BOND FUTURES: DESCRIPTION AND PRICING

OPENGAMMA QUANTITATIVE RESEARCH

Abstract. The descriptions of standard bond futures in major currencies are provided. The standard pricing approach based on cheapest-to-deliver is described. A method taking into account the delivery option, based on a one-factor HJM model, is also described.

1. Introduction

Bond futures are exchange-traded instruments, with an underlying that is a basket of deliverable bonds. For most bond futures, the short party has the option to deliver any of the instruments in the basket. The exception are the Australian and New Zealand futures, where the settlement is in cash using an averaging mechanism on the basket.

The basket is composed of government bonds from a unique issuer (country) with rules on remaining maturity, initial maturity and issue size to be eligible.

The bond futures are traded on different underlyings on different exchanges. In general, there are several maturity buckets for each country. A list is given in Table 1.

Underlying country	Currency	Exchange	Number of contracts
United States	USD	CBOT	5
Germany	EUR	Eurex	4
United Kingdom	GBP	Liffe	3
Japan	JPY	TSE	3
Italy	EUR	Eurex	2
Japan	JPY	LIFFE	1
Canada	CAD	MSE	3
Switzerland	CHF	Eurex	1
Australia	AUD	ASX	2
New-Zealand	NZD	ASX	2

Table 1. Main bond futures overview.

For all futures except Australian and New Zealand, the bonds in the basket are transformed to be comparable through a conversion factor mechanism. The factor is such that in a certain reference yield environment all the bonds have the same price. The reference yield acts somewhat like a strike for the delivery process.

Australian and New Zealand futures are settled in cash against a standardised bond. The standardized bond yield is computed as the average of actual bond yields for AUD and as a linear interpolation of actual bond yields for NZD.

There are other embedded options for some currencies. Some of those options are:

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Timing option: The delivery notice can be given in a period of time, not only on one date. This gives an American option flavour to the futures.

Wild card option: The underlying bonds can be selected after the price of the future has been fixed. During the delivery period, there is a daily option between the end of future trading at 2 p.m. and the end of bond trading at 6 p.m.. After the last trade, there can be a period (up to seven days) where the future price is fixed but the delivery notice has not yet been given.

2. Bond Futures

The texts in italic are quotes from the exchanges.

2.1. **USD.**

In USD, the futures are traded on the Chicago Board of Trade (CBOT)[1].

The description of the price used for delivery is: *The invoice price equals the futures settlement price times a conversion factor, plus accrued interest. The conversion factor is the price of the delivered bond (USD 1 par value) to yield 6 percent.*

The conversion factors are provided by the exchange and the values do not change through the life of the future.

Note that the last trading day and last delivery date are not the same for the different underlyings. The delivery take place one days after notice is given.

2.1.1. *Ultra T-Bond Futures.*

The Ultra T-Bond Futures, U.S. Treasury Bond Futures and 10-Year U.S. Treasury Note Futures have the same last trading date and last delivery day. The last trading day is the *Seventh business day preceding the last business day of the delivery month. Trading in expiring contracts closes at 12:01 p.m. on the last trading day.* The last Delivery Day is the *Last business day of the delivery month.*

Previously, U.S. Treasury Bond futures referred to all bonds with maturities above 15 years. That range has recently (March 2011) been divided into two different futures.

The underlying of the Ultra T-Bond Futures are *U.S. Treasury bonds with remaining term to maturity of not less than 25 years from the first day of the futures contract delivery month.*

2.1.2. *U.S. Treasury Bond Futures.*

Formerly called the 30 years future, the deliverable grade for T-Bond futures are *bonds with remaining maturity of at least 15 years, but less than 25 years, from the first day of the delivery month.*

The Treasury Bond futures are less liquid than 10 and 5 years note futures (see Table 2.

2.1.3. *10-Year U.S. Treasury Note Futures.*

U.S. Treasury notes with a remaining term to maturity of at least six and a half years, but not more than 10 years, from the first day of the delivery month.

