MASTER THESIS

Mathematical statistics

THE IMPACT OF DEFAULT RISK WHEN PRICING BERMUDAN BOND OPTIONS USING THE JARROW-TURNBULL APPROACH

by

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Abstract

The objective of this thesis is to investigate the impact of default risk when pricing Bermudan bond options using the Jarrow-Turnbull approach. This model incorporates credit ratings into the valuation of derivatives on corporate debt.

Bonds issued by three German banks with different credit ratings are used as the underlying. Bermudan option prices on these bonds are discussed and compared to Bermudan option prices on default-free bonds.

It is shown that ignoring the default risk of the underlying corporate bond, leads to Bermudan bond options being mispriced by up to 5-6 % of the notional amount.

This study, together with Otto Francke's, "The impact of default risk when pricing American bond options, using a Jarrow-Turnbull approach", was performed for Arthur Andersen, as part of a project on the effect of default risk, when pricing financial instruments.

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1 Introduction

Credit risk management has traditionally fallen between two camps. Traditionalists argued that credit risk assessment was more art than science, while financial engineers claimed that contingent claims pricing could explain all that was interesting about bond pricing. For pricing derivative securities subject to credit risk, two approaches have therefore been used.

The financial engineer’s have seen these derivatives as contingent claims not on the financial securities themselves, but as "compound options" on the assets underlying the financial securities. The problem that arise is that the underlying assets are often not tradable and their values not observable. This makes application of the theory and the estimation of the relevant parameters problematic.

The second approach, used by traditionalists, is to ignore the credit risk. The derivative securities involving credit risk are priced as interest rate default-free options. However, this is inconsistent with the existence of spreads between the yields on corporate debt and Treasuries.

The objective of this thesis is to study the impact of default risk when pricing Bermudan bond options using a Jarrow-Turnbull approach. The Jarrow-Turnbull approach takes into account the default risk of corporate bonds. Default risk is a form of credit risk in which payments are reduced or missed all together when the issuer of a contract defaults.

The Jarrow-Turnbull approach views risky debt as paying off a fraction of each promised payment in the event of default. The default process does not explicitly depend on the firm’s underlying assets and the model allows assumptions to be imposed only on observable parameters. In this work, we do not take into account the possible default risk introduced by the writer of the Bermudan option.

To model the evolution of the interest rates, we have used the Black-Dermand-Toy model (BDT). The BDT-model also serves to price Bermudan options with government bonds as the underlying. It is assumed that the government bonds cannot default. Therefore, we use these results to compare with the Bermudan options on corporate bonds priced according to the Jarrow-Turnbull approach.

The numerical results show that default risk plays an important roll when pricing Bermudan bond options. When these options were struck at par, the price difference reached 5-6 % of the notional amount.

This paper is organised as follows. Chapter 2 describes some terminology used in the thesis. In Chapter 3, the yield dynamics is investigated by using a principal component analysis of the US and the EUR-yield curve. This study is fundamental for the choice of interest rate model. The theory for the BDT-model is described in Chapter 4. In Chapter 5, the BDT-model is implemented in a lattice tree using forward induction. The BDT-model is calibrated to caps according to the procedure in Chapter 6.

In Chapter 7 the Jarrow-Turnbull approach is described. The Jarrow-Turnbull trees representing the evolution of corporate bonds are built up and then used to price Bermudan bond options. Option prices as well as corporate bond prices are presented and discussed in Chapter 8. Finally, conclusions are given in Chapter 9.

The program code was written in C++. Matlab was used to plot results and for the principal component analysis.

2 Notation and terminology

A Zero-coupon bond, ZCB is referred to as a single certain cash flow, occurring at a known time $T = m\Delta t$ in the future. Its price at time $t = n\Delta t$ $m \geq n$ is denoted by $P(t,T)$ in continuous time and by $P(n,m)$ in discrete time. In a lattice tree the price of a zero coupon bond at period n, state i, maturing at period m, is denoted $P(n,i,m)$.

A risky bond XYZ is referred to as a single *not certain* cash flow, due to the risk of default. The price of a risky bond XYZ at time $t = n\Delta t$ maturing at time $T = m\Delta t$ is denoted by $\upsilon(n,m)$. If default occurs the risky bond pays $\delta \cdot P(n,m)$ at time $T = m\Delta t$.

The payoff ratio, $e(n)$ represents the value at period n of one promised XYZ delivered immediately,

$$e(n) \equiv \upsilon(n,n) \tag{2.1}$$

If the XYZ is not in default $e(n)$ will be unity, but in the case of default, $e(n)$ will be the recovery rate of the XYZ denoted δ $(0 < \delta < 1)$.

A coupon bearing bond is referred to as a bond that promises a stream of payments at fixed times, referred to as coupons, and a usually larger payment on the date of maturity. A coupon-bearing bond can be valued as a collection of zero-coupon bonds since it is possible to construct an identical cash flow using a portfolio of zero-coupon bonds.

A discount factor, denoted $p(t,T)$ is a zero coupon bond with a notional of unity.

The annualised one-period "short rate" is an artificial rate, used in a BDT lattice. The short rate at period n and state i is denoted $r(n,i)$. The corresponding **discount factor** is denoted by $p(n,i)$. The relation between the "short rate" and the discount factor is,

$$p(n,i) = \frac{1}{1 + r(n,i)\Delta t} \tag{2.2}$$

The London Interbank Offer Rate (LIBOR) is a rate used for US dollar borrowing and lending between banks. The LIBOR rate is often used as the underlying rate for interest rate derivatives such as swaps and caps. There are rates of various maturities, which can be derived by zero coupon bonds with different maturities,

$$L(t,T_i) = \frac{1}{(T_{i+1} - T_i)}\left(\frac{P(t,T_i)}{P(t,T_{i+1})} - 1\right) \tag{2.3}$$

A common used rate is the six months LIBOR.

The continuously compounded discrete forward rate at time t spanning the period $[T, T+\Delta t]$, is defined by,

$$f(t,T,T+\Delta t) \equiv -\frac{\ln(P(t,T+\Delta t)) - \ln(P(t,T))}{\Delta t} \tag{2.4}$$

As $\Delta t \to 0$ the instantaneous forward rate is defined as

$$f(t,T) = -\frac{\partial \ln P(t,T)}{\partial T} \tag{2.5}$$

The link between the forward rate and the price of the zero coupon bond is

$$P(t,T) = \exp - \int_t^T f(t,s)ds \tag{2.6}$$

The simply compounded forward rate is defined as

$$F(t,T,T+\Delta t) = \frac{P(t,T)/P(t,T+\Delta t) - 1}{\Delta t} \tag{2.7}$$

3 Investigation of the yield curve

The US-yield curve is plotted for different capture dates in fig 3.1.

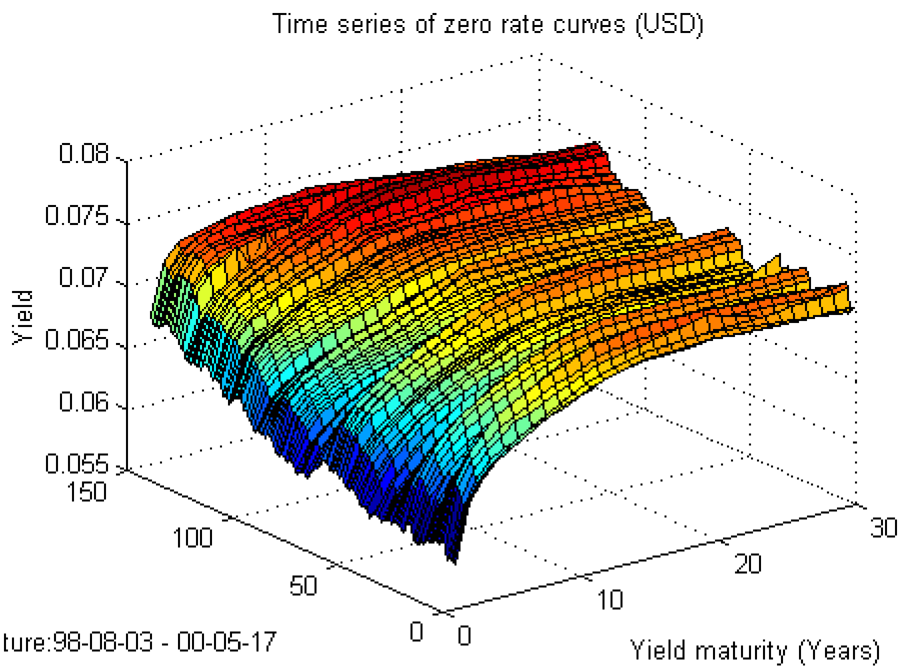

Fig 3.1

The yields correspond to zero coupon bonds ranging from 3 month to 30 years. We also plot the EUR- zero yield curve for 2 years to 10 years.

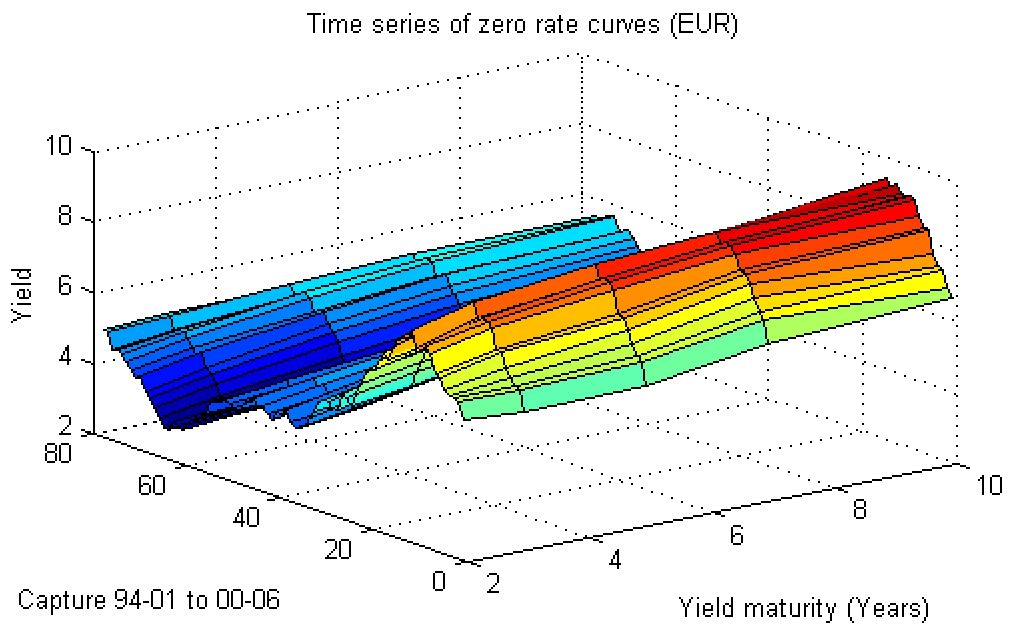

Fig 3.2

Before choosing an interest rate model to work with, it is important to investigate the dynamics of the yield curve. The correlation of interest rates will have great impact on how many variables that are needed to describe the yield curve. In order to define a set of variables that explain the movements of the yield curve, a *principal component analysis* is performed.

The principal components y_j, also called factors are linear combinations of relative rate changes g_k,

$$y_j = \sum_k a_{kj} g_k \tag{3.1}$$

which in vector form becomes

$$Y = A^T G \tag{3.2}$$

where A is a $n \times n$ matrix.

The column vectors a_j of the orthogonal matrix A, referred to as factor loadings, are normalised to unity and orthogonal to each other,

$$\sum_k a_{kj}^2 = 1 \tag{3.3}$$

$$\sum_k a_{kj} a_{kl} = 0 \quad , \quad l \neq j \tag{3.4}$$

Eq.(3.3) and Eq.(3.4) are the necessary conditions for the total variance being unchanged, in the transformation from G to Y, Eq.(3.2).

Let C denote the covariance matrix of the relative rate changes g_k. Finding the orthogonal axes of C is the same as finding its eigenvectors (factor loadings, a_j) and eigenvalues (variances of the principal components y_j).

The first principal component is the one corresponding to the largest eigenvalue of C, that is, the largest fraction of the total variance. This component will have the highest "explanatory power" of the yield curve movements. The second principal component corresponds to the second largest eigenvalue and so on.

If the relative rate changes g_k are linearly independent, there are as many principal components as there are rates. (If not there are as many as there are independent g_k's) However, in most currencies the "explanatory power" of the three first principal components, describe about 95-99 % of the total variance. This drastically reduces the number of factors needed to model the yield curve. The factor loadings for the three first principal components of the US-yield curve were calculated in Matlab.

