INTRODUCTION TO BLACK'S MODEL FOR INTEREST RATE DERIVATIVES

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Contents

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1. Introduction

We consider the Black Model for futures/forwards which is the market standard for quoting prices (via implied volatilities). Black [1976] considered the problem of writing options on commodity futures and this was the first "natural" extension of the Black-Scholes model. This model also is used to price options on interest rates and interest rate sensitive instruments such as bonds. Since the Black-Scholes analysis assumes constant (or deterministic) interest rates, and so forward interest rates are realised, it is difficult initially to see how this model applies to interest rate dependent derivatives.

However, if f is a forward interest rate, it can be shown that it is consistent to assume that

- The discounting process can be taken to be the existing yield curve.
- The forward rates are stochastic and log-normally distributed.

The forward rates will be log-normally distributed in what is called the T-forward measure, where T is the pay date of the option. This model is consistent is within the domain of the LIBOR market model. We can proceed to use Black's model without knowing any of the theory of the LMM; however, Black's model cannot safely be used to value more complicated products where the payoff depends on observations at multiple dates.

2. European Bond Options

The clean (quoted) price for a bond is related to the all-in (dirty, cash) price via:

$$\mathbb{A} = \mathbb{C} + \mathbb{I}_A(t) \tag{1}$$

where the accrued interest $\mathbb{I}_A(t)$ is the accrued interest as of date t, and is non-zero between coupon dates. The forward price is a carried all-in price, not a clean price. The option strike price $\mathbb{K}$ might be a clean or all-in strike; usually it is clean. If so, we change it to a all-in price by replacing $\mathbb{K}$ with $\mathbb{K}_\mathbb{A} = \mathbb{K} + \mathbb{I}_\mathbb{A}(T)$.

Applying Black's model to the price of the bond, the value of the bond option per unit of nominal is [Hull, 2005, §26.2]

$$V_\eta = \eta Z(0,T)[\mathbb{F}_\mathbb{A} N(\eta d_1) - \mathbb{K}_\mathbb{A} N(\eta d_2)] \tag{2}$$

$$d_{1,2} = \frac{\ln \frac{\mathbb{F}_\mathbb{A}}{\mathbb{K}_\mathbb{A}} \pm \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}} \tag{3}$$

where $\eta = 1$ stands for a call, $\eta = -1$ for a put. Here σ is the volatility measure of the fair forward all in price, and τ as usual is the term of the option in years with the relevant day-count convention applied.

Example 2.1. Consider a $10m$ European call option on a $1,000,000$ bond with 9.75 years to maturity. Suppose the coupon is 10% NACS. The clean price is 935,000 and the clean strike price is 1,000,000. We have the following yield curve information:

Term	Rate
3m	9%
9m	9.5%
10m	10%

The 10 month volatility on the bond price is 9%.
Firstly, $\mathbb{K}_{\mathbb{C}} = 1{,}000{,}000$ so $\mathbb{K}_{\mathbb{A}} = \mathbb{K}_{\mathbb{C}} + \mathbb{I}_{\mathbb{A}}(T) = 1{,}008{,}333.33$. (There will be one month of accrued interest in 10 months time.)
Secondly, $\mathbb{A} = \mathbb{C} + \mathbb{I}_{\mathbb{A}}(0) = 960{,}000$. (There is currently three months of accrued interest.)
Thirdly, $\mathbb{F}_{\mathbb{A}} = \left[960{,}000 - 50{,}000\left[e^{-\frac{3}{12}9\%} + e^{-\frac{9}{12}9.5\%}\right]\right] e^{\frac{10}{12}10\%} = 939{,}683.97$.
Hence $V_c = 7{,}968.60$ and $V_p = 71{,}129.06$.

2.1. **Different volatility measures.** The volatility above is a price volatility measure. However, quoted volatilities are often yield volatility measures. The relationship between the various volatilities of the bond is given via Itô's lemma as

$$\sigma_{\mathbb{A}} = -\frac{\sigma_y y \Delta}{\mathbb{A}} \tag{4}$$

$$\sigma_y = -\frac{\sigma_{\mathbb{A}} \mathbb{A}}{y\Delta} \tag{5}$$

$$\sigma_{\mathbb{A}} = \frac{\mathbb{C}}{\mathbb{A}} \sigma_{\mathbb{C}} \tag{6}$$

How do we see this? Note that

$$dy = \mu y \, dt + \sigma_y y \, dZ$$

is the geometric Brownian motion for the yield y. Now $\mathbb{A} = f(y)$ and so

$$d\mathbb{A} = \cdots \, dt + f'(y)\sigma_y y \, dZ := \cdots \, dt + \frac{\Delta \sigma_y y}{\mathbb{A}} \mathbb{A} \, dZ$$

But, also, a different Black model gives

$$d\mathbb{A} = \nu \mathbb{A} \, dt + \sigma_{\mathbb{A}} \mathbb{A} \, dZ$$

and so the result follows - except for a missing minus sign. Why? Note, the result is an approximation: the two models are not consistent with each other.

3. Caplets and Floorlets

See [Hull, 2005, §26.3]. Suppose that a market participant loses money if the floating rate falls. For example, they are long a FRA. The floorlet pays off if the floating rate decreases below a predetermined minimum value, which is of course called the strike. Thus, the floorlet 'tops-up' the payment received in the FRA, if required, so that the net rate received is at least the strike. The floating rate will be the prevailing 3-month LIBOR rate, which is set at the beginning of each period, with settlement at that time by discounting, much like a FRA. Indeed, it is a FRA position that this floorlet is hedging; the FRA date schedule is being applied.
Let the floorlet rate (strike) be L_K.
Let the $T_0 \times T_1$ FRA period be of length α as usual, where day count conventions are observed. Suppose the LIBOR rate for the period, observed at the determination date is L_1. Then the payoff at the determination date is $\frac{1}{1+L_1\alpha}\alpha \max(L_K - L_1, 0)$ per unit notional; a floorlet is like a put on the interest rate. Of course the valuation is the same whether settled in advance or arrears.
In an analogous fashion to a floorlet, we have a caplet, where the payout occurs when the floating rate rises above L_K, the cap rate.