2.1.4. *5-Year U.S. Treasury Note Futures.*

The last Trading Day is *Last business day of the calendar month.* The last Delivery Day is the *third business day following the last trading day.*

The eligible bonds are *U.S. Treasury notes with an original term to maturity of not more than five years and three months and a remaining term to maturity of not less than four years and two months as of the first day of the delivery month.*

2.1.5. *2-Year U.S. Treasury Note Futures.*

The eligible bonds are *U.S. Treasury notes with an original term to maturity of not more than five years and three months and a remaining term to maturity of not less than one year and nine months from the first day of the delivery month and a remaining term to maturity of not more than two years from the last day of the delivery month.*

[1] Part of CME group; www.cmegroup.com

Contract	Maturity	Nominal	Reference yield	Codes	Volume
Ultra T-Bond	> 25Y	100,000	6.00%	UB/UL/UBE	1,698,758
30-YR Bond	15Y to 25Y	100,000	6.00%	ZB/US/	9,231,332
10-YR Note	6.5Y to 10Y	100,000	6.00%	ZN/TY/	31,149,373
5-YR Note	4Y2M to 5Y3M	100,000	6.00%	ZF/FV/	17,858,805
2-YR Note	1Y9M to 2Y	100,000	6.00%	ZT/TU/	1,116,105

The codes are for CME Globex (Electronic Platform)/Open Outcry (Trading Floor)/Clearing Code. The volume is the monthly volume for May 2011.

Table 2. USD bond futures

2.2. **EUR.** In EUR, bond futures are traded on Eurex[2].

A delivery obligation arising out of a short position may only be fulfilled by the delivery of certain debt securities issued by the Federal Republic of Germany with a remaining term on the Delivery Day within the remaining term of the underlying. To be eligible, the debt securities must have a minimum issue amount of EUR 5 billion.

The delivery day is *the tenth calendar day of the respective quarterly month, if this day is an exchange day; otherwise, the exchange day immediately succeeding that day.* The last trading day is *two exchange days prior to the Delivery Day of the relevant maturity month.*

The maturity ranges for the eligible bonds are given in Table 3. The futures names are: Euro-Buxl®Futures, Euro-Bund Futures, Euro-Bobl Futures, and Euro-Schatz Futures.

Note that the reference yield for the Euro-Buxl, which is more recent than the others, is 4% (and not 6%, as for the majority of futures).

Contract	Maturity	Nominal	Reference yield	Code	Volume
Euro-Buxl	24Y to 35Y	100,000	4.00%	FGBX/UB	222,821
Euro-Bund	8.5Y to 10.5Y	100,000	6.00%	FGBL/RX	11,778,488
Euro-Bobl	4.5Y to 5.5Y	100,000	6.00%	FGBM/OE	7,252,498
Euro-Schatz	1.75Y to 2.25Y	100,000	6.00%	FGBS/DU	8,659,722

The volume is the monthly volume for December 2011.

Table 3. EUR bond futures

2.3. **GBP.** Futures are traded on Liffe[3]

First Notice Day: Two business days prior to the first day of the delivery month

Last Notice Day: First business day after the Last Trading Day

Last Trading Day: Two business days prior to the last business day of the delivery month

Delivery Day: Any business day in delivery month (at sellers choice)

Deliverable bonds subject to a coupon range of 3.00% around the reference yield.

2.4. **JPY.** Futures are traded on TSE[4].

The notional is JPY 100,000,000. The final settlement day is the 20th of each contract month

Last trading: 7th business day prior to each delivery date. Trading for the new contract month begins on the business day following the last trading day.

2.5. **AUD.** Futures are traded on ASX[5].

[2] www.eurexchange.com

[3] Part of NYSE Euronext; www.euronext.com

[4] www.tse.or.jp

[5] www.asx.com.au

Contract	Maturity	Nominal	Reference yield	Code	Volume
Long Gilt Futures	8Y9M to 13Y	100,000	6.00% / 4.00 %	G␣	476,025
Medium Gilt Futures	4Y to 6Y3M	100,000	6.00% / 4.00%	WX	183
Short Gilt Futures	1Y6M to 3Y3M	100,000	6.00% / 3.00%	WB	1,131

The volume is the monthly volume for December 2010. The change of coupon from 6% to a lower coupon will take place with the December 2011 contract.