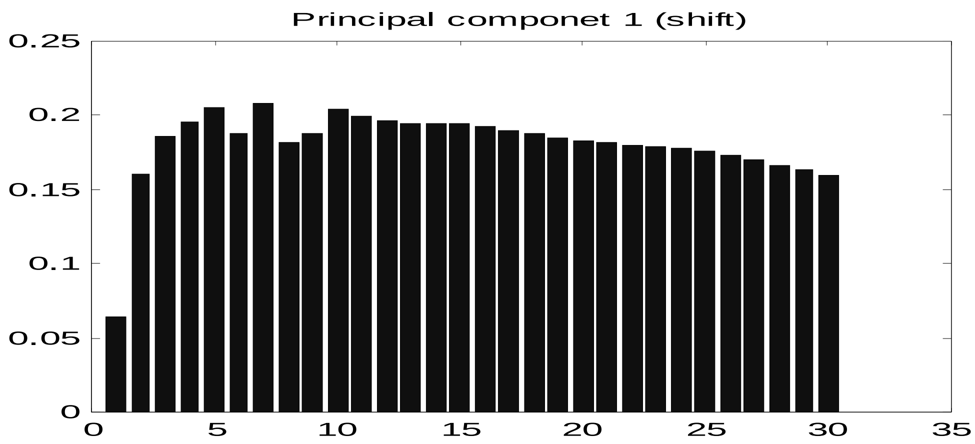

Fig 3.3

The *first principal component* is made up by factor loadings with approximately equal magnitude and positive signs. It corresponds to a parallel shift and can be interpreted as the "average level" of the yield curve.

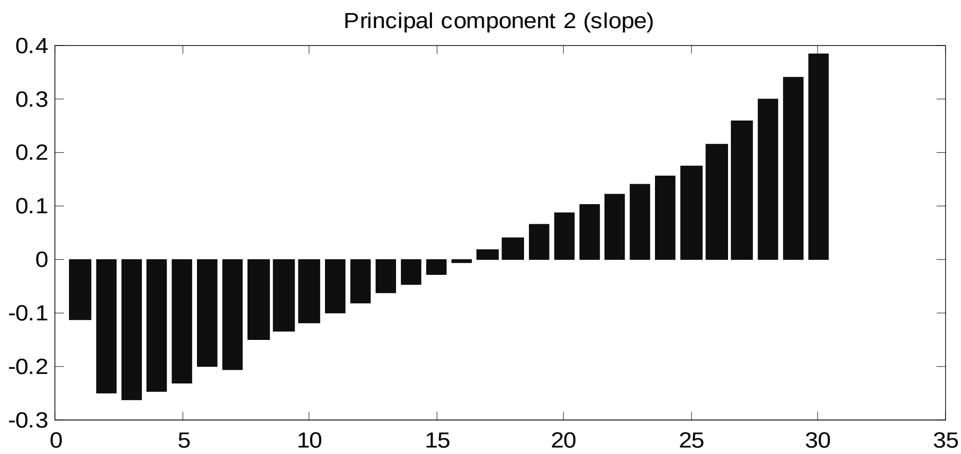

Fig 3.4

The *second principal component* has factor loadings with opposite signs, negative in the beginning, and positive in the end of the maturity spectrum. It can be interpreted as the slope of the yield curve.

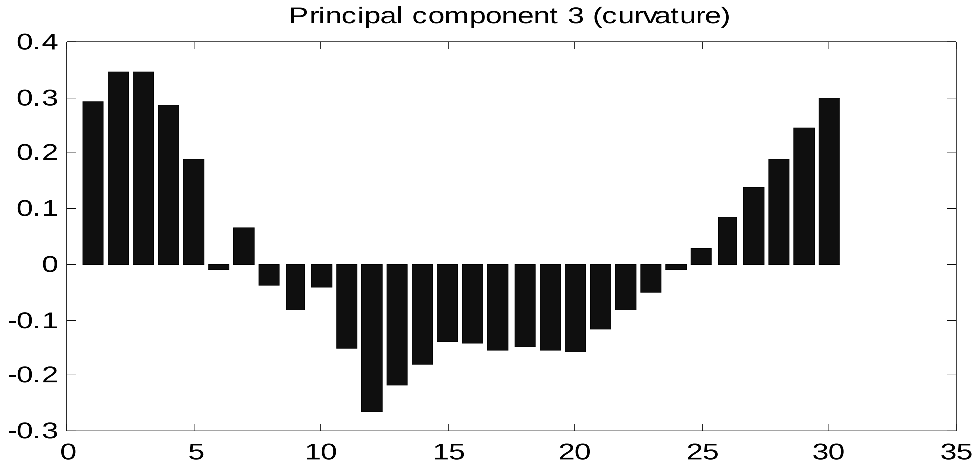

Fig 3.5

The *third principal component* is made up by factor loadings with positive signs at the extremums of the maturity spectrum. In the middle the sign is negative. It can therefore be interpreted as the curvature of the yield curve.

The "explanatory power" of the three principal components corresponding to the US-yield curve is plotted in fig 3.6.

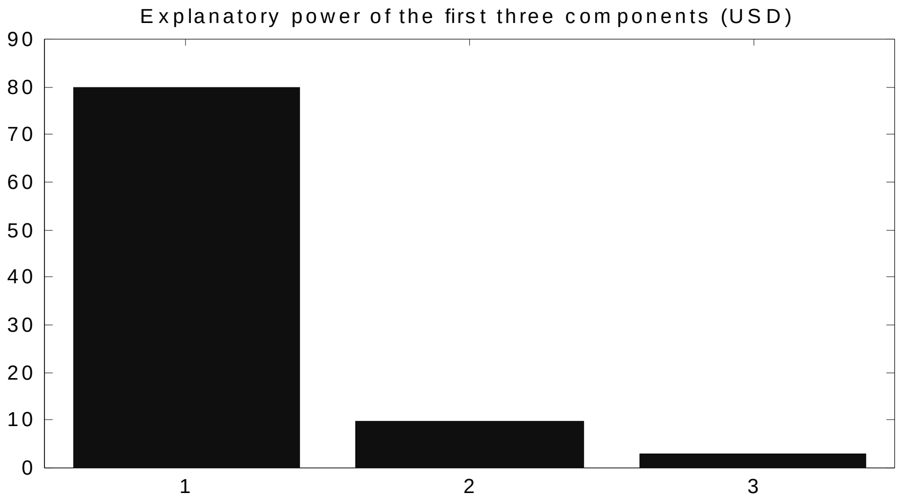

Fig 3.6

We also plot the explanatory power of the EUR-yield curve in figure 3.7.

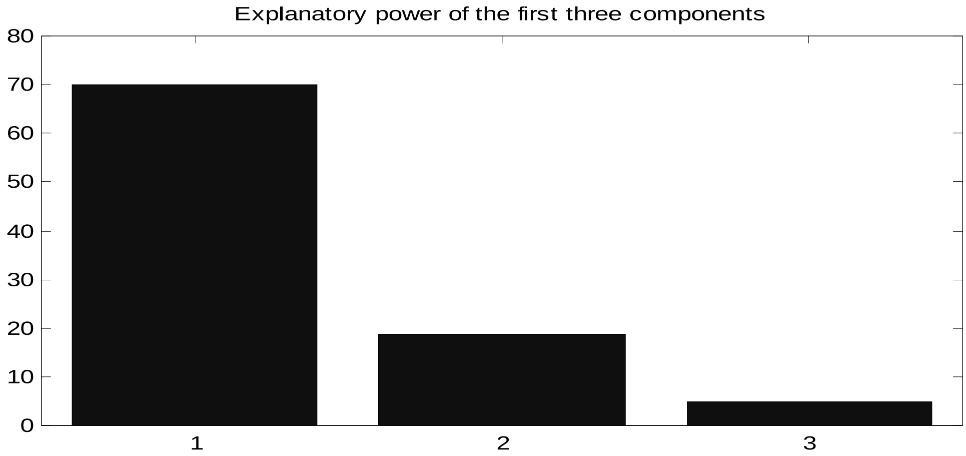

Fig 3.7

The first principal component "the shift" has an explanatory power of 80 % for the US- yield curve and 70% for the EUR-yield curve. The second component explains 10 % (US) and 20 % (EUR), while the third component only represent 3 % (US) and 5 % (EUR) of the explanatory power. It is clear that adding more than three factors will have very small impact in describing the yield curve movements for both the US-yield curve as well as the EUR-yield curve. In fact, using only one factor seems to be sufficient, due to the high explanatory power of the first principal component.

It must be stressed that using a one-factor model does not mean that the yield curve is forced to move in parallel. The individual rates can be affected by changes in the driving variable as much as the richness of the one factor model allows. The crucial point is that only one source of uncertainty is allowed to affect the different rates. In contrast to linearly independent rates, a one-factor model implies that all rates are perfectly correlated. Perfectly correlation means that all the relative rate changes, g_k are implied to be linearly dependent.

Of course, rates with different maturity are not perfectly correlated. A one-factor model is brutally simplifying real life, and if the derivative being priced depends on the imperfect correlations between rates, a two (or three)-factor model will be a better choice.

However, since the interest rate model will be calibrated to caps, not depending on the imperfect correlation of rates, the high explanatory power of the first principal component motivate the choice of a one-factor model. Mainly three advantages using a one-factor model rather than a two or three -factor model can be mentioned:

1. It is easier to implement.
2. It takes much less computer time.
3. It is much easier to calibrate.

The ease of calibration to caps is one of the advantages in the case of the Black-Dermand-Toy model [3]. This model is widely used and by many practitioners considered to outperform all other one-factor models. Therefore, the Black-Dermand-Toy model has been chosen to model the EUR-yield curve.

4 The Black-Derman-Toy model

The Black-Dermand-Toy model (BDT) is a one-factor model with the short rate as the fundamental variable. It is developed to match the observed term structure of yields on zero coupon bonds and their corresponding volatilities. Due to the behavior of the BDT model, which is well understood, many practitioners use the BDT-model for valuing interest rate derivatives such as caps and swaptions. The BDT model is a dynamic mean model and was first presented in an article published in The Financial Analyst Journal 1990 [3]

Several assumptions are made for the model to hold:

- Changes in all bond yields are perfectly correlated.
- Expected returns on all securities over one period are equal.
- The short rates are log-normally distributed. This prevents negative short rates and has the advantage of volatility input being in percentage form (specified by cap volatilities).
- There exists no taxes or transaction costs.

The continuous time BDT risk-neutral short-rate process has the form

$$r(t) = u(t)\exp(\sigma(t)z(t)) \tag{4.1}$$

where u(t) is the median of the short-rate distribution at time t, $\sigma(t)$ the short-rate volatility, and z(t) a standard Brownian motion. The dynamics of the logarithm of the short rate is given by, [Appendix A.1]

$$d\ln r(t) = \{\theta(t) + f'(t)[\Psi(t) - \ln r(t)]\}dt + \sigma(t)dz(t) \tag{4.2}$$

where

$$\theta(t) = \frac{\partial \ln u(t)}{\partial t}$$
$$f'(t) = -\frac{\partial \ln \sigma(t)}{\partial t}$$
$$\Psi(t) = \ln u(t)$$

A feature, which is specific to the behavior of interest rates as opposed to stock prices, is that interest rates tend to be pulled back to some long-term reversion level. This phenomenon is known as mean reversion, see fig 4.1.

Interest

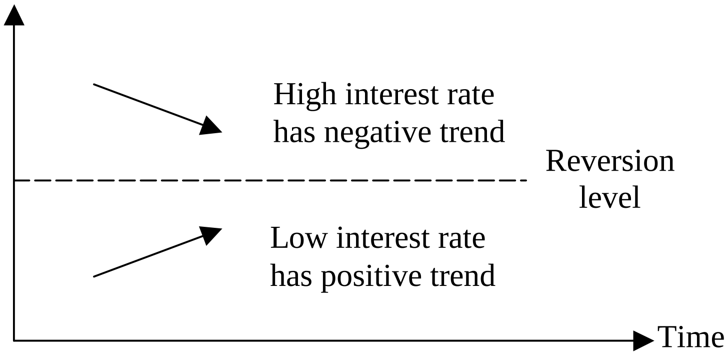

Fig 4.1

For constant volatility, $f'(t) = 0$, the BDT model does not display any mean reversion. In this case the process reduces to a lognormal version of the Ho-Lee model.

The short-rate evolves by diffusion with a drift that follows the logarithm of the median. If the volatility is decaying, the reversion speed $-f'(t)$ will be positive and the logarithm of the short rate will reverse to $\Psi(t)$.

This means that the short rates will not assume implausibly large values over long time horizons, which is along the lines of market observations. However, care must be taken to ensure that the model is viable. By specifying a volatility in discrete time $\sigma(k)$ in the BDT approach, the unconditional variance of the short rate is

$$Var(\ln r(k)) = \sigma^2(k) \cdot k\Delta t \tag{4.3}$$

A proof of Eq.(3.4) is included in appendix A.2. Eq.(4.3) states that the unconditional variance of the short rate depends neither on the volatility from time 0 to $T - \Delta t$ nor on the reversion speed $-f'(t)$. This fact is crucial because it is the reason why calibration to caps is so easy in the case of the BDT model. In fact, the almost exact BDT prices for caps can be found by assigning a value at expiry of the short rate volatility, equal to the Black "implied volatility".

