this instrument bring along some difficulties compared with other instruments. We have already seen that the future is a quite complicated instrument to price, which give us a clue that hedging it also could be troublesome.

The theoretical price of the future is a combination of the future nature and different optionally given by the contract specification. We do the same simplification as before in that the future effect contra forwards described earlier is excluded.

Since we will perform some differentiation’s we make two alterations to earlier notation. First we explicitly write the variables in parentheses. Secondly we change WLOG into continuously compounded interest rate, since this is easier to handle. The future price can be formulated as following:

$$F_{fut} = F_{fut}(t_m, t_e) \tag{5.1}$$

This gives us the variables for the future. We can divide the future price in two parts one for the CTD bond and one for the Delivery option.

$$F_{fut}(t_m, t_e) = F_{CTD}(t_m, t_e)\frac{1}{Pf_{CTD}} + opt(t_m, t_e) \tag{5.2}$$

Let us calculate the derivative, with respect to t_m, for this expression.

$$\frac{dF_{fut}(t_m, t_e)}{dt_m} = \frac{1}{Pf_{CTD}}\frac{dF_{CTD}(t_m, t_e)}{dt_m} + \frac{dopt(t_m, t_e)}{dt_m} \tag{5.3}$$

One large assumption here is that the delivery option is, in the short range, constant. This implies that the derivative of the option price is zero.

$$\frac{dopt(t_m, t_e)}{dt_m} = 0 \tag{5.4}$$

From earlier papers we have that the forward price for bond *i* is given by:

$$F_i(t_m, t_e) = \left[P_i(t_m) + A_i(t_m)\right] Exp \int_{t_m}^{t_e} r(v)dv - C_{mei} Exp \int_{t_c}^{t_e} r(v)dv - A_i(t_e) \tag{5.5}$$

We should be ashamed of ourselves since the problem with the C_{me} term has not been properly handled. The problem here is that C_{me} is not a continuous function, which implies that the derivative is not defined for all times. To solve this in a mathematically stringent way would complicate the discussion without introducing “anything interesting” from a financial viewpoint. We simply say, “We divide the time into two parts, before and after the coupon has been separated, and we can in this context consider C_{me} as a constant.” We could have the same discussion for the accrued interest, since $A_i(t_m)$ also follows a “step function”. But we skip this for the same reason as before. With these matters clarified we calculate the derivative of (5.5):

$$\frac{dF_i(t_m, t_e)}{dt_m} = -r(t_m)\left[P_i(t_m) + A_i(t_m)\right] Exp \int_{t_m}^{t_e} r(v)dv + \left(\frac{dP_i(t_m)}{dt_m} + \frac{dA_i(t_m)}{dt_m}\right) Exp \int_{t_m}^{t_e} r(v)dv \tag{5.6}$$

Now it is time to discuss what we mean by hedging. In this case we say that by selling *x* contracts of the future, we hedge our bond. We start by saying that we want to hedge the CTD bond. One annoying fact with futures and forwards is that even if we discuss prices, the actual **value** of the instrument is something else. The value of the future is zero at t_m. The value of the total position at t_m is:

$$P_{CTD}(t_m) + A_{CTD}(t_m) \tag{5.7}$$

Now we require that the position give us risk free interest.

$$\frac{d}{dt_m}\left[P_{CTD}(t_m) + A_{CTD}(t_m) - xF_{fut}(t_m, t_e)\right] = r(t_m)\left[P_{CTD}(t_m) + A_{CTD}(t_m)\right] \tag{5.8}$$

From now on we take (5.8) to be the interesting case. Now we perform some calculations:

$$\frac{dP_{CTD}(t_m)}{dt_m}+\frac{dA_{CTD}(t_m)}{dt_m}-x\frac{dF_{fut}(t_m,t_e)}{dt_m}=r(t_m)\left[P_{CTD}(t_m)+A_{CTD}(t_m)\right] \tag{5.9}$$

Combine (5.9), (5.3) and (5.6).