Table 4. GBP bond futures

Contract	Maturity	Nominal	Reference yield	Code	Volume
20-year JGB Futures	15Y to 21Y	100,000,000	6.00%	?	?
10-year JGB Futures	7Y to 10Y	100,000,000	6.00%	JB	?
5-year JGB Futures	4.0Y to 5.25Y	100,000,000	3.00%	JJ	?

TSE has decided to halt trading on new contract months for 20-year JGB futures beginning with the December 2002 contract.

Table 5. JPY bond futures

3. Preliminaries (non-AUD/NZD)

The price at t of a zero coupon bond with maturity u is denoted $P(t,u)$.

Suppose there are N bonds in the basket. Each of them ($1 \leq i \leq N$) has n_i coupons after the common delivery date t_0; the cash flows amount are $c_{i,j}$ and are paid on $t_{i,j}$. Let $\text{AccruedInterest}_i(t)$ denote the accrued interest of bond i for delivery date t. The conversion factor associated with each bond is denoted K_i. The bond future notice takes place on $\theta \leq t_0$. The time t *futures price*[6] is denoted by F_t. At delivery, the short party can choose the bond to be delivered (i) and receives the amount $F_\theta.K_i + \text{AccruedInterest}_{i^*}(t_0)$ on the delivery date.

Let i^* be the index of the cheapest-to-deliver bond at maturity. The future price at maturity will reflect the delivery option of the short party, with

$$\text{DirtyPrice}_{i^*,\theta}(t_0) = \sum_{j=1}^{n_{i^*}} c_{i^*,j}\frac{P(\theta,t_{i^*,j})}{P(\theta,t_0)} = F_\theta.K_{i^*} + \text{AccruedInterest}_{i^*}(t_0).$$

The other bonds are more expensive and for all $1 \leq i \leq N$.

$$\sum_{j=1}^{n_i} c_{i,j}\frac{P(\theta,t_{i,j})}{P(\theta,t_0)} \geq F_\theta.K_i + \text{AccruedInterest}_i(t_0).$$

The equality and the inequalities are summarised by

$$\max_{1\leq i\leq N}\left(F_\theta.K_i + A_i - \sum_{j=1}^{n_i} c_{i,j}\frac{P(\theta,t_{i,j})}{P(\theta,t_0)}\right) = 0.$$

This is equivalent to

$$F_\theta = \min_{1\leq i\leq N}\left(\sum_{j=1}^{n_i}\frac{c_{i,j}}{K_i}\frac{P(\theta,t_{i,j})}{P(\theta,t_0)} - \frac{\text{AccruedInterest}_i(t_0)}{K_i}\right).$$

[6] The term *price* is the standard jargon for futures, but it would be more correct to speak of *number* or *reference index*. The future price is never actually paid. It is only a reference number for subsequent payment computation. The price could be shifted by an arbitrary amount without impact on the economy. Also, the futures price can be *negative* in very exceptional circumstances.

4. Pricing with cheapest-to-deliver (non AUD/NZD)

In this section, the futures are priced as if they were forwards. The cheapest forward is used as future price.

4.1. **Price.** Let $\text{DirtyPrice}_{i,t}(t_0)$ be the forward dirty price on t of the bond i for delivery on t_0. The future price is computed as

$$F_t = \min_{1\leq i\leq N} \frac{1}{K_i}\left(\text{DirtyPrice}_{i,t}(t_0) - \text{AccruedInterest}_i(t_0)\right).$$

4.2. **Gross basis.** The gross basis is the difference between the bond price and the future adjusted by the conversion factor.

$$\text{GrossBasis}_{i,t} = \text{CleanPrice}_{i,t}(\text{Spot}(t)) - F_t K_i).$$

Note that the bond price is the spot delivery clean price but the future price is for delivery according to the future description.

Note also that the gross basis is usually quoted in 32nds of percent (i.e. a screen figure like 20.0 means $20.0/32/100 = 0.00625$).

4.3. **Net basis.** The net basis for a bond in the delivery basket is

$$\text{NetBasis}_{i,t} = \text{DirtyPrice}_{i,t}(t_0) - (F_t K_i + \text{AccruedInterest}_i(t_0)).$$

Note that the net basis requires a forward bond price (delivery in t_0) and thus can not be computed from the quoted spot price.