If decrease in volatility is too sharp, Eq.(4.3) implies that we have more information about a future time period k than about an earlier time period m, which is not correct. Therefore, the following relationship must hold

$$k \cdot \sigma^2(k) > m \cdot \sigma^2(m) \quad , \quad k > m \tag{4.4}$$

Note that volatility term structures are not the same for different markets. The BDT model is for example used in the US- and the European market where cap volatilities are declining rather smoothly possibly after an initial hump. This is not the case for the Japanese cap volatilities, which decrease to quickly with option maturity. See fig 4.2.

JP-Cap Volatilities

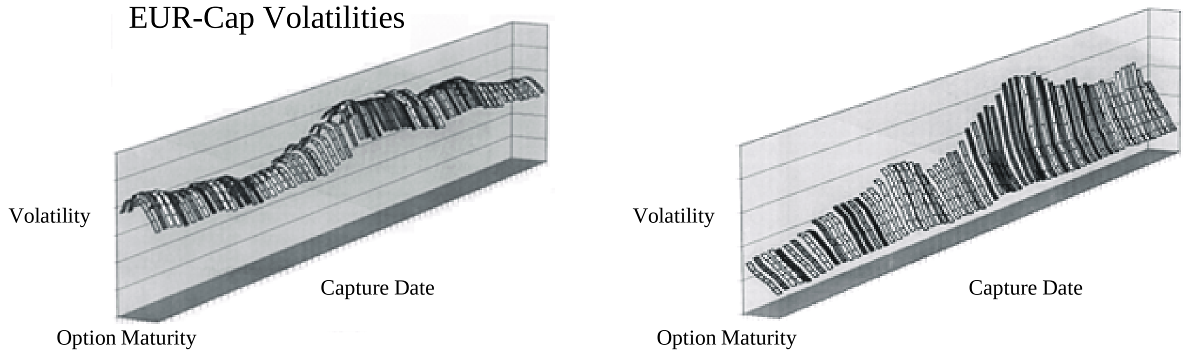

Fig 4.2

Figure 4.2 highlights the different cap volatility term structure for the European and the Japanese markets. It would not be appropriate to use the BDT model for the Japanese market, since Eq.(4.4) is not satisfied. In the European (and the US) market Eq.(4.4) holds and the BDT-model is viable.

Consider again Eq.(4.2). There is no explicit solution to this stochastic differential equation. The BDT model is developed algorithmically using for example Monte Carlo simulation, Finite Difference methods or a binomial lattice. We have chosen to work with a binomial lattice which is also the method suggested in the original article [3].

As already stated, the reversion speed $-f'(t)$ that determines the volatilities of rates with different maturities, depends only on the short rate volatility. Yield volatilities of different maturities depend on the future volatility of the short rate. In the BDT model, as for all one-factor models, this implies perfect correlation between different rates.

Due to this fact, the BDT model cannot be calibrated simultaneously to caps and swaptions. Using cap volatilities would overstate the swap rate volatility and vice versa. Therefore, the BDT model should always be used to price derivatives within the set of securities to which the model has been calibrated. We will now describe the implementation of the model using a binomial lattice.

5 Binomial Lattice model

The unit time is divided into M periods of length $\Delta t = 1/M$ each. At each period n, corresponding to time $t = n/M = n\Delta t$, there are $n+1$ states. These states range from $i = -n, -n+2, ..., n-2, n$. At the present period $n = 0$, there is a single state $i = 0$. see fig 5.1.

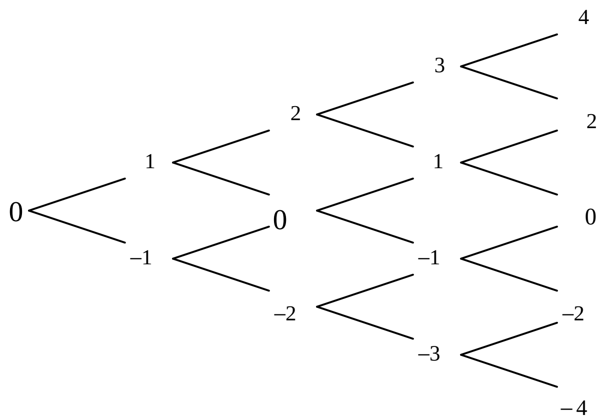

Fig 5.1

Let $r(n,i)$ denote the annualised one-period rate at period n and state i.
Denote the discount factor at period n and state i by

$$p(n,i) = \frac{1}{1 + r(n,i)\Delta t} \tag{5.1}$$

For example, consider the five-step tree of discount factors in fig 5.2.

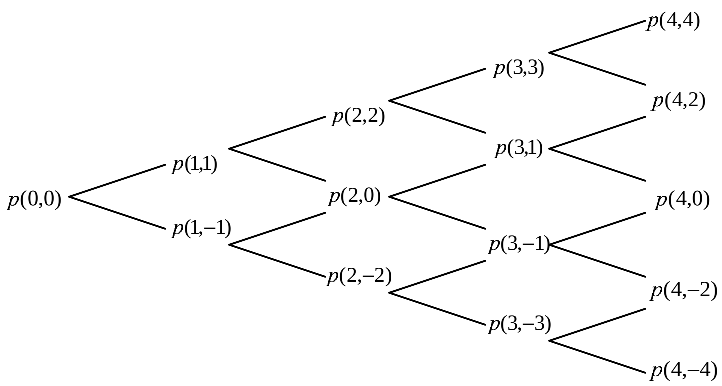

Fig 5.2

The tree in fig 5.2 is a discrete time representation of the stochastic process for the short rate, where we have used Eq. (5.1) to express the discount factors instead of the one period rate at each node. The probability for an up or down move in the three is chosen to be $\frac{1}{2}$.

The attraction of the binomial lattice model lies in the fact that, once the one period discount factors $p(n,i)$ are determined, securities are evaluated easily by backward induction. For example let $C(n,i)$ denote the price of a security at period n and state i. This price is obtained from its prices at the up and down nodes in the next period by the backward equation.

$$C(n,i) = \frac{1}{2} p(n,i)\left[C(n+1,i+1) + C(n+1,i-1)\right] \tag{5.2}$$

$$\begin{tikzcd}[column sep=small]
 & C(n+1,i+1) \\
C(n,i) \arrow[ur, no head] \arrow[dr, no head] & \\
 & C(n+1,i-1)
\end{tikzcd}$$

Fig 5.3

This iteration is continued backward all the way to period $n = 0$. The price of the security today is $C(0,0)$.

Table 5.1 gives the assumed market term structure.

Table 5.1: Market term structure

Maturity: m (years)	**ZCB : P(0,m)** (Paying \$1 in m years)	**Yield: y(m)** (%)	**Yield volatility:** $\sigma_{term}(m)$ (%)
1	0.9091	10	20
2	0.8116	11	19
3	0.7118	12	18
4	0.6243	12.5	17
5	0.5428	13	16

We want to find the discount factors in fig 5.2 that assure matching between the model’s term structure and the market term structure in table 5.1. First we present the solution.

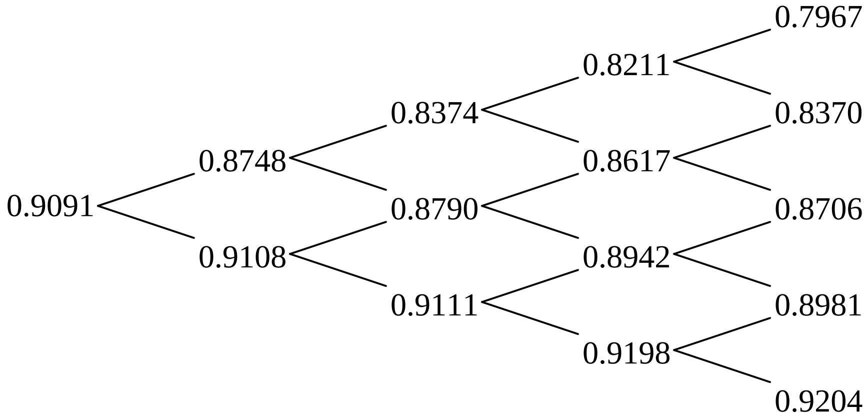

Fig 5.4

The discount factors in the tree was found by using forward induction, a method first introduced by Jamshidian[6]. Forward induction is an efficient tool in the generation of yield curve binomial trees. It is an application of the binomial formulation of the Fokker-Planck forward equation. The forward induction method will be described for the general class of Brownian-Path Independent interest models that includes the BDT model.

5.1 Brownian-Path Independent Interest Models

An interest rate model is referred to as Brownian Path Independent (BPI) if there is a function $r(z(t),t)$ such that $r(t) = r(z(t),t)$, where $z(t)$ is the Brownian motion. The instantaneous interest rate and hence the entire yield curve depends, at any time t, on the level $z(t)$ but not on the prior history $z(s)$, $s < t$ of the Brownian motion. Two BPI families are of major interest:

Normal BPI $\quad r(t) = U_N(t) + \sigma_N(t)z(t) \quad$ (5.3)

Lognormal BPI $\quad r(t) = U_L(t)\exp(\sigma_L(t)z(t)) \quad$ (5.4)

where $\sigma_N(t)$ and $\sigma_L(t)$ represent, respectively the absolute and the percentage volatility of the short rate $r(t)$. $U_N(t)$ is the mean and $U_L(t)$ the median of $r(t)$.

The advantages with the lognormal BPI are positive interest rates and natural unit of volatility in percentage form, consistent with the way volatility is quoted in the market place. However, unlike the normal BPI, the lognormal BPI does not provide a closed form solution. It is possible to fit the yield curve by trial and error but this is inefficient. In fact, the total computational time needed to calculate all the discount factors of a tree with N periods is proportional to N^3. Since N should be at least 100, too many iterations are needed. Forward induction efficiently solves the yield curve fitting problems, but before describing the procedure a discrete version of the BPI models is necessary.

Consider again fig 5.1. Define the variable $X_k = \sum_{j=1}^{k} y_j$ where,

$y_j = 1$ if an up move occurs at period k

$y_j = -1$ if a down move occurs at period k.

The variable X_k gives the state of the short rate at period k. At any period k, the X_k has a binomial distribution with mean zero and variance k. Now, let us investigate the mean and variance of $X_k\sqrt{\Delta t}$.

$$E(X_k\sqrt{\Delta t}) = \sqrt{\Delta t}E(X_k) = 0$$
$$Var(X_k\sqrt{\Delta t}) = \Delta t Var(X_k) = k\Delta t = t$$

It follows that $X_k\sqrt{\Delta t}$ has the same mean and variance as the Brownian motion $z(t)$. Since the normal distribution is a limit of binomial distributions, and the binomial process X_k has independent increments, the binomial process $X_k\sqrt{\Delta t}$ converges to the Brownian motion $z(t)$ as Δt approaches zero.

The state of the short rate was denoted by i. Replacing X_k by i will lead to having $z(t)$ approximated as $i\sqrt{\Delta t}$. Now, replacing t by n (with $n = \frac{t}{\Delta t}$) in Eq.(5.3) and Eq.(5.4) gives the discrete version of the normal and lognormal BPI families.

Additive: $\quad r(n,i) = U_N(n) + \sigma_N(n)i\sqrt{\Delta t} \quad i = -n, -n+2, .., n-2, n \qquad$ (5.3')

Multiplicative : $\quad r(n,i) = U_L(n)\exp(\sigma_L(n)i\sqrt{\Delta t}) \quad i = -n, -n+2, .., n-2, n \qquad$ (5.4')

5.2 Forward induction method

In discrete-time finance, the Green function is known as Arrow-Debreu prices where it represents prices of primitive securities. Let $G(n,i;m,j)$ denote the price at period n and state i of a security that has a cash flow of unity at period m $(m \geq n)$ and state j. Note that $G(m,j;m,j) = 1$ and that $G(m,i;m,j) = 0$ for $i \neq j$. Following relationship holds for $j = -m+1, -m+3, ..., m-3, m-1$:

$$G(n,i;m+1,j) = \frac{1}{2}\left[p(m,j-1)G(n,i;m,j-1) + p(m,j+1)G(n,i;m,j+1)\right]$$

(5.5)

For j= ± m $\quad G[n,i;m+1,\pm(m+1)] = \frac{1}{2}p(m,\pm m)G(n,i;m,\pm m)$

Eq.(5.5) is the binomial forward equation. By intuitive reasoning, it states that we discount a cash flow of unity for receiving it one period later on. This is simply the dual of the backward binomial equation. Note that when $j = \pm m$, there is only one node (bottom or top), which gives a modified expression for these two cases.