4. Caps and Floors

A cap is like a strip of caplets which will be used to hedge a swap. However, because swaps are settled in arrears so too is each payment in the cap strip, unlike an individual caplet. Moreover, the cap strip will have the swap date schedule applied and not the FRA date schedule. So, caps stand to swaps exactly as caplets stand to FRAs.

A cap might be forward starting or spot starting (that is, starting immediately). However, in the latter case, there is no payment in 3 months time - it is excluded from the computations and from any payments because there is no optionality. Thus, for example, a 2y cap actually has seven payments, not eight. Alternatively, one might consider that a spot starting cap is actually a forward starting cap starting in 3m time.

Let the cap rate (strike) be L_K. Let the i^{th} reset period from t_{i-1} to t_i be of length α_i as usual, where day count conventions are observed. Suppose the LIBOR rate for the period, observed at time t_{i-1}, is L_i. Then the payoff at time t_i is $\alpha_i \max(L_i - L_K, 0)$ per unit notional.

In an analogous fashion to writing a cap, we can write a floor, where the payout occurs when the floating rate drops below L_K, the floor rate.

We value each caplet or floorlet separately off the yield curve using the implied forward rates at $t = 0$, for each time period t_i. Then,

$$V = \sum_{i=1}^{n} V_i \tag{7}$$

$$V_i = Z(0,t_i)\alpha_i\eta\left[f(0;t_{i-1},t_i)N\left(\eta d_1^i\right) - L_K N\left(\eta d_2^i\right)\right] \tag{8}$$

$$d_{1,2}^i = \frac{\ln\frac{f(0;t_{i-1},t_i)}{L_K} \pm \frac{1}{2}\sigma_i^2 t_{i-1}}{\sigma_i\sqrt{t_{i-1}}} \tag{9}$$

where $\eta = 1$ stands for a cap(let), $\eta = -1$ for a floor(let), and where the floating rate (swap) curve is being used. $f(0;t_{i-1},t_i)$ is the simple forward rate for the period from t_{i-1} to t_i. Cap prices are quoted with $\sigma_i = \sigma$ for $i = 1, 2, \ldots, n$. On the other hand, if we are assembling a set of caplets into a cap, then the σ_i will be different.

Note that the i^{th} caplet is being valued in the t_i forward measure.

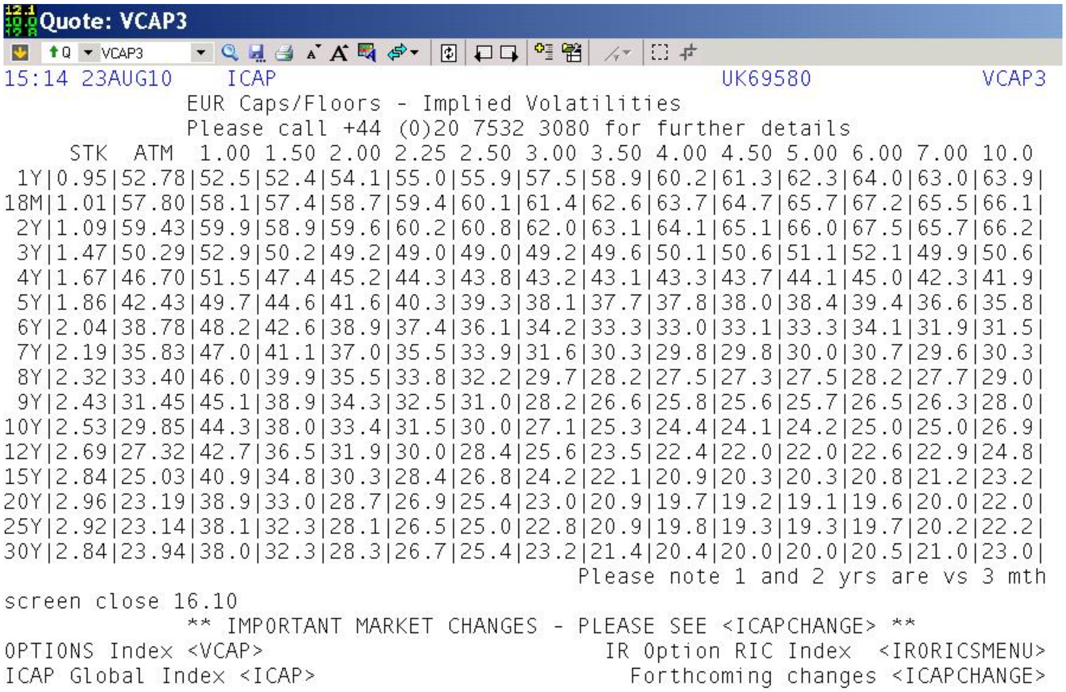
For examples, see the sheets CapFloorBasic.xls and CapsandFloors.xls

4.1. **A call/put on rates is a put/call on a bond.** A caplet (a call on an interest rate) is actually a put on a floating-rate bond whose yield is the LIBOR floating rate (a put option because of the inverse relationship between yield and bond price).

To see this, each caplet has a payoff in arrears of $\alpha_i \max(L_i - L_K, 0)$. The value of this in advance is

$$\begin{aligned}
V(t_{i-1}) &= (1+\alpha_i L_i)^{-1}\alpha_i \max(L_i - L_K, 0) \\
&= \max\left(\frac{\alpha_i L_i - \alpha_i L_K}{1+\alpha_i L_i}, 0\right) \\
&= \max\left(1 - \frac{1+\alpha_i L_K}{1+\alpha_i L_i}, 0\right) \\
&= (1+\alpha_i L_K)\max\left(\frac{1}{1+\alpha_i L_K} - \frac{1}{1+\alpha_i L_i}, 0\right)
\end{aligned}$$

which at time $t = t_0$ is $1 + \alpha_i L_K$ many puts on a zero coupon bond maturing at time t_i with the option exercise at t_{i-1}, strike $\frac{1}{1+\alpha_i L_K}$.