4.4. **Price from net basis.** Given a net basis, one can compute the *net basis adjusted cheapest-to-deliver future price*. It is given by

$$F_t = \min_{1\leq i\leq N} \frac{1}{K_i}\left(\text{DirtyPrice}_{i,t}(t_0) - \text{AccruedInterest}_i(t_0) - \text{NetBasis}\right).$$

5. Pricing with delivery option (non AUD/NZD)

5.1. **Theoretical description.** The algorithm described here is in a separable one factor HJM model and is from Henrard [2006].

Let $\sigma(t,s)$ be the Hull-White one factor short rate volatility. The bond volatility is denoted

$$\nu(t,u) = \int_t^u \sigma(t,s)ds.$$

Let $N_t = \exp(\int_0^t r_s ds)$ be the cash-account numeraire with the short rate $(r_s)_{0\leq s\leq T}$ given by $r_t = f(t,t)$. The equations of the model in the numeraire measure associated to N_t are

$$df(t,u) = \sigma(t,u)\nu(t,u)dt + \sigma(t,u)dW_t.$$

As in the case of swaptions, (see Henrard [2003]), a separability condition is used to obtain explicit results. The condition is similar to the condition to have Markovian short rates obtained by Carverhill [1994].

H: The function σ satisfies $\sigma(t,u) = g(t)h(u)$ for some positive function g and h.

The condition is satisfied by the Hull-White one factor (extended Vasicek) model.

The discount factor ratio in the futures price can be written as

$$\frac{P(\theta,t_{i,j})}{P(\theta,t_0)} = \frac{P(t,t_{i,j})}{P(t,t_0)}\beta_{i,j}\exp(-\alpha_{i,j}^2 - \alpha_{i,j}X)$$

with

$$\beta_{i,j} = \beta_{i,j}(t,\theta) = \exp\left(-\int_t^\theta \nu(s,t_0)(\nu(s,t_{i,j}) - \nu(s,t_0))ds\right),$$

$$\alpha_{i,j}^2 = \alpha_{i,j}^2(t,\theta) = \int_t^\theta (\nu(s,t_{i,j}) - \nu(s,t_0))^2 ds,$$

and X a standard normal random variable. The separability condition (H) is used to prove that the same variable X can be used for all bonds and coupons.

Using the notation

$$d_{i,j} = d_{i,j}(t,\theta) = \frac{c_{i,j}}{K_i}\beta_{i,j},$$

$e_i = \text{AccruedInterest}_i(t_0)/K_i$, $t_{i,0} = t_0$ and $d_{i,0}(t,\theta) = -e_i$, the future price in θ is

$$F_\theta = \min_{1\leq i\leq N}\left(\sum_{j=0}^{n_i} d_{i,j}(t,\theta)\frac{P(t,t_{i,j})}{P(t,t_0)}\exp(-\frac{1}{2}\alpha_{i,j}^2(t,\theta) - \alpha_{i,j}(t,\theta)X)\right).$$

The functions in the minimum are denoted

$$f_i(x) = f_i(x;t,\theta)) = \sum_{j=0}^{n_i} d_{i,j}\frac{P(t,t_{i,j})}{P(t,t_0)}\exp(-\frac{1}{2}\alpha_{i,j}^2 - \alpha_{i,j}x).$$

The pricing can be done with an (semi-)explicit formula. There is no explicit description of the crossing points between the different bonds. However, suppose for the moment that those points are known.

Let $-\infty = \kappa_0 < \kappa_1 < \cdots < \kappa_{k-1} < \kappa_k = +\infty$ be the ends of the intervals on which one bond is the cheapest-to-deliver. On the interval $[\kappa_{i-1},\kappa_i]$ $(1 \leq i \leq k)$, the bond m_i is the cheapest-to-deliver:

$$\min_{1\leq l\leq N}(f_l(x)) = f_{m_i}(x) \quad \text{for} \quad x \in [\kappa_{i-1},\kappa_i].$$

The ends satisfy the equations

$$f_{m_i}(\kappa_i) = f_{m_{i+1}}(\kappa_i) \quad (1 \leq i \leq k-1).$$

The κ_i are $\mathcal{F}_t$-measurable random variable.