The term structure $P(0,m)$, which represents today price of a bond that pays unity at period m, can be obtained for all values of m, by the maximum smoothness criterion, see appendix [B.1]. Arrow-Debreu prices are the building blocks of all securities. The price of a zero coupon bond which matures at period $m+1$ can be expressed in terms of the Arrow-Debreu prices and the discount factors in period m.

$$P(0,m+1) = \sum_j G(0,0;m,j)\,p(m,j) \qquad j=-m,-m+2,\dots,m-2,m \tag{5.6}$$

The term structure can be fit to any Brownian path independent models of the general form $r(m,j) = f(j\sqrt{\Delta t}, B(m), \sigma(m))$ using forward induction. First, assume that $\sigma(m)=\sigma$ is a given constant and the problem is to solve for $B(m)$ as to match the given discount function $P(0,m)$.

Let $m \geq 1$ and assume that $B(m-1)$ and $G(0,0,m-1,j)$ have been found. (the induction starts with $G(0,0,0,0)=1$ and $B(0) = f^{-1}[r(0)]$).

1. Compute all the discount factors $p(m-1,j)$ from the $B(m-1)$ for all j.
2. Use the binomial forward Eq.(5.5) (with $n=i=0$, m replaced by $m-1$) to compute $G(0,0,m,j)$ for all j.
3. Substituting $p(m,j) = 1/[1+f[j\sqrt{\Delta t}, B(m), \sigma]\Delta t]$ into Eq.(5.6) leads to a non linear equation in one unknown $B(m)$, which can be solved by Newton-Raphson iteration.
4. Proceed inductively forward until the whole tree is constructed.

Next consider the case matching both the yield and volatility curves in BPI models of the same general form. This requires solving jointly for $B(m)$ and $\sigma(m)$. Let $P(1,\pm 1,m)$ represent the prices at period $n=1$, state $i=\pm 1$ of m-maturity zero coupons bonds. See fig 5.5.

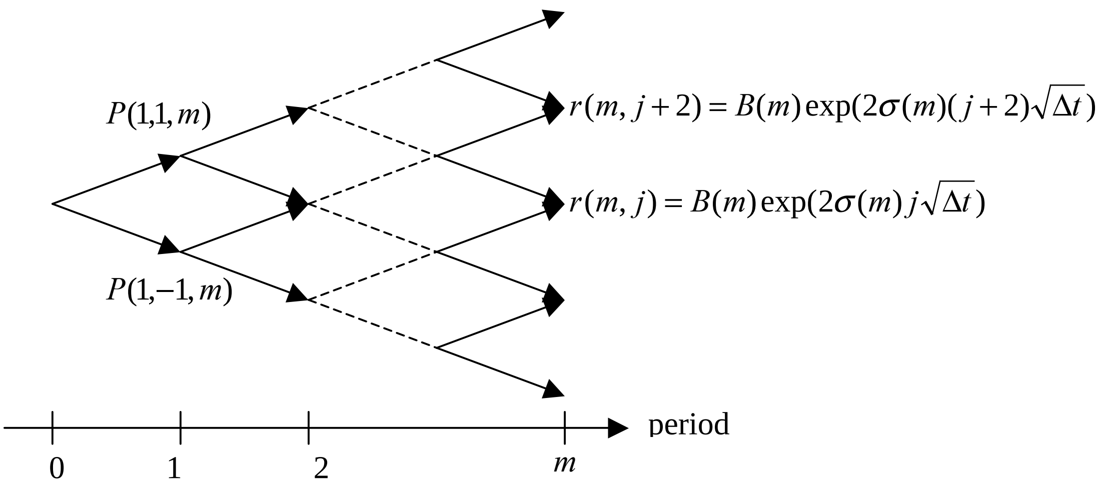

Fig 5.5

In fig. 5.5 we have used the short rate formula for the BDT model. Note that an up move of the short rate differs from a down move of the short rate by a factor $\exp(2\sigma(m)\sqrt{\Delta t})$. Therefore we have,

$$\frac{y_+(m)}{y_-(m)} = \exp(2\sigma_{term}(m)\sqrt{\Delta t}) \tag{5.7}$$

where $y\pm(m)$ denote the yield corresponding to $P(1,\pm1,m)$. Solving for $\sigma_{term}\sqrt{\Delta t}$ in Eq.(5.7) yields,

$$\sigma_{term}(m)\sqrt{\Delta t} = \frac{1}{2}\ln\left(\frac{\left[\left(P(1,1,m)\right)^{-\frac{1}{m-1}} - 1\right]}{\left[\left(P(1,-1,m)\right)^{-\frac{1}{m-1}} - 1\right]}\right) \qquad m \geq 2 \tag{5.8}$$

A second equation for $P(1,\pm1,m)$ is found by discounting back to the origin.

$$P(0,m) = P(0,0,m) = \frac{1}{2}\left[P(1,1,m) + P(1,-1,m)\right]p(0,0) \qquad m \geq 2 \tag{5.9}$$

The term structure of zero coupon bonds at period $n = 0$, $P(0,m)$ and the yield volatility term structure $\sigma_{term}(m)$ at period $n = 0$ are known. $P(1,\pm1,m)$ can be found for all periods m by using Eq.(5.8) in conjunction with Eq.(5.9) in a Newton-Raphson iteration. This provides two equations similar to Eq.(5.5)

$$P(1,\pm1,m+1) = \sum G(1,\pm1;m,j)\,p(m,j) \qquad j = -m,-m+2,..,m-2,m \tag{5.10}$$

Given $B(m-1)$ and $\sigma(m-1)$ we can now construct the quantities $\sigma(m)$, $B(m)$ and $G(1,\pm1,m,j)$ inductively by going forward as in the previous case.

1. Compute all discount factors $p(m-1,j)$ using $B(m-1)$ and $\sigma(m-1)$ for all j.
2. Compute the $G(1,\pm1,m,j)$ using the binomial forward equation for all j.
3. Substituting $p(m,j) = 1/[1 + f[j\sqrt{\Delta t}, B(m), \sigma(m)]\Delta t]$ into (5.9) leads to a non linear equation in two unknowns $\sigma(m)$ and $B(m)$, which again can be solved with Newton's method because the Jacobian is available.
4. Proceed inductively forward until the whole tree is constructed.

The method of forward induction requires only n arithmetic operations to construct the discount factors at period n. The iterations are made only on the nodes at the given period , not the whole way back to the root of the tree as in "trial and error". In a tree with N periods, the computational time is proportional to the number of nodes N^2, which is of the same order as when closed formed solutions for the discount factors are available.

6 Calibrating the BDT-tree

In this chapter we will describe how to fit the BDT-tree to caps. Due to the fact that the BDT model is a one-factor model, it is in general not possible to match at the time both cap and swaption prices. The reason is that in any such model all rates are perfectly correlated. If the short rate process were assigned the volatilities needed to price the market caps, the forward equilibrium swap rate volatility would be overstated.

Caps are securities with payoffs depending on forward interest rates. They are important securities as they provide the means for controlling the risks involved with a fluctuating interest rate. For example, a company can buy a cap to provide insurance against the rate of interest on an underlying floating loan. If the floating rate rises above a certain level, called the cap rate, the insurance falls out. The company receives money for the cost of paying more interest than the cap rate on the loan. The holder of the cap still has the possibility to profit on the floating rate knowing that the interest rate will not exceed the cap rate on the loan.

A cap can be expressed as a bond option, which is made up by a collection of caplets. Each individual caplet can be seen as a call on the τ -maturity forward rate with reset date t_i and payment date $t_{i+1} = t_i + \tau$. It is common in the market to express the value of a cap in terms of the "implied flat volatility". "Implied" corresponds to that the volatility is extracted from Black's formula to match the market price of the cap. Flat volatility means that the same volatility is used for all caplets in Black's formula. The price of a caplet at time t_0 with strike K is thus

$$Caplet = Max\left[\left\{F(t_0,t_i,t_{i+1})N(h_1) - KN(h_2)\right\},0\right]p(t_0,t_{i+1}) \qquad (6.1)$$

where $F(t_0,t_i,t_{i+1})$ is today's implied forward rate between t_i and t_{i+1}, σ is the "implied flat volatility" of the forward rate and $p(t_0,t_{i+1})$ represents the discount factor between t_0 and t_{i+1}. The cumulative normal distribution N is given by,

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}dy$$

where,

$$h_{1,2} = \frac{\ln(\frac{F(t_0,t_i,t_{i+1})}{K}) \pm \frac{1}{2}\sigma^2(t_i - t_0)}{\sigma\sqrt{t_i - t_0}}$$

Note that practitioners analyse the spot volatility. The "implied flat volatility" is only used for quoting purposes in Black's formula.

6.1 Calibration to Cap prices

The market cap prices are calculated for the quoted volatilities using Black's formula. To calibrate the BDT-tree to these prices, a procedure to obtain the cap price within the BDT-tree is needed.

A cap is a portfolio of independent options (caplets). The payoff of each individual caplet is τ Max[R-X,0] where τ is the tenor of the underlying reference rate R, (e.g. the 6 month LIBOR rate for a semi-annual cap) resetting at period n and maturing at period $m = n + \tau / \Delta t$. X is the strike. At period n, there are $n+1$ possible rates denoted $R(n,i)$, $i = -n, -n+2, .., n-2, n$ in the BDT-tree.

These reference rates are given by the ratio of the floating to the fixed leg originating from node (n,i).

$$R(n,i) = \frac{P(n,i,n) - P(n,i,n+\tau/\Delta t)}{P(n,i,n+\tau/\Delta t)\tau} = \frac{1}{\tau}\left[\frac{1}{P(n,i,n+\tau/\Delta t)} - 1\right] \tag{6.2}$$

$P(n,i,m)$ denotes the value of a m-maturity zero coupon bond at period n in state i. The $P(n,i,m)$'s are found by placing a unity cash flow at all nodes in period $m = n + \tau / \Delta t$, and then discounting backwards to period n. Note that the actual payoff will take place at period m which makes it necessary to discount each payoff back to period n. The caplet price at expiry, period n at state i, is therefore

$$caplet(n,i) = \tau \cdot Max\left[\frac{\frac{1}{P(n,i,n+\tau/\Delta t)} - 1}{\tau} - X, 0\right] P(n,i,n+\tau/\Delta t) \tag{6.3}$$

When the caplet prices $caplet(n,i)$ for all states i at expiry, period n are known, we discount back to the origin to obtain the present value of the caplet. The price of a cap is the sum of all caplet prices making up the cap.

6.1.1 Constant volatility

In the case where a constant volatility is used to generate the BDT-tree, this unique volatility will also price the cap within the tree denoted $P^{BDT}(\sigma)$. Denote the market price (Black's price) of the same cap P^m. The calibration part is now to

$$\text{Minimize } (P^m - P^{BDT}(\sigma))^2 \qquad \text{subject to } \sigma \tag{6.4}$$

A volatility σ^* that minimizes (6.4) is found by using Brent's method described in appendix B.2 .

For pricing of a single cap this calibration procedure is useful. However, care must be taken when using these "average volatilities" if several caps are being priced. For example, assume that a four period cap is priced by a constant volatility of 20% and that a five period cap is priced by a constant volatility of 18%. Now, the implied forward volatility of the short rate from period 4 to period 5 is easily derived. But for low enough values of the market price of the five period cap, no solution might be found. This implies that constant volatilities cannot decline too sharply.

6.1.2 Term structure volatility

When using term structure volatility, the BDT-tree is calibrated to caps with different maturity. If the time-step in the BDT-tree Δt is smaller than the tenor τ of the cap, a linear interpolation is made, to obtain exactly one volatility for each period in the lattice.

$$\sigma^2(l)l = \sigma^2(k)k + \frac{\sigma^2(m)m - \sigma^2(k)k}{m-k}(l-k) \qquad k < l < m \qquad (6.5)$$

Assume that market prices for caps of a tenor $\tau = 6$ months are available. The $n+1$ cap is assumed to have exactly the same reset dates as the nth cap, plus an additional one. For simplicity, let the time-step Δt be equal to the tenor.