Quote: VCAP3

15:14 23AUG10 ICAP UK69580 VCAP3

EUR Caps/Floors - Implied Volatilities

Please call +44 (0)20 7532 3080 for further details

	STK	ATM	1.00	1.50	2.00	2.25	2.50	3.00	3.50	4.00	4.50	5.00	6.00	7.00	10.0
1Y	0.95	52.78	52.5	52.4	54.1	55.0	55.9	57.5	58.9	60.2	61.3	62.3	64.0	63.0	63.9
18M	1.01	57.80	58.1	57.4	58.7	59.4	60.1	61.4	62.6	63.7	64.7	65.7	67.2	65.5	66.1
2Y	1.09	59.43	59.9	58.9	59.6	60.2	60.8	62.0	63.1	64.1	65.1	66.0	67.5	65.7	66.2
3Y	1.47	50.29	52.9	50.2	49.2	49.0	49.0	49.2	49.6	50.1	50.6	51.1	52.1	49.9	50.6
4Y	1.67	46.70	51.5	47.4	45.2	44.3	43.8	43.2	43.1	43.3	43.7	44.1	45.0	42.3	41.9
5Y	1.86	42.43	49.7	44.6	41.6	40.3	39.3	38.1	37.7	37.8	38.0	38.4	39.4	36.6	35.8
6Y	2.04	38.78	48.2	42.6	38.9	37.4	36.1	34.2	33.3	33.0	33.1	33.3	34.1	31.9	31.5
7Y	2.19	35.83	47.0	41.1	37.0	35.5	33.9	31.6	30.3	29.8	29.8	30.0	30.7	29.6	30.3
8Y	2.32	33.40	46.0	39.9	35.5	33.8	32.2	29.7	28.2	27.5	27.3	27.5	28.2	27.7	29.0
9Y	2.43	31.45	45.1	38.9	34.3	32.5	31.0	28.2	26.6	25.8	25.6	25.7	26.5	26.3	28.0
10Y	2.53	29.85	44.3	38.0	33.4	31.5	30.0	27.1	25.3	24.4	24.1	24.2	25.0	25.0	26.9
12Y	2.69	27.32	42.7	36.5	31.9	30.0	28.4	25.6	23.5	22.4	22.0	22.0	22.6	22.9	24.8
15Y	2.84	25.03	40.9	34.8	30.3	28.4	26.8	24.2	22.1	20.9	20.3	20.3	20.8	21.2	23.2
20Y	2.96	23.19	38.9	33.0	28.7	26.9	25.4	23.0	20.9	19.7	19.2	19.1	19.6	20.0	22.0
25Y	2.92	23.14	38.1	32.3	28.1	26.5	25.0	22.8	20.9	19.8	19.3	19.3	19.7	20.2	22.2
30Y	2.84	23.94	38.0	32.3	28.3	26.7	25.4	23.2	21.4	20.4	20.0	20.0	20.5	21.0	23.0

Please note 1 and 2 yrs are vs 3 mth

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** IMPORTANT MARKET CHANGES - PLEASE SEE <ICAPCHANGE> **

OPTIONS Index <VCAP> IR Option RIC Index <IRORICSMENU>

ICAP Global Index <ICAP> Forthcoming changes <ICAPCHANGE>

FIGURE 1. A Reuter's page indicating cap/floor volatilities. The STK column indicates the at the money cap level and the ATM column is the at the money volatility. The other strikes (1% to 10%) indicate the volatilities at those strikes i.e. they give the implied volatility surface.

Likewise, floors - which are European put options on rates - are actually European call options on the (underlying) floating-rate bond.

Example 4.1. Suppose we have a given term structure

Term	Rate	Z
0.75	11.00%	0.92081
1	11.30%	0.89315

and consider a 9×12 caplet with strike $L_K = 12.1818\%$ and yield volatility $\sigma_y = 10\%$. Then the forward period is $\alpha = 0.25$ and the forward NACQ rate is given by $L_F = \left(\frac{0.92081}{0.89315} - 1\right) 0.25 = 12.3880\%$. We find $d_1 = 0.23708$ and $d_2 = 0.15048$ using (9), and calculate $N(d_1) = 0.59370$ and $N(d_2) = 0.55981$. Thus, using (8), we obtain the value of the caplet as $V_c = 0.0011953$.

Now we consider the put on a bond. We find the strike $K = \frac{1}{1+\alpha L_K} = 0.97045$, price volatility $\sigma = \sigma_y L_F \alpha = 0.3097\%$ (using (4) - α is the duration of the forward bond). From (3), calculate $d_1 = -0.18509$ and $d_2 = -0.18778$, and hence find $N(d_1) = 0.57342$ and $N(d_2) = 0.57447$. Then the value of the bond option from (2) is $V_b = 0.00116$. However, we still need to take into account the size which is given by $1 + \alpha L_K = 1.03045$. Multiplying the size with V_b yields 0.0011952.

Differences between these two values typically occur at the 5^{th} decimal place. It is impossible, mathematically, to have these two values equal; in the caplet model, rates are lognormal and in the bond option model, bond prices are lognormal. These two models cannot be made compatible.

4.2. **Greeks.** Let $\bar{r}$ denote the entire yield curve.