Theorem 1. *In the HJM one-factor model with volatility satisfying the separability condition (H), the price of the bond futures is given by*

$$F_t = \frac{1}{P(t,t_0)}\sum_{i=1}^{k}\sum_{j=0}^{n_{m_i}} d_{m_i,j}P(t,t_{m_i,j})(N(\kappa_i + \alpha_{m_i,j}) - N(\kappa_{i-1} + \alpha_{m_i,j})). \tag{1}$$

where $d_{m_i,j}$ and $\alpha_{m_i,j}$ are to be taken in (t,θ) and κ_i, k and m_i are the $\mathcal{F}_t$-measurable functions described above.

In a typical bond future, the number of bonds entering into the valuation formula (k) is two to five, even if the basket is larger. There are one to four non-trivial κ's to estimate. Once the estimation of the interval ends is done, the pricing of the futures is similar to the one of a swaption. To estimate those ends, a numerical estimate of the potential CTD bonds is done through a procedure similar to a numerical integration. This can be done with few points (like 100). The goal is not to find the intersection points precisely but only to find their *existence*. Once the bonds (m_i) and a rough estimate of interval ends are available, a numerical solution of the intersection between two curves is done. The equation to solve is $f_{m_i}(\kappa) = f_{m_{i+1}}(\kappa)$. This can be done quite efficiently as the functions f_m are simple sums of exponentials. Those numerically estimated κ's are used to feed Theorem 1.

5.2. **Numerical example.** The difference in curve sensitivity between the cheapest-to-deliver and the delivery option approaches is depicted in Figure 1. The graph is for the December 2011 Short Gilt Futures (WBZ1) contract, valued in August 2011. Note that for that future, the reference rate is 3%. The old style future (with a reference rate of 6%) is also displayed.

The curves used are flat with rates between 1 and 7% (continuously compounded). The difference between the standard cheapest-to-deliver (dotted lines) and delivery option (plain curves) approaches is obvious in the 2 to 4% region for the 3% reference rate future and in the 5 to 7% region for the 6% reference rate. At the time of writing (August 2011), the UK Government rates are at about 1% (left of the graph).