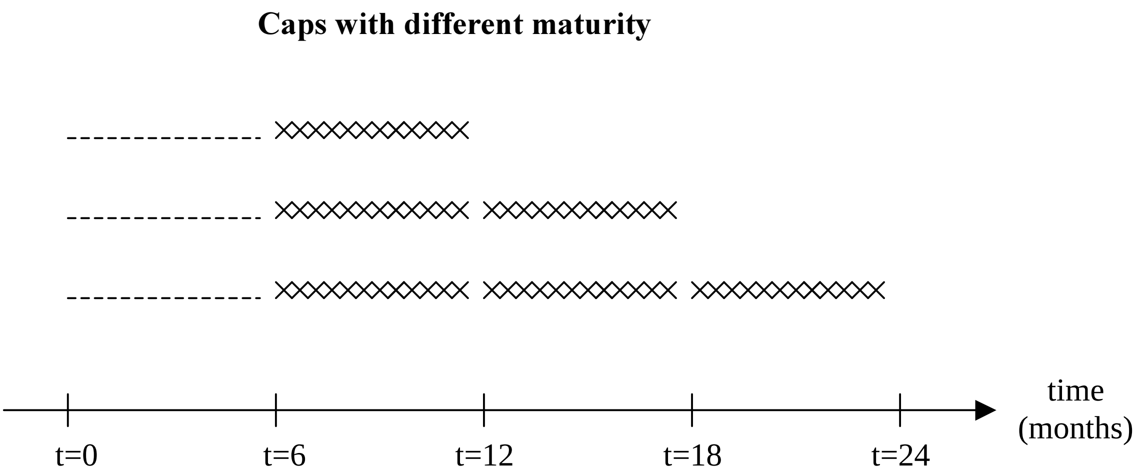

Fig 6.1. All caps have the same option life (---) , but are built up by different numbers of caplets (XXX) .

The first cap consists of only one caplet. We want to find the volatility of the short rate σ_1 so that the
BDT-tree prices this first cap(let) correctly. In other words, we solve

$$\min_{\sigma_1} \ (P_1^m - P_1^{BDT}(\sigma_1))^2 \qquad (6.6)$$

by using the Brent's root finder, see appendix B.2. Having priced the first cap(let), we move on to the second cap. Holding σ_1 fixed we now search for σ_2 so that the BDT price $P_2^{BDT}(\sigma_2)$ equals the market price of the second cap P_2^m. The procedure can be extended to any cap maturity. For each new cap, a single new short rate volatility is to be determined.

The BDT-model assumes lognormal short rates and displays almost lognormal forward rates. Since caps are priced by Black's formula, which assumes lognormal forward rates, the market assumptions are very similar to the BDT framework. Therefore, almost exact BDT prices for caps can be found by assigning a value at expiry of the short rate volatility, equal to the Black "implied volatility".

7 The Jarrow-Turnbull approach

To take credit risk into account, we extend the BDT-model using a method introduced by Robert A. Jarrow and Stuart M.Turnbull in 1995[7]. This approach views risky debt as paying of a fraction of each promised payment in the event of bankruptcy. The time of bankruptcy is given as an exogenous process, which not explicitly depends on the firms underlying assets. We will here concentrate on the discrete time case since it gives a good insight in how we have implemented the theory.

In the studied economy there exists two classes of zero coupon bonds, default free zero coupon bonds of all maturities $P(n,m)$ and risky XYZ zero coupon bonds of all maturities $\upsilon(n,m)$, both promising unity at period $m \geq n$. The current period is denoted by n. Define the payoff ratio as

$$e(n) \equiv \upsilon(n,n) \tag{7.1}$$

which represents the value at period n of one promised XYZ delivered immediately. If the XYZ is not in default $e(n)$ will be unity, but in the case of default, $e(n)$ will be the recovery rate of the XYZ denoted δ $(0 < \delta < 1)$. A hypothetical zero-coupon bond can now be defined

$$P^*(n,m) \equiv v(n,m) / e(n) \tag{7.2}$$

Note that these XYZ paying zero-coupon bonds are default-free. The decomposition of the XYZ zero-coupon bonds is by rearranging expression (7.2)

$$v(n,m) = P^*(n,m)e(n) \tag{7.3}$$

This decomposition is useful for modeling purposes since the term structure of XYZs can be characterized separately in terms of $P^*(n,m)$ and the payoff ratio $e(n)$. The payoff ratio process is shown for three periods in fig 7.1.

Payoff ratio process

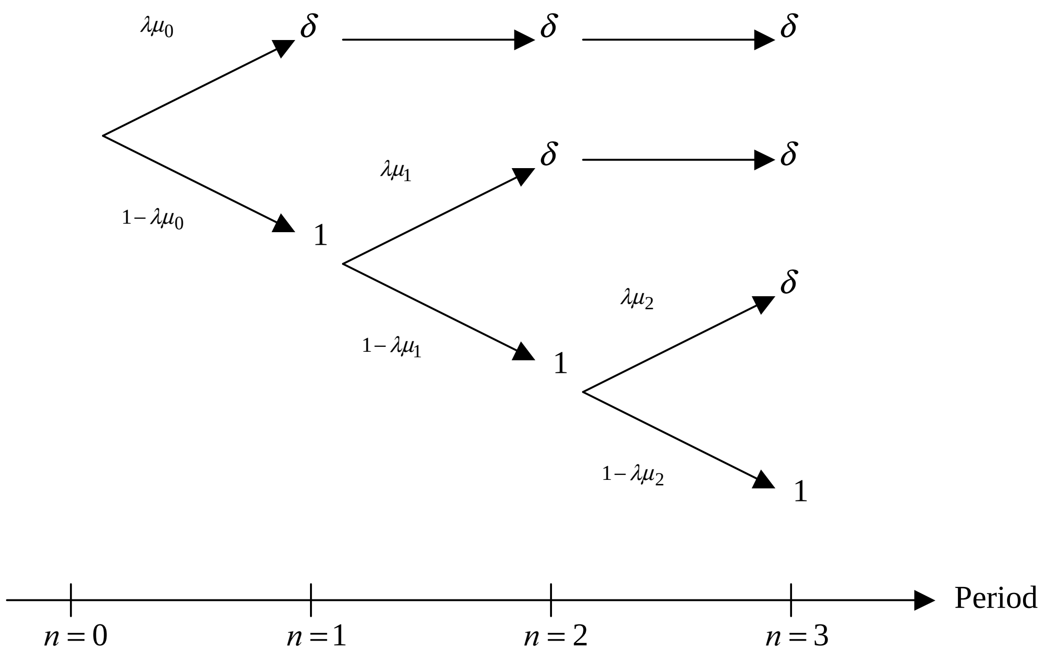

Fig 7.1

The probabilities of default, which ensures that the Jarrow-Turnbull model is free of arbitrage, are referred to as pseudoprobabilities. The pseudoprobabilities that default occurs at period n = 0,1,2 are denoted $\lambda\mu_0$, $\lambda\mu_1$ and $\lambda\mu_2$, while the recovery rate δ is constant for all time periods.. The discrete-time binomial process is selected to approximate a continuous-time Poisson bankruptcy process. *It is assumed that the short rates in the BDT-tree and the bankruptcy process are independent under the pseudoprobabilities.*

We will now describe how to obtain the pseudoprobabilities of default $\lambda\mu_n$ for multiple periods via the default-free bond market and the risky debt market.

The three expected payoff ratios at future dates are calculated as follows (see fig 7.1)

$$\tilde{E}_0(e(1)) = \lambda\mu_0\delta + (1-\lambda\mu_0) \tag{7.4}$$

$$\tilde{E}_0(e(2)) = \lambda\mu_0\delta + (1-\lambda\mu_0)[\lambda\mu_1\delta + (1-\lambda\mu_1)] \tag{7.5}$$

$$\tilde{E}_0(e(3)) = \lambda\mu_0\delta + (1-\lambda\mu_0)[\lambda\mu_1\delta + (1-\lambda\mu_1)[\lambda\mu_2\delta + (1-\lambda\mu_2)]] \tag{7.6}$$

where $\tilde{E}_n(\cdot)$ denote the conditional expected payoff under the pseudoprobabilities at period n. Expression (7.4) states that the expectation of the payoff ratio one period from today is the sum of having δ with probability $\lambda\mu_0$ (default) and receiving unity with probability $1-\lambda\mu_0$ (no default). To calculate the expected payoff ratio two periods from today, we take the weighted average of the payment δ, from going bankrupt in period n=1, and the expected payoff at period n=2. In the same way the expected payoffs can be found for n=3,4,…,N, including one more expected payoff for each period. Eq.(7.6) represent the expected payoff when n=3.

With Eq.(7.3) in mind we now express the price of a XYZ zero coupon bond as

$$v(n,m) = P(n,m)\tilde{E}_n(e(m)) \qquad (7.7)$$

Eq.(7.7) states that the XYZ zero-coupon bond price is its discounted expected payoff at period m. As the expected payoff ratio is strictly less than one ($\tilde{E}_n(e(m)) < 1$), the XYZ zero coupon bond is strictly less valuable than a default free zero coupon bond of equal maturity.

Consider the case when the recovery rate δ is known. Given term structures of default-free bonds $P(0,m)$ and risky XYZ bonds $v(0,m)$, the following recursive estimation procedure for the pseudoprobabilities is introduced:

1. Calculate the expected payoff $\tilde{E}_0(e(1))$ using Eq.(7.7) for m=1.
2. Use $\tilde{E}_0(e(1))$ and the recovery rate δ in Eq.(7.4) to estimate the first probability of default $\lambda\mu_0$.
3. Go back to Eq.(7.7) and calculate $\tilde{E}_0(e(2))$ for m=2.
4. Calculate $\lambda\mu_1$ using $\tilde{E}_0(e(2))$, δ, and $\lambda\mu_0$ in Eq.(7.5).
5. Continue in the same manner until all the pseudoprobabilities have been found.

Note that not all the XYZ zero-coupon bond must trade to apply the model. For example the pseudoprobabilities $\lambda\mu_n$ can be constant over intervals where market data of risky bonds is not available. It is also possible to estimate the recovery rate δ. In fact, if the pseudo-probabilities are constant $\lambda\mu_n = \lambda\mu$ only two traded XYZ zero-coupon bonds are needed to deduce δ and $\lambda\mu$. This result is very useful in applications where there is sparsity of XYZ zero-coupon bonds trading.

We will now study the XYZ zero coupon bond price process for the three period economy building up a “Jarrow Turnbull” tree. The probability for an up or down movement of the short rate is constant, $\frac{1}{2}$ for all periods in the tree. For each up and down movement of the short rate there are two nodes denoting, default occurred (circle) and no default occurred (rectangle). Suppose that the recovery rate δ is known and that the pseudo-probabilities of default $\lambda\mu_0$, $\lambda\mu_1$, $\lambda\mu_2$ have been found. The Jarrow-Turnball tree is shown in fig 7.2.

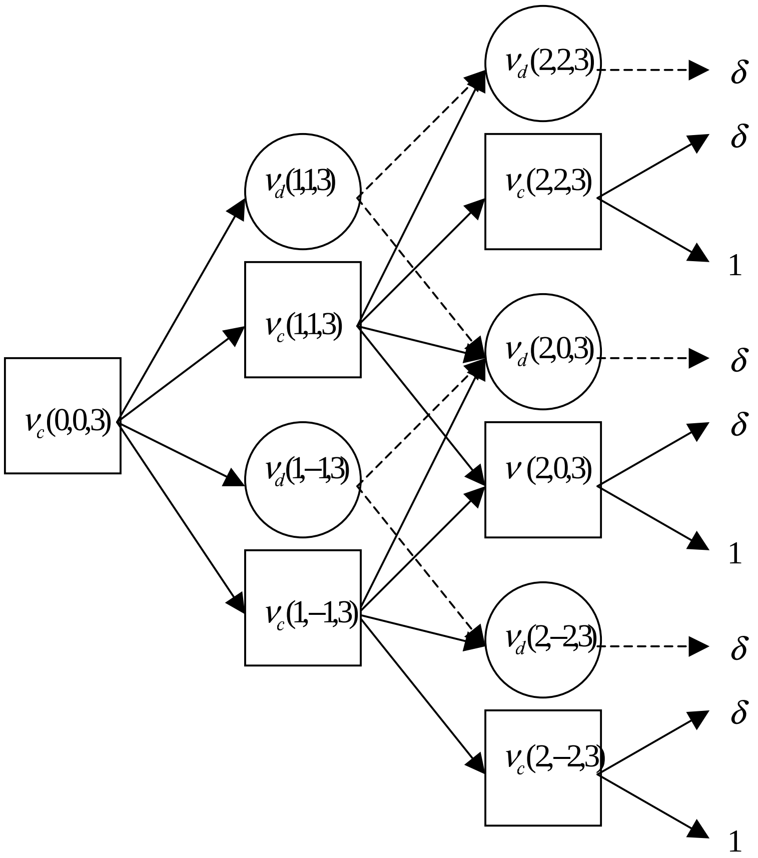

Fig 7.2

where,

$v_c(n,i,m)$ - value at period n, state i of a m maturing XYZ, c = currently not in default

$v_d(n,i,m)$ - value at period n, state i of a m maturing XYZ, d = default has occurred,

At period $n=0$, state $i=0$ the price of a XYZ zero coupon bond maturing in period m=3 is $v_c(0,0,3)$. The “Jarrow Turnbull-tree" describes the value process of this bond due to changes in the short rate and the possible event of default.