We need the following preliminary calculations:

$$\frac{\partial}{\partial \bar{r}} f(0; t_{i-1}, t_i) = 1 \tag{10}$$

$$\frac{\partial}{\partial r} Z(0, t_i) = -t_i Z(0, t_i) \tag{11}$$

$$\frac{\partial}{\partial \tau} Z(0, t_i) = -r_i Z(0, t_i) \tag{12}$$

$$\frac{\partial}{\partial \bar{r}} d_1^i = \frac{\partial}{\partial \bar{r}} d_2^i = \frac{1}{f(0; t_{i-1}, t_i)\sigma_i \sqrt{t_{i-1}}} \tag{13}$$

$$\frac{\partial}{\partial \sigma} d_1^i = \frac{\partial}{\partial \sigma} d_2^i + \sqrt{t_{i-1}} \tag{14}$$

$$\frac{\partial}{\partial t_{i-1}} d_1^i = \frac{\partial}{\partial t_{i-1}} d_2^i + \frac{\sigma}{2\sqrt{t_{i-1}}} \tag{15}$$

4.2.1. *Delta and pv01.*

$$\begin{aligned}
\frac{\partial V_i}{\partial \bar{r}} &= -t_i V_i + Z(0, t_i)\alpha_i \eta \left[\frac{\partial}{\partial \bar{r}} f(0; t_{i-1}, t_i) N(\eta d_1^i) \right. \\
&\quad \left. + f(0; t_{i-1}, t_i) N'(\eta d_1^i) \frac{\partial}{\partial \bar{r}} d_1^i - r_X N'(\eta d_2^i) \frac{\partial}{\partial \bar{r}} d_2^i \right] \\
&= -t_i V_i + Z(0, t_i)\alpha_i \eta N(\eta d_1^i) \left[1 + \alpha_i f(0; t_{i-1}, t_i) \right] && (16)
\end{aligned}$$

$$\frac{\partial V}{\partial \bar{r}} = \sum_{i=1}^{n} \frac{\partial V_i}{\partial \bar{r}} \tag{17}$$

Then pv01 $= \frac{1}{10000} \frac{\partial V}{\partial \bar{r}}$.

4.2.2. *Gamma.*

$$\begin{aligned}
\frac{\partial^2 V_i}{\partial \bar{r}^2} &= -t_i \frac{\partial V_i}{\partial \bar{r}} + \alpha_i \left[Z(0, t_i) N'(d_1^i) \frac{\partial}{\partial \bar{r}} d_1^i - t_i Z(0, t_i) N(d_1^i) \right] \\
&= -t_i \frac{\partial V_i}{\partial \bar{r}} + \alpha_i Z(0, t_i) \left[\frac{N'(d_1^i)}{f(0; t_{i-1}, t_i)\sigma_i \sqrt{t_{i-1}}} - t_i N(d_1^i) \right] && (18)
\end{aligned}$$

$$\frac{\partial^2 V}{\partial \bar{r}^2} = \sum_{i=1}^{n} \frac{\partial^2 V_i}{\partial \bar{r}^2} \tag{19}$$

4.2.3. *Vega.* Vega is $\frac{\partial V}{\partial \sigma}$. It is the Greek w.r.t. the cap volatility; it is not a Greek with respect to the caplet volatilities, so $\sigma_i = \sigma$ for $i = 1,\ 2,\ \ldots,\ n$.

$$\begin{aligned}
\frac{\partial V_i}{\partial \sigma} &= Z(0, t_i)\alpha_i \left[f(0; t_{i-1}, t_i) N'(d_1^i) \frac{\partial}{\partial \sigma} d_1^i - r_X N'(d_2^i) \frac{\partial}{\partial \sigma} d_2^i \right] \\
&= Z(0, t_i)\alpha_i f(0; t_{i-1}, t_i) N'(d_1^i) \sqrt{t_{i-1}} && (20)
\end{aligned}$$

$$\frac{\partial V}{\partial \sigma} = \sum_{i=1}^{n} \frac{\partial V_i}{\partial \sigma} \tag{21}$$

4.2.4. *Bucket (Caplet/Floorlet) Vega.* For sensitivities to caplet volatilities, we would be looking at a bucket risk type of scenario. This would be $\frac{\partial V}{\partial \sigma_i}$, and this only makes sense of course in the case where we have individual forward-forward volatilities rather than just a flat cap volatility.

$$\frac{\partial V_i}{\partial \sigma_i} = Z(0, t_i)\alpha_i f(0; t_{i-1}, t_i) N'(d_1^i) \sqrt{t_{i-1}}$$

where now it is the caplet volatility σ_i being used, not the cap volatility σ.

4.2.5. *Theta.* Theta can only be calculated w.r.t. some explicit assumptions about yield curve evolution. The assumption can be that when time moves forward, the bootstrapped continuous curve will remain constant. By full revaluation we can then calculate the theta of the position with one day of time decay.

4.2.6. *Delta hedging caps/floors with swaps.* We require the 'with delta' value of a cap or floor. Often the client wants to do the deal 'with delta', which means that the 'linear' hedge comes with the option trade. Thus we quote delta based on hedging the cap/floor with a (forward starting) swap with the same dates, basis and frequency. Then

$$\Delta = \frac{\frac{\partial V(\text{cap})}{\partial \underline{r}}}{\frac{\partial V(\text{swap})}{\partial \underline{r}}} = \frac{\text{pv01(cap)}}{\text{pv01(swap)}} \tag{22}$$

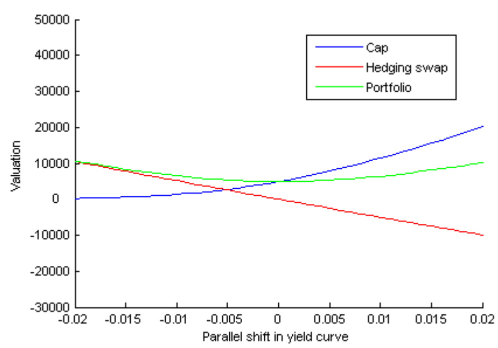

FIGURE 2. A cap with its forward swap hedge

The numerator is found in (17). For the denominator: the value of the swap fixed receiver, fixed rate R having been set, is

$$V(\text{swap}) = R\sum_{i=1}^{n}\alpha_i Z(0,t_i) - Z(0,t_0) + Z(0,t_n) \tag{23}$$

as seen in (32), (33). So the derivative is

$$\frac{\partial V(\text{swap})}{\partial \underline{r}} = -R\sum_{i=1}^{n}\alpha_i t_i Z(0,t_i) + t_0 Z(0,t_0) - t_n Z(0,t_n) \tag{24}$$