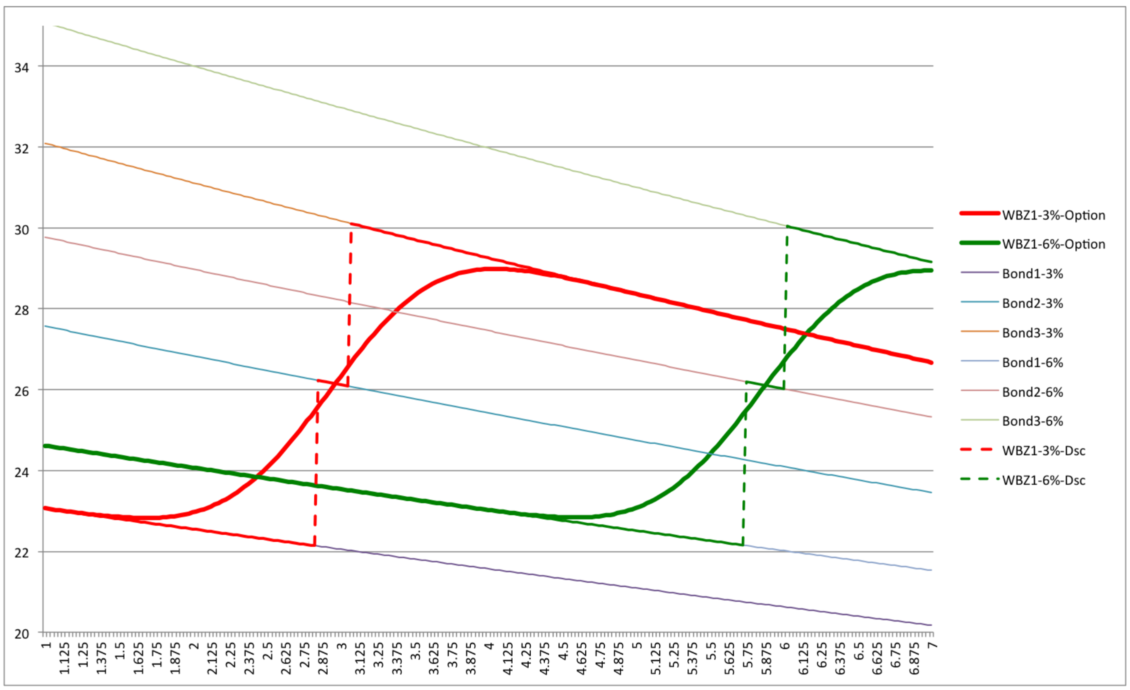

FIGURE 1. Curve sensitivity with cheapest-to-deliver (discounting) and delivery option pricing. Result for the new nominal rate (3%) and the old (6%).

6. PRELIMINARIES (AUD)

Suppose there are N bonds underlying the future. Each of them ($1 \leq i \leq N$) has n_i coupons after the delivery date t_0; the cash flows amount are $c_{i,j}$ and are paid in $t_{i,j}$. As the bonds are standard bonds, $c_{i,j} = C_i$ ($1 \leq j \leq n_i$) and $c_{i,n_i} = 1 + C_i$.

The price in t of a zero-coupon bond with maturity u is denoted $P(t, u)$. The time t *futures price* is denoted by Φ_t.

All the margining payments related to SFE bond futures are done according to a reference bond price R_t computed from the future index in the following way. Let $m = 6$ for the three year futures and $m = 20$ for the 10 year futures.

$$
\begin{aligned}
Y_t &= 1 - \Phi_t \\
v_t &= \frac{1}{1 + Y_t/2} \qquad (2) \\
R_t &= 3\frac{1 - v_t^m}{Y_t/2} + v_t^m. \qquad (3)
\end{aligned}
$$

In practice the reference price is multiplied by the notional, which is AUD 100,000 by contract.

The formula for R may seem artificial. It is simply the value of a semi-annual three (or ten) year bond with a C=6% coupon at a semi-annual yield of Y_t. The value is

$$\sum_{i=1}^{m} C/2\frac{1}{(1+Y/2)^i} + \frac{1}{(1+Y/2)^m} = C/2\frac{v-v^{m+1}}{1-v} + v^m = C/2\frac{1-v^m}{Y/2} + v^m.$$

The contract settles in cash. The settlement is done versus the average of the yield of the underlying bonds. Let $Y_{i,\theta}$ $(1 \leq i \leq N)$ be the yields on the fixing date for the underlying bonds. The reference yield for the settlement is

$$Y_\theta = \frac{1}{N}\sum_{i=1}^{N} Y_{i,\theta}. \tag{4}$$

From the yield the final future index and equivalent bond price are computed as above.

The future equivalent bond price on which the future margining is done is given by (see Hunt and Kennedy [2004] for the generic pricing formula)

$$R_0 = \mathrm{E}^0\left[R_\theta|\mathcal{F}_0\right] \tag{5}$$

with R_θ linked to the yield curve through (2), (3), (4) and $Y_{i,\theta}$. The $Y_{i,\theta}$ are the conventional yields of the underlying bonds, i.e.

$$B_{i,t} = {v_{i,t}}^{q_i}\left(c_i/2\frac{1-v_{i,t}^{N_i}}{Y_{i,t}/2} + {v_{i,t}}^{N_i}\right) \quad \text{and} \quad v_{i,t} = \frac{1}{1+Y_{i,t}/2}. \tag{6}$$

The B_i are the bond prices, i.e.

$$B_{i,t} = \sum_{j=1}^{n_i} c_{i,j} P(t, t_{i,j}).$$

7. Implementation

The bond future security (non-AUD and NZD) is described in the class `BondFutureSecurity` and the transaction in `BondFutureTransaction`.

The pricing without delivery option (discounting of the cheapest-to-deliver) is in the method `BondFutureTransactionDiscountingMethod`. The pricing with delivery option in the Hull-White one factor model is in the class `BondFutureTransactionHullWhiteMethod`.

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