The BDT-tree fig 5.2 with discount factors $p(n,i)$ is needed to discount the values at each node in the Jarrow Turnbull tree when we are calculating backwards. We present the backward equations for the general case.

$$
\begin{aligned}
v_d(m-1,i,m) &= p(m-1,i)\delta \\
v_c(m-1,i,m) &= p(m-1,i)\left[\lambda\mu_{m-1}\delta + (1-\lambda\mu_{m-1})\right] \\
v_d(n,i,m) &= \frac{1}{2}p(n,i)\left[v_d(n+1,i+1,m) + v_d(n+1,i-1,m)\right] \\
v_c(n,i,m) &= \frac{1}{2}p(n,i)\left[\lambda\mu_n v_d(n+1,i+1,m) + (1-\lambda\mu_n)v_c(n+1,i+1,m) + \lambda\mu_n v_d(n+1,i-1,m) + (1-\lambda\mu_n)v_c(n+1,i-1,m)\right]
\end{aligned}
\tag{7.8}
$$

The two first equations in Eq.(7.8) are used one period before maturity of the bond, $n=m-1$, while the last two equations in Eq.(7.8) are used for $n<m-1$. Note that in default the payoff

is deterministic (receiving δ at time T) and the only change in price of the XYZ zero coupon bonds is due to changes of the short rate. At period n there are $n+1$ states for the short rate, $i=-n,-n+2,...,n-2,n$, and therefore a total of $2n+2$ outcomes in the Jarrow-Turnbull tree.

7.1 Coupon bearing risky bonds

Until now the Jarrow-Turnbull tree has been built up to represent zero-coupon bonds subject to credit risk. Later on, bond options on risky coupon bearing bonds will be priced. However, since each coupon payment can be seen as a zero coupon bond, a coupon bearing bond is nothing more than a collection of zeroes.

$$\upsilon_c^k(0,0,m)=\sum_{j=1}^{m} kv_c(0,0,j)+\upsilon_c(0,0,m) \tag{7.9}$$

where,

$\upsilon_c^k(0,0,m)$ - today's value of a m maturing coupon bearing bond with k % coupon

At period m, the coupon bearing bond pays one coupon payment plus the notional.
A Jarrow-Turnbull tree representing a coupon bearing bond with m coupon payments is therefore the sum of $m+1$ trees, representing zeroes with different maturity. There are m trees with notional $k\cdot 1$ and one with notional=1.

7.2 Pricing bond options in the Jarrow-Turnbull tree.

Assume that we want to price a European call with the right to sell one m-maturing risky bond, with strike price K at period n. First, the Jarrow-Turnbull tree is built up to cover the whole life of the risky bond (up to period m).

At period n, we have to check the value of the call for the $n+1$ states of default and the $n+1$ states of no default.

$$\begin{aligned} C_c(n,i) &= \max[\upsilon_c(n,i,m)-K,0] \qquad i=-n,-n+2,...,n-2,n \\ C_d(n,i) &= \max[\upsilon_d(n,i,m)-K,0] \qquad i=-n,-n+2,...,n-2,n \end{aligned} \tag{7.10}$$

Since all probabilities of default are calculated earlier when building up the tree, the only thing left to do is discounting back to the origin. The first step is shown in fig 7.3.

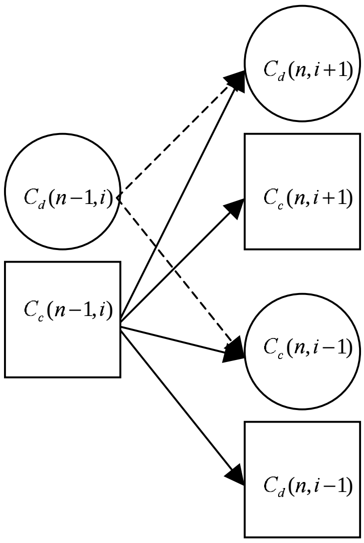

Fig 7.3

$$C_c(n,i)=\frac{1}{2}p(n-1,i)\left[\lambda\mu_n C_d(n,i+1)+(1-\lambda\mu_n)C_c(n,i+1)+\lambda\mu_n C_d(n,i-1)+(1-\lambda\mu_n)C_c(n,i-1)\right]$$
$$C_d(n-1,i)=\frac{1}{2}p(n-1,i)\left[C_d(n,i+1)+C_d(n,i-1)\right] \qquad (7.11)$$

The value of the call today is $C_c(0,0)$.

Bermudan and American options with the risky bond as the underlying can be valued with little extra effort. The holder of a Bermudan option has the right to exercise at specific dates in the future, whereas an American option can be exercised at any time.

The value of an American option at each node in the Jarrow-Turnbull tree is therefore the greater of its value if held or its value if exercised. The valuation of a Bermudan option is the same as in the case of a European call but with some periods where exercise is possible. At these periods the greater of the option value if held or exercised is chosen.

8 Application to Bermudan bond options

In this section we present the results of pricing Bermudan bond options. The underlying bonds are the German government bond and three different corporate bonds issued by German banks. The banks belong to different credit ratings, AAA, AA3, A1, where AAA is the highest and A1 is the lowest rating.

For simplicity reasons, we assume that if a corporate bond defaults, the holder of the bond will lose all the remaining coupon payments. The notional amount can never be lost. In the Jarrow-Turnbull model, this corresponds to a recovery of 0% for the coupons and 100 % recovery for the notional amount, which is set to 100.

Bermudan bond options will be priced for underlying bonds maturing in 6 and 10 years. Also the effect of changing the early exercise dates will be accounted for. The strike price of all the Bermudan options is set to par, that is 100. The probabilities of default were calculated using the interest rate spread, corresponding to each credit rating, see table 8.1 and 8.2.

Interest rate spreads

Table 8.1	AAA	AA3	A1
year			
2	0.00356091	0.00465071	0.00709045
3	0.0037244	0.00488407	0.00750897
4	0.00393287	0.00516165	0.00797151
5	0.00415556	0.00545251	0.00844645
6	0.00438563	0.00575091	0.00893098
7	0.00455541	0.00596954	0.00929179
8	0.00478705	0.00626921	0.00978046
9	0.00501954	0.00657037	0.01027441
10	0.00524987	0.00686937	0.01093889

Default probabilities

Table 8.2	AAA	AA3	A1
year			
2	0.00713526	0.00930879	0.0141574
3	0.00404498	0.00533904	0.00831583
4	0.00457163	0.0060076	0.00936374
5	0.0050259	0.00658404	0.0102777
6	0.00552385	0.0072208	0.011296
7	0.00600211	0.00783493	0.0122906
8	0.00649285	0.00846865	0.0133288
9	0.00698229	0.00910419	0.0143845
10	0.00738763	0.00963396	0.0152885

Since the interest rate spread was known only for 2-10 years, with one year time interval, we could only calculate nine probabilities of default. These were linearly interpolated to obtain values for each period in the Jarrow-Turnbull tree. The time period Δt was set equal to 3 months. The prices of bonds maturing in 6 years are shown in fig 8.1.

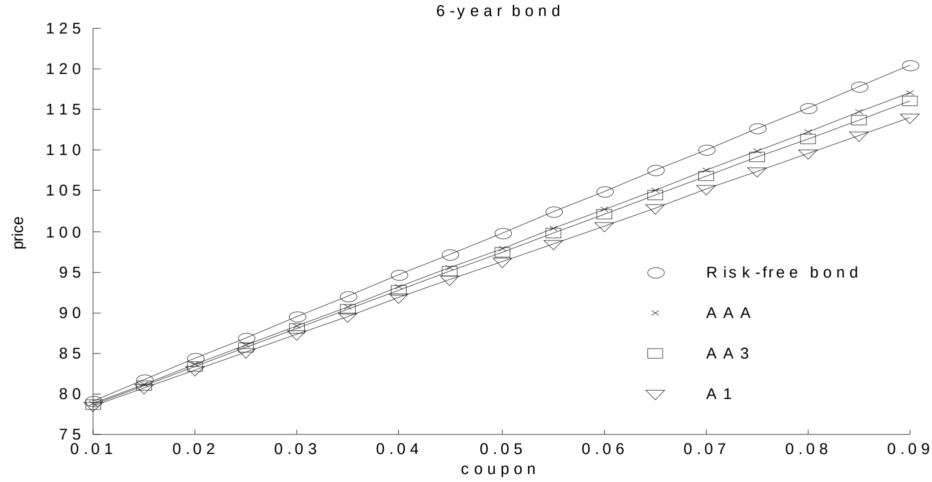

Fig 8.1

The bond prices increases linearly with higher coupon payments. We also see that the difference in price for bonds with different credit ratings increases with the coupon payments. This is due to the fact that higher coupons correspond to the risk of loosing more money in the case of default. In fig 8.2 the corporate bonds are plotted as relative prices of the risk-free government bond.

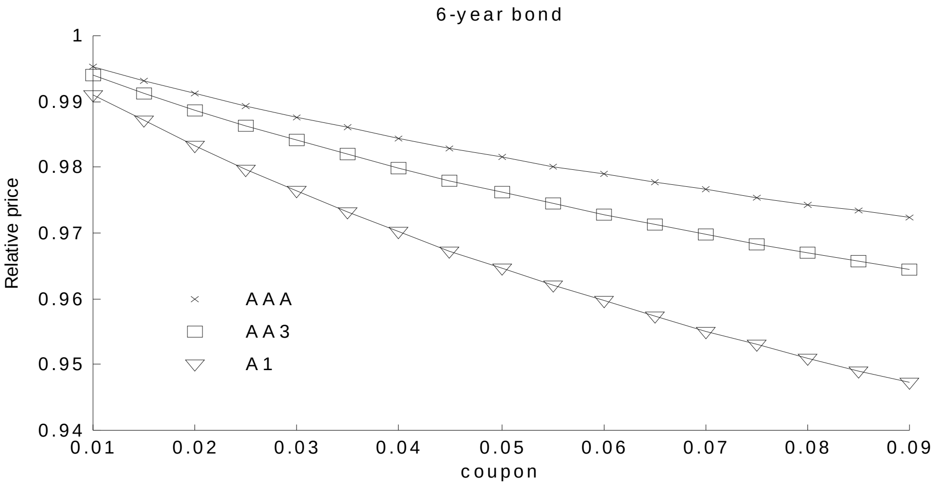

Fig 8.2

The value of a 9 % bond issued by an A1 rated bank is 95 % of the risk-free bond. Note that the difference in price between different credit ratings relative the risk-free bond is up to 3 %.

Bermudan call options prices with the bonds in fig 8.1 as the underlying are shown in fig 8.3. The holder of the Bermudan option has the right to exercise after 3,4, and 5 years.

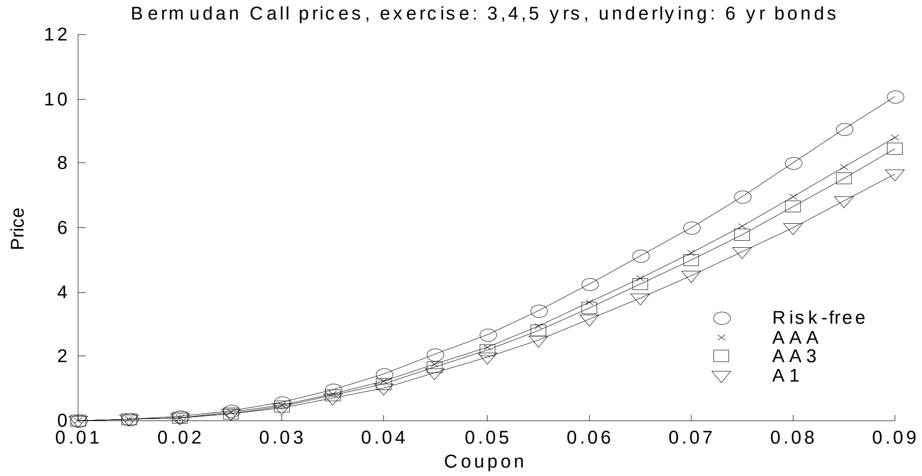

Fig 8.3

The prices of Bermudan call options in fig 8.3 increase exponentially with the coupon payment. Since the strike is set equal to the notional amount which is constant (100), higher coupons will rapidly increase the possible states in the Jarrow-Turnbull tree, where the options are in the money.

Approximately for coupon payments higher than 3.5 % of the notional, the value of the Bermudan options reaches 1 % of the notional. It is also from here, we can start to see a difference in price of the Bermudan options due to the default risk of the underlying bonds.

When the coupon reaches 9 %, the Bermudan option with an A1 rated bond as the underlying is worth 8 % of the notional while the Bermudan option on the risk-free bond is worth 10 %. It is clear that the default risk of the underlying corporate bond does have impact on the price of the Bermudan option. Ignoring the default risk, as in the naive approach, could in this case differ as much as 2 % of the notional of the underlying bond.

Bermudan put options on the 6-year bonds are plotted in fig 8.4.