5. Stripping Black caps into caplets

Since each caplet is valued separately we expect a different volatility measure for each. Cap volatilities are always quoted as flat volatilities where the same volatility is used for each caplet, which in some sense will be a weighted average of the individual caplet volatilities. Thus, we use $\sigma = \sigma_i$ for $i = 1,\ 2,\ \ldots,\ n$. Most traders work with independent volatilities for each caplet, though, and these are called forward-forward vols. There exists a hump at about 1 year for the forward-forward vols (and, consequently, also for the flat vol, which can be seen as a cumulative average of the forward-forward vols). This can be observed or backed out of cap prices: see Figure 3.

Thus, from a set of caplet volatilities $\sigma_1,\ \sigma_2,\ \ldots,\ \sigma_n$ we may need to determine the corresponding cap volatility σ. This of course is a uniquely determined implied volatility problem. It only makes sense if the strikes of all the caplets are equal to the strike of the cap; this won't be the case in general (typically quoted volatilities will be at the money).

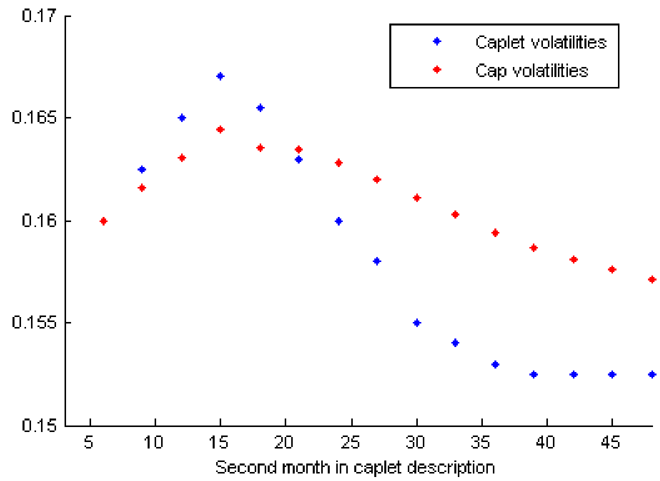

Figure 3. Caplet (fwd-fwd) and cap (flat fwd-fwd) volatility term structure

A far more difficult problem is the specification of the term structure of caplet volatilities given an incomplete term structure of cap volatilities. This is a type of bootstrap problem: there is insufficient information to determine a unique solution. For example, only 3x6, 6x9 and 9x12 caplets and 1y, 2y, 3y , 4y caps might be available. This critical point is generally not really well dealt with in the textbooks.

Instantaneous forward rate volatilities will be specified. Suppose the instantaneous volatility of $F_k(t)$ is modelled as $\sigma_k(t)$. Having done so, one now has the implied volatility of the $T_{k-1} \times T_k$ caplet given by

$$\sigma_{k,\mathrm{imp}} = \sqrt{\frac{1}{T_{k-1}} \int_0^{T_{k-1}} \sigma_k(t)^2 \, dt} \tag{25}$$

Some forms will allow us to calibrate to given cap or caplet term structures exactly. Clearly, and as emphasised in [Brigo and Mercurio, 2006, §6.2], the pricing of caps is independent of the joint dynamics of forward rates. However, that does not mean that calibration should also be an independent process. There are such straightforward formulations of the calibration of caps, so then the only parameters left to tackle swaptions calibration are the instantaneous correlations of forward rates, and this will typically be inadequate for use in the LMM.

[Brigo and Mercurio, 2006, §6.3.1] discuss seven different formulations for calibrating cap term structures which allow for more or less flexibility later in the swaption calibration.

The approach suggested in [Rebonato, 2002, Chapter 6], [Rebonato, 2004, Chapter 21] is to specify a parametric form for the instantaneous volatility such as

$$\sigma_k(t) = (a + b(T_{k-1} - t))e^{-c(T_{k-1}-t)} + d \tag{26}$$

This form is flexible enough to reproduce the typical shapes that occur in the market. It can accommodate either a humped form or a monotonically decreasing volatility. The model is time homogeneous, and the parameters have some economic interpretation, as described in Rebonato [2002]. For example, the time-0 volatility is $a + d$ and the long run limit is d. Furthermore, within the context of models such as LMM, calculus is easy enough: for example

$$\begin{aligned}
&\int \left((a+b(T_i-t))e^{-c(T_i-t)}+d\right)\left((a+b(T_j-t))e^{-c(T_j-t)}+d\right)\,dt \\
&= \frac{ad}{c}\left(e^{c(t-T_i)}+e^{c(t-T_j)}\right)+d^2t-\frac{bd}{c^2}\left(e^{c(t-T_i)}\left[c(t-T_i)-1\right]+e^{c(t-T_j)}\left[c(t-T_j)-1\right]\right) \\
&+\frac{e^{c(2t-T_i-T_j)}}{4c^3}\left[2a^2c^2+2abc(1+c(T_i+T_j-2t))+b^2(1+2c^2(t-T_i)(t-T_j)+c(T_i+T_j-2t))\right] \\
&=: I(t,T_i,T_j) \qquad (27)
\end{aligned}$$

as in [Rebonato, 2002, §6.6 - correcting for the typo], [Jäckel, 2002, (12.13)]. Then as in (25)

$$\sigma_{k,\mathrm{imp}} = \sqrt{\frac{1}{T_{k-1}}\left(I(T_{k-1},T_{k-1},T_{k-1})-I(0,T_{k-1},T_{k-1})\right)} \qquad (28)$$

Now the problem has gone from being under-specified to over-specified; an error minimisation algorithm will be used. Financial constraints are that $a+d>0$, $c>0$, $d>0$. What we do here is, for any choices of a, b, c and d,

- determine the caplet volatilities for every caplet using (28).
- find the model price of all the caplets using the caplet pricing formula.
- find the model price of the caps that are trading in the market (for example, the 1y, 2y, ... caps).
- We can then formulate an error function which measures the difference between model and market prices. This function will be something like

$$\mathrm{err}_{a,b,c,d} = \sum_i |V_i(a,b,c,d)-P_i| \qquad (29)$$

  The i varies only over those caps that actually trade (are quoted) in the market.
- Minimise the error. Solver might be used, but it might need to be trained to find a reasonable solution. Use of Nelder-Mead is suggested.