$$\frac{dP_{CTD}(t_m)}{dt_m}+\frac{dA_{CTD}(t_m)}{dt_m}-\frac{x}{Pf_{CTD}}\left\{-r(t_m)\left[P_{CTD}(t_m)+A_{CTD}(t_m)\right]Exp\left(\int_{t_m}^{t_e}r(v)dv\right)+\right.$$

$$\left.+\left(\frac{dP_{CTD}(t_m)}{dt_m}+\frac{dA_{CTD}(t_m)}{dt_m}\right)Exp\left(\int_{t_m}^{t_e}r(v)dv\right)\right\}=r(t_m)\left[P_{CTD}(t_m)+A_{CTD}(t_m)\right] \tag{5.10}$$

$$\frac{x}{Pf_{CTD}}Exp\left(\int_{t_m}^{t_e}r(v)dv\right)\left\{\frac{dP_{CTD}(t_m)}{dt_m}+\frac{dA_{CTD}(t_m)}{dt_m}-r(t_m)\left[P_{CTD}(t_m)+A_{CTD}(t_m)\right]\right\}=$$

$$=\frac{dP_{CTD}(t_m)}{dt_m}+\frac{dA_{CTD}(t_m)}{dt_m}-r(t_m)\left[P_{CTD}(t_m)+A_{CTD}(t_m)\right] \tag{5.11}$$

$$\frac{x}{Pf_{CTD}}Exp\left(\int_{t_m}^{t_e}r(v)dv\right)=1 \tag{5.12}$$

$$x=Pf_{CTD}Exp\left(-\int_{t_m}^{t_e}r(v)dv\right) \tag{5.13}$$

5.2 Analytical approach to hedging a non-CTD bond

This is the same as asking how we can hedge the CTD bond with some other bond. Let us construct a portfolio of these bonds, buy CTD and sell the other. Let us introduce $PD_i(t_m)$ as the dirty price for the bond to facilitate notation. The value of the total position at t_m is:

$$PD_{CTD}(t_m)-y_i PD_i(t_m) \tag{5.14}$$

To be analytically correct we demand that this portfolio yields risk free interest as the case for the CDT bond.

$$\frac{d\left[PD_{CTD}(t_m)-y_i PD_i(t_m)\right]}{dt}=r\left[PD_{CTD}(t_m)-y_i PD_i(t_m)\right] \tag{5.15}$$

It is easy to check that if this equation holds we can construct a new portfolio in the same manner as before. What we want to do now is of course to get a “easy” expression for y_i in (5.15).

$$y_i=\frac{dPD_{CTD}(t_m)-rPD_{CTD}(t_m)dt}{dPD_i(t_m)-rPD_i(t_m)dt} \tag{5.16}$$

When discussing hedging between different bonds we are in to the area of approximations since bonds are defined at different intervals of the yield curve. This leads to the fact that **perfect** hedges, between different bonds, never can be constructed. We therefor behave in a somewhat more careless fashion. In

(5.16) we conclude that movements for the bonds are much bigger in absolute numbers compared with the risk free earning part, which leads us to this approximation:

$$y_i \approx \frac{dPD_{CTD}(t_m)}{dPD_i(t_m)} \tag{5.17}$$

Since we promote simplicity in this paper we want (5.17) in a more recognisable shape. Most people that deal with interest rate products know their way around with modified duration. The definition of this is:

$$Dur_i(t_m) = -\frac{1}{\left(1+R_i\right)}\frac{d\left[PD_i(t_m)\right]}{dR_i(t_m)}\left[PD_i(t_m)\right]^{-1} \tag{5.18}$$

We indicate that we talking about the yearly compounded effective rate, which are an alternative notation for bonds, and not a continuous one by changing to upper case. We now assume that:

$$dR_i(t_m) = dR_k(t_m) \quad \forall\, i,k \tag{5.19}$$

This means that the rate notation increases and decreases with perfect correlation. This is what some people a little careless refer to as parallel shifts in interest rates. We actually say that the prices of the bonds, expressed in interest rate, moves with perfect correlation and by the same size. If this is true for all the bonds, we can facilitate (5.17) to:

$$y_i = \frac{dPD_{CTD}(t_m)}{dPD_i(t_m)} \approx \frac{d\left[PD_{CTD}(t_m)\right]/dR_{CTD}}{d\left[PD_i(t_m)\right]/dR_i} = \frac{PD_{CTD}(t_m)Dur_{CTD}(t_m)}{PD_i(t_m)Dur_i(t_m)} \tag{5.20}$$

This means that one unit of CTD bond in (4.3) can be approximated by *y* number of another one in the delivery basket. We have this portfolio:

$$P_{CTD} - xF_{fut} \rightarrow y_i P_i - xF_{fut} \rightarrow P_i - \frac{x}{y_i}F_{fut} \tag{5.21}$$

From this follows that (5.13) can be expressed like:

$$x_i = \frac{x}{y_i} = \frac{PD_i(t_m)Dur_i(t_m)}{PD_{CTD}(t_m)Dur_{CTD}(t_m)} Pf_{CTD} Exp\left(-\int_{t_m}^{t} r(v)dv\right) \tag{5.22}$$

Where x_i is the amount of futures you should buy to hedge a non-CTD bond with the future.