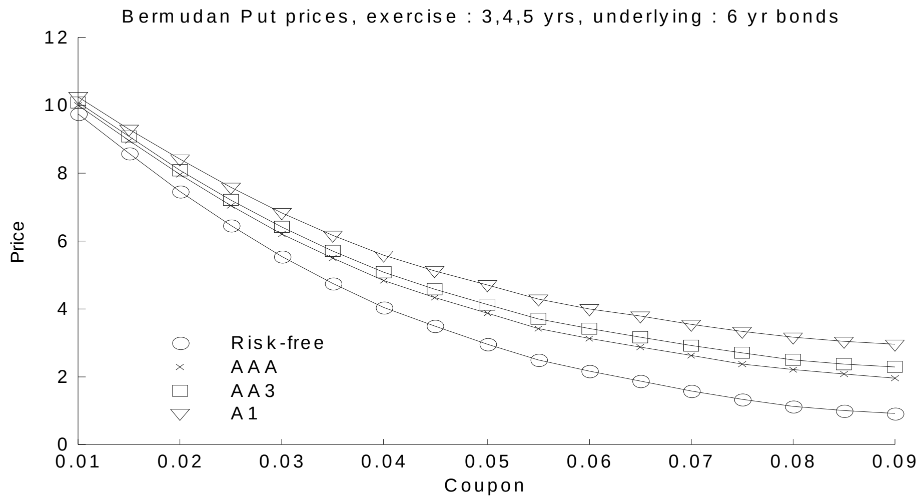

Fig 8.4

An event of default would drastically reduce the value of the corporate bond. Therefore it is now the price of the Bermudan put option with the lowest rated bond (A1) as the underlying, which is the highest. Note that for low coupon payments, the value of the Bermudan put options differ more than in the case of the Bermudan call (fig 8.3). This is due to the fact that the put option is much more worth for lower coupons than the call, which in turn affects the price spreads. However, for higher coupon payments, the difference of Bermudan puts with the lowest credit bond as the underlying, compared to the Bermudan put on the risk-free bond, differs with approximately the same as in the case of the call, that is 2%.

Until now, we have only analysed Bermudan bond options with 6-year bonds as the underlying. The prices of ten-year bonds with different ratings are plotted in fig 8.5.

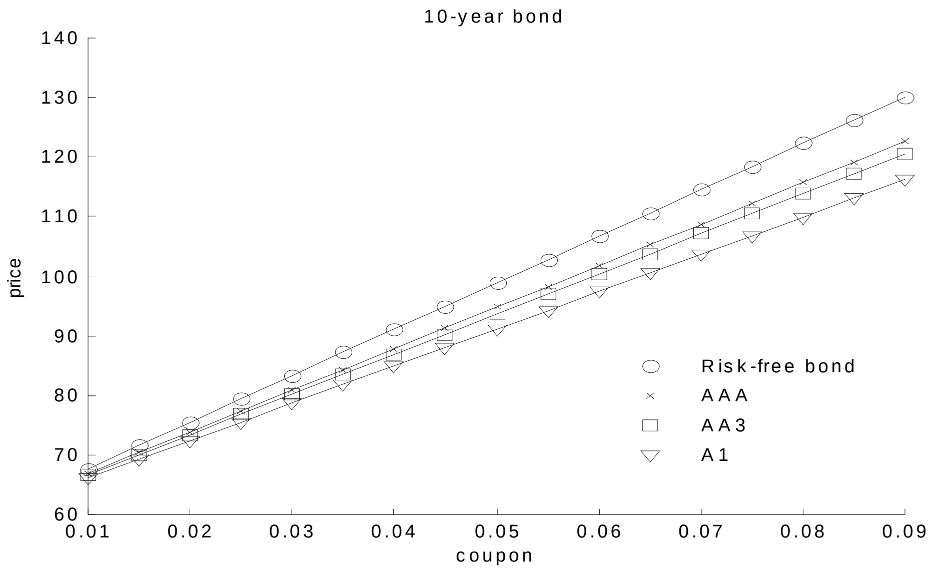

Fig 8.5

Compared to the prices of 6-year bonds in fig 8.1, the 10-year bond prices are more inclined. The notional payment is of course less worth if received after 10 years instead of 6 years. Since the notional will make up the most of the bond's value when coupon payments are small, the 6-year bond will have a higher value. When coupon payments increase, the value of a ten-year bond increases faster than for the 6-year bond since interest will be paid for a longer time.

The price of the corporate 10-year bonds relative the risk-free bond price is plotted in fig 8.6.

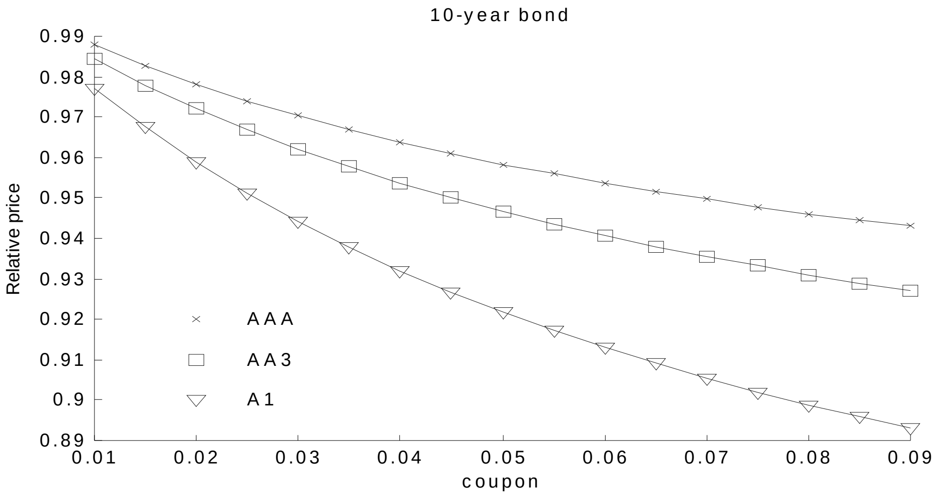

Fig 8.6

The relative corporate bond prices for the ten-year bond are lower than in the case of the 6-year bonds in fig 8.2. The reason is that during 10-years, 16 more coupon payments are to be made than for the 6-year bond. Since default implies that the coupon payments are cancelled, the 10-year bond is subject to more risk than the 6-year bond.

Bermudan call prices with 10-year bonds as the underlying are plotted in fig 8.7. The early exercise dates are set to 2,4, and 6 years.

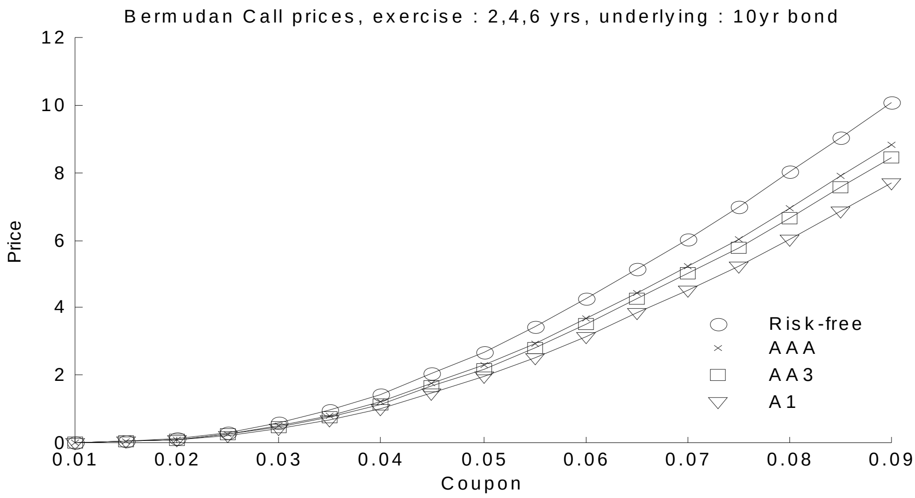

Fig 8.7

Fig 8.7 looks almost identical to fig 8.3. The exercise dates are rather close to before, which might be an explanation to the poor difference in price of the Bermudan calls, when the maturity of the underlying has changed. The fact that the default risk plays an important roll remains. When coupon payments are 9 %, the difference in price of the Bermudan call with the A1 rated bond compared to the risk free bond as the underlying, is still about 2 % of the notional.

To see how default risk is affected by the exercise dates, a plot of Bermudan call options are made, but now the early exercise dates are set equal to 5,7 and 9 years.

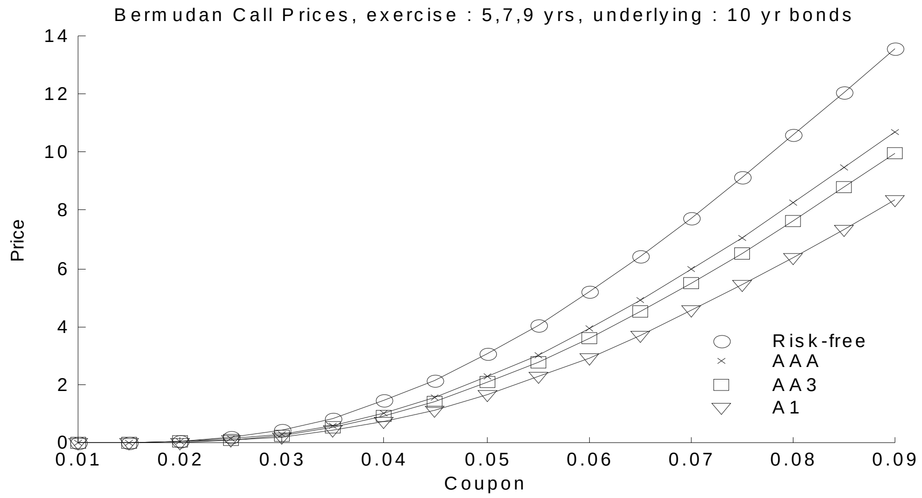

Fig 8.8

The effect of exercise dates being longer in the future is clearly shown in fig8.8. The much higher price of the call with the risk-free bond as the underlying, is the result of the increasing volatility. Having, the right to exercise later on, is the same as buying time value which will make the price of the call go up. The same reasoning can be made for the Bermudan call options with the corporate bonds as the underlying.

However, buying these call options always correspond to a higher risk If the holder of the call on the corporate bond does not have the right to exercise until after 5 years, this risk becomes even greater. The reason is that the event of default is much more likely to appear in the next 5-years than in 2-years time. Therefore the higher value due to the increasing volatility is, smaller for the Bermudan call options on the corporate bonds. In fact, the option price on the A1 rated bond does not show any significant change at all.

It is important to note that the difference in price of the Bermudan call options has increased. The Bermudan call with the risk-free bond as the underlying is up to 5-6 % of the notional amount more expensive, than the call on the A1 rated bond. We conclude that default risk will have even more impact when exercise is allowed further on in the future.

9 Conclusions

This thesis has focused on the theory behind and the implementation of an approach for pricing derivative securities subject to default risk presented by Robert A Jarrow and Stuart M Turnbull [7]. Their approach uses as input a given stochastic term structure of interest rates and interest rate spreads for different credit ratings observable in the market.

The risk free interest rates, corresponding to government zero coupon bonds are modelled using the Black-Derman-Toy model (BDT) which is calibrated to caps. It is assumed that the government bonds cannot default. The aim of the paper was further to compare the prices of Bermudan bond option prices generated by the Jarrow-Turnbull approach to the prices generated by the BDT-model in order to analyse the impact of ignoring the default risk of the underlying bond.

The numerical results from the Jarrow-Turnbull approach shows that ignoring the default risk of the underlying bonds can lead to Bermudan bond option prices being mispriced by up to 5-6 % of the notional amount of the bond. If the right to exercise was set later on in the future, the difference in price was even bigger. Of course, the difference in price of the underlying bond and the option prices are strongly related to how much money that is lost in the event of default.

In the application performed in this paper we assumed that only coupons were subject to default risk. This assumption is often fulfilled rather well since the notional amount of a corporate bond issued by a bank often is protected by legal systems. If this is not the case, it is also very easy to introduce a recovery rate for the notional, as described in chapter 7. The recovery rate of the coupons was set equal to 0 %. It is important to note that a higher recovery rate, would imply higher default probabilities of the underlying bond, since the expected loss of money should be reflected in the interest rate spread observed in the market.

The Jarrow-Turnbull approach handles the default risk in a new way. A big advantage is that all parameters used are observable. This is not the case in earlier approaches to model credit risk, where derivatives have been priced using the assets of a financial security as the underlying. The analytic results are also very elegant. In particular, Jarrow and Turnbull show that the price of a risky coupon bond, is the sum of the present value of its coupons, with explicit formulas for the discount factors.
It is therefore easy to incorporate different recovery rates on coupon payments or notional amounts of the bonds in the event of default.

One problem that was not resolved in the Jarrow and Turnbull's article [7], is the importance on the assumption on the assumed independence of the payout on default from the timing of default. Some statistics (Moody's special report, 1992) show that 1-year default rates for investment grade bonds have had little variation over the time period 1970-1991. Although speculative grade 1-year default rates exhibit more variation. However, this assumption needs to be analysed further.