Some care needs to be taken here. The inputs will be at the money cap volatilities. For each cap, the at the money level (the forward swap rate) will probably be different. These are the forwards that need to be used in the cap pricing formula. The output is a parametric form for at the money caplet volatilities, and for each of these the at the money level will be different. This difference is usually ignored, as we are pricing without any skew anyway.

Having found the parameters, one still wants to price instruments that trade in the market exactly. Thus, after the parameters $a,\ b,\ c,\ d$ have been found, the model is re-specified as

$$\sigma_k(t) = K_k\left[(a+b(T_{k-1}-t))e^{-c(T_{k-1}-t)}+d\right] \qquad (30)$$

Equivalently,

$$\sigma_{k,\mathrm{imp}} = K_k\sqrt{\frac{1}{T_{k-1}}\left(I(T_{k-1},T_{k-1},T_{k-1})-I(0,T_{k-1},T_{k-1})\right)} \qquad (31)$$

We assume that K_k is a piecewise constant function, changing only at the end of each cap i.e. as a cap terminates and a new calibrating cap is applied. For example, if we have a 1y and a 2y cap (and others of later tenor) then $K_2=K_3=K_4$, and $K_5=K_6=K_7=K_8$.

With these assumptions, K_k is found uniquely. For caplets, the value of K_k is found directly. For caps, we note that there is one root find for each set of equal K_k's; we proceed from smallest to largest k. Thus

- For any given K_k, calculate the volatility using (31).
- price all the caplets using the caplet pricing formula.
- find the model price of the cap.
- vary K_k to match this model price with the market price. As the model price is an increasing function of K_k, the root is unique. We use a root finder such as Brent's method.

The model is no longer time homogeneous, and the deviation from being so is in some sense measured by how far the K_k deviate from 1. The better the fit of the model, the closer these values are to 1, one would hope to always have values of K_k between 0.9 and 1.1 say. This correction is discussed in [Rebonato, 2004, §21.4].

6. Swaptions

A swaption is an option to enter into a swap. We consider European swaptions. (Bermudan swaptions also exist.) Thus, at a specified time t_0, the holder of the option has the option to enter a swap which commences then (the first payment being one time period later, at t_1, and lasts until time t_n).

Of course, we have two possibilities

(a) a payer swaption, which gives the holder the right but not the obligation to receive floating, and pay a fixed rate L_K (a call on the floating rate).
(b) a receiver swaption, which gives the holder the right but not the obligation to receive a fixed rate L_K, and pay floating (a put on the floating rate).

Let f be the fair (par) forward swap rate for the period from t_0 to t_n. The date schedule for swaptions is the swap schedule. The time of payments of the forward starting swap are $t_1,\ t_2,\ \ldots,\ t_n$, where $t_0,\ t_1,\ \ldots,\ t_n$ are successive observation days, for example, quarterly, calculated according to the relevant day count convention and modified following rules. As usual, let $t_i - t_{i-1} = \alpha_i$, measured in years, for $i = 1,\ 2,\ \ldots,\ n$.

Note that if $n = 1$ then we have a one period cap (a payer swaption) or a one period floor (a receiver swaption). Thus, modulo the date schedule and the advanced/arrears issue, a caplet or floorlet.

Note that in general a swap (forward starting or starting immediately; in the later case $t_0 = 0$) with a fixed rate of R has the fixed leg payments worth

$$V_{\text{fix}} = R \sum_{i=1}^{n} \alpha_i Z(0, t_i) \tag{32}$$

while the floating payments are worth

$$V_{\text{float}} = Z(0, t_0) - Z(0, t_n) \tag{33}$$

Hence the fair forward swap rate, which equates the fixed and floating leg values, is given by

$$f = \frac{Z(0, t_0) - Z(0, t_n)}{\sum_{i=1}^{n} \alpha_i Z(0, t_i)} \tag{34}$$

Of course, these values are derived from the existing swap curve. Thus, the fair forward swap rate is dependent upon the bootstrap and interpolation method associated with the construction of the yield curve. Nevertheless, empirically it is found that the choice of interpolation method will only affect the result to less than a basis point, and typically a lot less. Also, let

$$L = \sum_{i=1}^{n} \alpha_i Z(0, t_i) \tag{35}$$

L is called the level, or the annuity.