5.3 Numerical approach to hedging

As we can see from the previous sections, analytical hedging requires us to make some more ore less accurate simplification regarding the behaviour of the future. Whether or not this is “good enough” for us is a matter of opinion. There is other, more direct, ways to perform hedging. This includes a regression analysis of the future-bond relationship. In this way you can easily calculate the *x* value in (5.13). Or if you consider the future-CTD relation to be strong you can perform the regression between the different bonds and calculate *y* in (5.16). This is not meant to be an instruction in regression analysis but in short you take an appropriate number of observations of the future and the bond you are interested in (what appropriate number means is of course up to the user) and construct a numerical approximation of the derivative:

$$\Delta P_i(t_{mm}) = a\Delta F_{fut}(t) \tag{5.23}$$

Calculate a “best fit” of this. You can now use *a* as an approximation of *x*. Do observe that (5.23) constitute a very rudimentary regression function. One can of course extend the analysis with a more complex function incorporating more parameters.

5.4 Basis trades in the OM context

When talking about basis trades we mean selling the future and buying the bond (or reversed). In most cases we talk about the deliverable bonds (but there are of course nothing that stops us from using whichever bond we like). One important aspect of these trades is that they should be "risk neutral" (RN). What this means and why one should use this disgusting expression, is something that could be discussed for a long time. One suggestion is that RN is a perfect hedge, but this of course brings up the matter of when this hedge is perfect! You could think of the delivery moment, which implies that, an equal amount of futures and bonds should be the correct hedge. But one could also make the assumption that a perfect hedge should be constructed for the immediate market. This very much aligns to the previous discussion about hedging bonds with the future. The reason why RN is a poor expression is of course that if the definition of instantaneously hedge is chosen, which seams to be standard, nothing like a perfect hedge exist.

From now on we **define** a RN basis trade as a trade consisting of some amount of bought/sold future and an opposite position of **one** type of bond. The quote of the number of bonds and futures should be in the "neighbourhood" of the hedge ratio for the future and that bond.

Let us calculate an easy rule for determining if something is a basis trade. We turn to previous discussion of hedging. Equation (5.22) gives us a hedge approximation for different bonds. Take the trade quote of the basis trade, Bas_i , to be the same as the hedge ratio.

$$Bas_i = \frac{PD_i(t_m)Dur_i(t_m)}{PD_{CTD}(t_m)Dur_{CTD}(t_m)} Pf_{CTD} Exp\left(-\int_{t_m}^{\tau} r(v)dv\right) \tag{5.24}$$

We do of course have this discussion with some sort of supervision in mind. To check if something is a basis trade should be straightforward and easy. In (5.24) we have both duration and bond prices which somewhat complicate matters. With the fact that (5.24) itself is an approximation, we ask ourselves if we can make any further approximations, which would make life even easier?

The discounting factor in (5.24) is "very close" to one.

$$Exp\left(-\int_{t_m}^{\tau} r(v)dv\right) \approx 1 \tag{5.25}$$

The dirty price of the bonds are calculated at yields which are "relatively similar" which indicate that the relation between prices and price factors should be fairly similar:

$$\frac{PD_i(t_m)}{PD_{CTD}(t_m)} \approx \frac{Pf_i}{Pf_{CTD}} \tag{5.26}$$

So what have we left?

$$Bas_i = \frac{D_i(t_m)}{D_{CTD}(t_m)} Pf_i \tag{5.27}$$

We really have difficulties in assuming that the modified duration quote equals one since we have quite large differences in coupons and length of the bonds. But, however reluctant we are, we do this approximation and gets.

$$Bas_i = Pf_i \tag{5.28}$$

Which constitute the actual rules for basis trades.

5.5 Conclusion

In other countries the bonds coupon is very standard in the sense that the rate is similar for a very large amount of bonds. The time intervals for excepted bonds have generally a narrower interval than ours (do to a larger amount of bonds to choose from). Booth these differences causes the approximations (5.26) and (5.27) to be less accurate in the Swedish market then in many other markets. This implies that trading rules can not easily be compared with each other.

The conclusion is that because of these differences, OM should take a more liberal standpoint, if the trading rules would be an issue in the future, with basis trade rules then other exchanges to avoid problems for the market.

A very **easy** way to excuse the present rule is to say that we can not distinguish which bond are CTD which is the same as saying that the price factor is the best approximation for the hedge factor.