Quote: KEPLEROPTZAR

09:25 10AUG10 Kepler Equities (Suisse) SA Geneva CH02387 KEPLEROPTZAR

SOUTH AFRICAN RAND INTEREST RATE OPTIONS

SWAPTION VOLATILITIES

	3M OPT		6M OPT		12M OPT		2YR OPT		3YR OPT		4YR OPT		5YR OPT	
1Y	12.3	16.3	12.5	16.5	13.2	17.2	12.7	16.7	11.8	15.8	10.9	14.9	10.0	14.0
2Y	12.5	16.5	12.7	16.7	13.2	17.2	12.1	16.1	11.4	15.4	10.7	14.7	9.9	13.9
3Y	13.8	17.8	13.6	17.6	13.0	17.0	11.6	15.6	11.0	15.0	10.4	14.4	9.7	13.7
4Y	14.0	18.0	14.2	18.2	13.5	17.5	11.8	15.8	11.4	15.4	11.0	15.0	10.2	14.2
5Y	14.3	18.3	14.8	18.8	14.0	18.0	12.0	16.0	11.8	15.8	11.5	15.5	11.0	15.0
6Y	14.3	18.3	14.5	18.5	13.8	17.8	11.8	15.8	11.5	15.5	11.2	15.2	11.0	15.0
7Y	14.3	18.3	14.3	18.3	13.5	17.5	11.5	15.5	11.3	15.3	11.0	15.0	10.5	14.5
8Y	14.5	18.5	14.0	18.0	13.0	17.0	11.2	15.2	11.0	15.0	10.5	14.5	9.8	13.8
9Y	14.5	18.5	13.5	17.5	13.0	17.0	11.0	15.0	10.5	14.5	10.0	14.0	9.5	13.5
10Y	14.5	18.5	13.1	17.1	12.8	16.8	10.8	14.8	10.2	14.2	10.0	14.0	9.3	13.3

CAPS/FLOORS VS 3M JIBAR

1Y	11.8	14.8
2Y	13.8	16.8
3Y	14.0	17.0
4Y	14.0	17.0
5Y	14.0	17.0
6Y	14.0	17.0
7Y	17.0	17.0
8Y	14.0	17.0
9Y	14.0	17.0
10Y	14.0	17.0

IRG VOL

3's	6's	11.5	14.5
6's	9's	12.5	15.5
9's	12's	13.5	13.5

Figure 4. A Reuter's page for at the money volatility quotes for caps/floors, caplet/floorlets (IRG means interest rate guarantee), and swaptions. In the swaption matrix the columns indicate the tenor of the option, the rows indicate the tenor of the into swap that is created if the swaption is exercised. If we had different strikes as well, then we would have the 'vol cube': option tenors, into tenors, strikes, and then the implied volatility at each point in the cube.

6.1. **Valuation.** The value of the swaption per unit of nominal is [Hull, 2005, §26.4]

$$V_\eta = L\eta[fN(\eta d_1) - L_K N(\eta d_2)] \tag{36}$$

$$d_{1,2} = \frac{\ln \frac{f}{L_K} \pm \frac{1}{2}\sigma^2 t_0}{\sigma\sqrt{t_0}} \tag{37}$$

where $\eta = 1$ stands for a payer swaption, $\eta = -1$ for a receiver swaption. Here σ is the volatility of the fair forward swap rate, and is an implied variable quoted in the market.

For an example, see the sheet Swaption.xls

6.2. **Greeks.** We will use the following:

$$\begin{aligned}
\frac{\partial}{\partial \overline{r}} Z(0,t) &= -tZ(0,t) \\
\frac{\partial L}{\partial \overline{r}} &= \sum_{i=1}^{n} -t_i \alpha_i Z(0,t_i) \\
\frac{\partial^2 L}{\partial \overline{r}^2} &= \sum_{i=1}^{n} t_i^2 \alpha_i Z(0,t_i) \\
\frac{\partial L}{\partial \tau} &= \sum_{i=1}^{n} -r_i \alpha_i Z(0,t_i) \\
\frac{\partial}{\partial \overline{r}} d_1 = \frac{\partial}{\partial \overline{r}} d_2 &= \frac{\partial f}{\partial r} \frac{1}{f\sigma\sqrt{t_0}} \\
\frac{\partial}{\partial \sigma} d_1 &= \frac{\partial}{\partial \sigma} d_2 + \sqrt{t_0} \\
\frac{\partial}{\partial \tau} d_1 &= \frac{\partial}{\partial \tau} d_2 + \frac{\sigma}{2\sqrt{t_0}}
\end{aligned}$$

Also, if we have a function $\frac{g(\overline{r},\tau)}{h(\overline{r},\tau)}$, then

$$\begin{aligned}
\left(\frac{g}{h}\right)' &= g'h^{-1} - gh^{-2}h' \\
\left(\frac{g}{h}\right)'' &= -2g'h^{-2}h' + g''h^{-1} + 2gh^{-3}(h')^2 - gh^{-2}h''
\end{aligned}$$

where the differentiation is with respect to $\overline{r}$.

6.2.1. *Delta.*

$$\Delta = \eta N(\eta d_1) \tag{38}$$

6.2.2. *pv01.*

$$\begin{aligned}
\frac{\partial V}{\partial \overline{r}} &= \eta \frac{\partial L}{\partial \overline{r}} \left[fN(\eta d_1) - r_K N(\eta d_2) \right] + \eta L \left[\frac{\partial f}{\partial \overline{r}} N(\eta d_1) + fN'(\eta d_1)\eta \frac{\partial d_1}{\partial r} - r_K N'(\eta d_2)\eta \frac{\partial d_2}{\partial r} \right] \\
&= \eta \frac{\partial L}{\partial \overline{r}} \left[fN(\eta d_1) - r_K N(\eta d_2) \right] + \eta L \frac{\partial f}{\partial \overline{r}} N(\eta d_1) \qquad (39)
\end{aligned}$$

Then pv01 $= \frac{1}{10000} \frac{\partial V}{\partial \overline{r}}$.

6.2.3. *Gamma.*

$$\begin{aligned}
\frac{\partial^2 V}{\partial \overline{r}^2} &= \eta \frac{\partial^2 L}{\partial \overline{r}^2} \left[fN(\eta d_1) - r_K N(\eta d_2) \right] + \eta \frac{\partial L}{\partial \overline{r}} \left[\frac{\partial f}{\partial \overline{r}} N(\eta d_1) + fN'(\eta d_1)\eta \frac{\partial d_1}{\partial \overline{r}} - r_K N'(\eta d_2)\eta \frac{\partial d_2}{\partial \overline{r}} \right] \\
&\quad + \eta \frac{\partial L}{\partial \overline{r}} \frac{\partial f}{\partial \overline{r}} N(\eta d_1) + \eta L \left[\frac{\partial^2 f}{\partial \overline{r}^2} N(\eta d_1) + \frac{\partial f}{\partial \overline{r}} N'(\eta d_1)\eta \frac{\partial d_1}{\partial \overline{r}} \right] \\
&= \eta \frac{\partial^2 L}{\partial \overline{r}^2} \left[fN(\eta d_1) - r_K N(\eta d_2) \right] + 2\eta \frac{\partial L}{\partial \overline{r}} \frac{\partial f}{\partial \overline{r}} N(\eta d_1) + \eta L \left[\frac{\partial^2 f}{\partial \overline{r}^2} N(\eta d_1) + \frac{\partial f}{\partial \overline{r}} N'(\eta d_1)\eta \frac{\partial d_1}{\partial \overline{r}} \right] \qquad (40)
\end{aligned}$$

6.2.4. *Vega.*

$$\begin{aligned}
\frac{\partial V}{\partial \sigma} &= L\eta \left[fN'(\eta d_1)\eta \frac{\partial d_1}{\partial \sigma} - r_K N'(\eta d_2)\eta \frac{\partial d_2}{\partial \sigma} \right] \\
&= LfN'(d_1)\sqrt{\tau} \qquad (41)
\end{aligned}$$

6.2.5. *Theta.* As before.

7. Why Black is useless for exotics

In each Black model for maturity date T, the forward rates will be log-normally distributed in what is called the T-forward measure, where T is the pay date of the option. The fact that we can 'legally' discount the expected payoff under this measure using today's yield curve is a consequence of some profound academic work of Geman et al. [1995], which establishes the existence of alternative pricing measures, and the ways that they are related to each other. This paper is very significant in the development of the Libor Market Model.

We can use Black's model without knowing any of the theory of Geman et al. [1995]; however, the Black model cannot safely be used to value more complicated products where the payoff depends on observations at multiple dates. For this an alternative model which links the behaviour of the rates at multiple dates will need to be used.

For this, the most universal approach is the Libor Market Model. Here inputs are all the Black models as well as a correlation structure between all the forward rates. From a properly calibrated LMM, one recaptures (up to the calibration error) the prices of traded caplets, caps, and swaptions. However, this calibration can be difficult. It then requires Monte Carlo techniques to value other derivatives.

An intermediate approach is the use of a more parsimonious model with just one driving factor - the so-called single-factor models.

8. Exercises

(1) A company caps three-month JIBAR at 9% per annum. The principal amount is R10 million. On a reset date, namely 20 September 2004, three-month JIBAR is 10% per annum.
   (a) What payment would this lead to under the cap?
   (b) When would the payment be made?
   (c) What is the value of the payment on the reset date?

(2) Use Black's model to value a one-year European put option on a bond with 9 years and 11 months to expiry. Assume that the current cash price of the bond is R105, the strike price (clean) is R110, the one-year interest rate is 10%, the bond's price volatility measure is 8% per annum, and the coupon rate is 8% NACS.

(3) Consider an eight-month European put option on a Treasury bond that currently has 14.25 years to maturity. The current cash bond price is \$910, the exercise price is \$900, and the volatility measure for the bond price is 10% per annum. A semi-annual coupon of \$35 will be paid by the bond in three months. The risk-free interest rate is 8% for all maturities up to one year.
   (a) Use Black's model to determine the price of the option. Consider both the case where the strike price corresponds to the cash price of the bond and the case where it corresponds to the clean price.
   (b) Calculate delta, gamma (both with respect to the bond price B) and vega in the above problem when the strike price corresponds to the quoted price. Explain how they can be interpreted.

(4) Using Black's model calculate the price of a caplet on the JIBAR rate. Today is 20 September 2004 and the caplet is the one that corresponds to the 9x12 period.

    The caplet is struck at 9.4%. The current JIBAR rate is 9.3%, the 3x6 FRA is 9.4% and the 6x9 FRA is 9.34%, and the 9x12 FRA is 9.20%.

Interest-rate volatility is 15%.

Also calculate delta, gamma (both with respect to the 9x12 forward rate) and vega in the above problem.

(5) Suppose that the yield, R, on a discount bond follows the process

$$dR = \mu(R,t)dt + \sigma(R,t)dz$$

where dz is a standard Wiener process under some measure. Use Ito's Lemma to show that the volatility of the discount bond price declines to zero as it approaches maturity, irrespective of the level of interest rates.

(6) The price of a bond at time T, measured in terms of its yield, is $G(y_T)$. Assume geometric Brownian motion for the forward bond yield, y, in a world that is forward risk-neutral with respect to a bond maturing at time T. Suppose that the growth rate of the forward bond yield is α and its volatility is σ_y.
   (a) Use Ito's Lemma to calculate the process for the forward bond price, in terms of α, σ_y, y and $G(y)$.
   (b) The forward bond price should follow a martingale in the world we are considering. Use this fact to calculate an expression for α.
   (c) Assume an initial value of $y = y_0$. Now show that the expected value of y at time T can be directly calculated from the above expression.

(7) Consider the following quarterly data:

	1	2	3	4	5	6	7	8
zero	8.00%	8.50%	8.30%	8.00%	8.00%	8.20%	8.25%	8.40%
vol	17.00%	16.00%	17.00%	16.00%	15.00%	15.00%	14.00%	

Time here is modified following quarterly.

The volatility data represents the volatility of the all in price for a bond option on the r153 that expires at that time.

The details of the r153 are as follows:

Bond Name	Maturity	Coupon	BCD1	BCD2	CD1	CD2
R153	2010/08/31	13.00%	821	218	831	228

(The BCD details of all r bonds have changed.)

Today is 29 August 2004 and the all in price is 1.1010101.

Price a vanilla European bond option, with a clean price strike, with expiry 30 May 2005, according to Black's model.

Construct your yield curve using raw interpolation.

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