In this paper the impact of default risk of corporate bonds has been taken into account when pricing Bermudan bond options. As a suggestion for future work, a study of changes in credit ratings, would be interesting. For example, how would the price of a Bermudan call be affected when the issuer of the underlying corporate bond is downgraded? In fact, Jarrow, Lando, and Turnbull presents an extension to incorporate these features in their article from 1997 [8]. This article takes the Jarrow-Turnbull model and characterises the bankruptcy process as a finite Markov chain in the firm's credit rating. It should be stated that in this approach more data is necessary, as for example the transition matrix with probabilities of jumps in the credit rating.

Appendix A

A1. Derivation of the dynamics of the BDT model

The continuous time BDT risk-neutral short-rate process has the form:

$$r(t) = u(t)\exp(\sigma(t)z(t)) \tag{A1}$$

Taking the logarithm of (A1)

$$\ln r(t) = \ln u(t) + \sigma(t)z(t) \tag{A2}$$

and derivating yields,

$$d\ln r(t) = \frac{\partial \ln u(t)}{\partial t}dt + \sigma(t)dz(t) + z(t)\sigma'(t)dt \tag{A3}$$

Solving for $z(t)$ in (A1)

$$z(t) = \frac{1}{\sigma(t)}\left(\ln r(t) - \ln u(t)\right) \tag{A4}$$

and replacing with (A4) in (A3),

$$d\ln r(t) = \left\{\frac{\partial \ln u(t)}{\partial t} + \frac{\sigma'(t)}{\sigma(t)}\left[\ln r(t) - \ln u(t)\right]\right\}dt + \sigma(t)dz(t) \tag{A5}$$

which can be written as,

$$d\ln r(t) = \left\{\theta(t) + f'(t)\left[\Psi(t) - \ln r(t)\right]\right\}dt + \sigma(t)dz(t) \tag{A6}$$

where

$$\theta(t) = \frac{\partial \ln u(t)}{\partial t}$$
$$f'(t) = -\frac{\partial \ln \sigma(t)}{\partial t}$$
$$\Psi(t) = \ln u(t)$$

A2. Proof of the unconditional variance of the short rate in the BDT model

A BDT lattice is fully described by a vector of medians $u(m)$ and a vector of short rate volatilities $\sigma(m)$ for periods $m = 0,1,..,k$ by the following formula.

$$r(m, j) = u(m)\exp\{2\sigma(m) j\sqrt{\Delta t}\} \tag{A7}$$

where Δt is the time step in years. At period m there are $m+1$ states $j = -m, -m+2, ..., m-2, m$. Define k random variables $y_1, y_2, ..., y_k$ by

$y_k = 1$ if an up move occurs at period k
$y_k = -1$ if a down move occurs at period k.

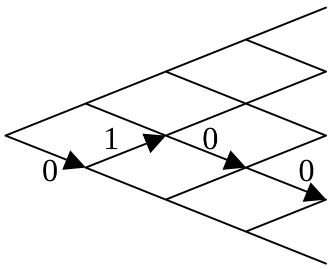

Fig A1.

Assume that the variables y_k are independent, and that the probability $P[y_k = 1] = P[y_k = -1] = \frac{1}{2}$. Now define the variable $X_k = \sum_{j=1}^{k} y_k$, which gives the level of the short rate at time $k\Delta t$,

$$r(k, X_k) = u(k)\exp\{2\sigma(k) X_k\sqrt{\Delta t}\} \tag{A8}$$

We want to calculate the expectation and the variance of the logarithm of this quantity, denoted $E[\ln r(k, X_k)]$ and $Var[\ln r(k, X_k)]$. By construction, the probability of X_k assuming value j is given by the equiprobable binomial distribution, and therefore

$$P[X_k = j] = C_k^j / 2^k \tag{A9}$$

with,

$$C_k^j = k!/((k-j)!\,j!) \tag{A10}$$

But the probability of the short rate assuming the jth value at time $k\Delta t$ is also given by

$$P[r(k,X_k)=u(k)\exp\{2\sigma(k)j\sqrt{\Delta t}\}]=P[X_k\ =j]=C_k^j\ /\ 2^k \tag{A11}$$

The expectation of the logarithm of the short rate is thus ($r(k,X_k)$ is abbreviated as r_k)

$$\begin{aligned}E[\ln r_k]&=\sum_{j=0}^{k}\left(\frac{1}{2}\right)^k C_k^j(\ln r_k+2\sigma(k)j\sqrt{\Delta t}\\&=\ln r_k\left(\frac{1}{2}\right)^k 2^k+\left(\frac{1}{2}\right)^k 2\sigma(k)\sqrt{\Delta t}\sum_{j=1}^{k}jC_k^j\end{aligned} \tag{A12}$$

Given the definition of Eq.(A10),

$$jC_k^j\ =kC_{k-1}^{j-1} \tag{A13}$$

and after substituting in (A12) one obtains,

$$E[\ln r_k]=\ln r_k\ +k\sigma(k)\sqrt{\Delta t} \tag{A14}$$

For the variance of the logarithm of the short rate we have,

$$\begin{aligned}Var[(\ln r_k)^2]&=\sum_{j=0}^{k}\left(\frac{1}{2}\right)^2 C_k^j(\ln r_k+2\sigma(k)j\sqrt{\Delta t})^2\\&=(\ln r_k)^2+2k\ln r_k\sigma(k)\sqrt{\Delta t}+4\sigma^2(k)\Delta t\sum_{j=1}^{k}j^2C_k^j\end{aligned} \tag{A15}$$

But the last term is simply equal to

$$j^2C_k^j\ =k(k-1)C_{k-2}^{j-2}+kC_{k-1}^{j-1} \tag{A16}$$

and therefore the last summation adds up to

$$\sum_{j=0}^{k}j^2C_k^j\ =k(k-1)2^{k-2}+k2^{k-1} \tag{A17}$$

It follows that the unconditional variance is given by

$$\begin{aligned}Var[\ln r_k]&=E[(\ln r_k)^2]-(E[\ln r_k])^2\\&=(\ln r_k)^2+2k\sigma(k)\sqrt{\Delta t}+\sigma^2(k)\ln r_k k(k+1)\Delta t-(\ln r_k+k\sigma(k)\sqrt{\Delta t})^2\\&=\sigma^2(k)k\Delta t\end{aligned} \tag{A18}$$

The expression (A18) shows that the unconditional variance of the logarithm of the short rate in the BDT model depends only on the final value of the instantaneous volatility of the short rate.

APPENDIX B

B.1 Fitting yield curves and forward rate curves with maximum smoothness

Fitting yield curves to current market data can be done by single-factor term structure models such as Hull and White (1993). This method will generally match the observable yield curve data very well but however, between observable data points, yield curve smoothing technique is necessary. Kenneth J. Adams and Donald R. Van Deventer provide an approach to yield curve fitting by introducing the ”maximum smoothness criterion”.

The objective is to fit observable points on the yield curve with the function of time that produces the smoothest possible forward rate curve. To do this a technique from numerical analyses is used. The smoothest possible forward rate curve on an interval (0,T) is defined as one that minimizes the functional

$$Z = \int_0^T \left[f''(0,s)\right]^2 ds \qquad \text{(B1)}$$

Subject to

$$\exp\left\{-\int_0^{t_i} f(0,s)ds\right\} = P(0,t_i) \qquad i = 1,2,..,m \qquad \text{(B2)}$$

where $P(0,t_i)$ represent the observed prices of zero coupon bonds with maturities t_i. Expressing the forward rate curve as a function of a specified form with a finite number of parameters may approach this problem. The maximum smoothness term structure can then be found within this parametric family, that is, it will be more smooth than that given by any other mathematical expression of the same degree and same functional form.

However, it would be more useful to determine the maximum smoothness term structure within all possible functional forms. This is possible due to the theorem provided by Oldrich Vasicek and stated in an article by Adams and Van Deventer [1].

Theorem

The term structure $f(0,t)$, $0 \le t \le T$ of forward rates that satisfies Eq.(B1) and Eq.(B2) is a fourth order spline with the cubic term absent given by

$$f(0,t) = c_i t^4 + b_i t + a_i \qquad t_{i-1} \le t < t_i \qquad i = 1,2,..,m+1 \qquad \text{(B3)}$$

where the maturities satisfy $0=t_0<t_1<\ldots<t_{m+1}<T$. The coefficients a_i, b_i, c_i, $i = 1,2,..,m+1$ satisfy the equations

$$c_i t_i^4 + b_i t_i + a_i = c_{i+1} t_i^4 + b_{i+1} t_i + a_{i+1} \qquad i = 1,2,..,m \qquad \text{(B4)}$$

$$4c_i t_i^3 + b_i = 4c_{i+1} t_i^3 + b_{i+1} \qquad i = 1,2,..,m \qquad \text{(B5)}$$

$$\frac{1}{5} c_i (t_i^5 - t_{i-1}^5) + \frac{1}{2} b_i (t_i^2 - t_{i-1}^2) + a_i (t_i - t_{i-1}) = -\log\left(\frac{P(0,t_i)}{P(0,t_{i-1})}\right) \qquad i = 1,2,..,m \qquad \text{(B6)}$$

$$c_{m+1} = 0 \qquad \text{(B7)}$$

For a proof of this theorem we refer to the article written by Kenneth J. Adams and Donald R. Van Deventer [1]. It is seen that the theorem specifies $3m+1$ equations for the $3m+3$ unknown parameters a_i, b_i, c_i , $i = 1,2,..,m+1$. The maximum smoothness solution is unique and can be obtained analytically as follows.

The objective function (B1) is proportional to

$$\mathrm{Z} = \sum_{i=1}^{m} c_i^{\,2} (t_i^{\,5} - t_{i-1}^{\,5}) \qquad \text{(B8)}$$

according to the term structure $f(t)$ stated by Oldrich Vasicek´s theorem. This function is quadratic in the parameters while the conditions Eq.(B4), Eq.(B5), Eq.(B6) and Eq.(B7) are all linear in the parameters. We have an unconstrained quadratic problem of the form:

min $x^T Dx$

subject to (QP)

Ax=b

(QP) has the solution $(I-A^T(AA^T)^{-1}A)Dx=0$ (B9)

Any two of these equations provide the remaining conditions on the parameters a_i, b_i, c_i. Two additional requirements may be stated:

1. $f'(T) = 0$ for the asymptotic behavior of the term structure

2. a_0=r which means that the instantaneous forward rate at time zero is equal to an observable rate r.

If both of the additional requirements are used, no equation from Eq.(B9) is needed.

B.2 Brent’s root finding algorithm

Brent’s method is an algorithm for finding the roots of a function without the use of its derivative. It combines root bracketing, bisection and inverse quadratic interpolation to achieve fast convergence and a root approximation that is within a positive tolerance 2δ.

While bisection assumes approximately linear behaviour between two prior root estimates, inverse quadratic interpolation uses three prior points to fit an inverse quadratic function (x as a function of y). The value at y = 0 is taken as the next estimate of the root x. The algorithm

makes sure that the new root falls inside the current brackets and that no illegal operation (such as division by zero) is performed.
When the quadratic interpolation gives a root, which lands outside of the given bounds, or when it doesn't converge fast enough a bisection step is taken.

The error of this algorithm never exceeds $6 \cdot \varepsilon |x| + 2t$, where ε is the relative machine precision, x is the value for which f(x) = 0 and t is a positive absolute tolerance.
Note that the algorithm only guarantees to find a root to the computed function and that this root may be nowhere near the mathematically defined function that the user is really interested in.

The algorithm

$$x = \frac{[y - f(x_1)][y - f(x_2)]x_3}{[f(x_3) - f(x_1)][f(x_3) - f(x_2)]} + \frac{[y - f(x_2)][y - f(x_3)]x_1}{[f(x_1) - f(x_2)][f(x_1) - f(x_3)]} +$$

$$+ \frac{[y - f(x_3)][y - f(x_1)]x_2}{[f(x_2) - f(x_3)][f(x_2) - f(x_1)]}$$

Here x_2 is the current best root estimate, x_1 is the last best estimate and x_3 is a point with sign ($f(x_3)$) = -sign ($f(x_2)$). The root is narrowed down to the boundary between x_3 and x_2.

Subsequent root estimates are obtained by setting y=0, giving $x = x_2 + \frac{P}{Q}$,

where
P = S[R(R-T)($x_3 - x_2$)- (1-R)($x_2 - x_1$)]
Q= (T-1)(R-1)(S-1)
with

$$R \equiv \frac{f(x_2)}{f(x_3)}$$

$$S \equiv \frac{f(x_2)}{f(x_1)}$$

$$T \equiv \frac{f(x_1)}{f(x_3)}$$

$$m = \frac{1}{2}(x_3 - x_2)$$

If $|x_2 - x_1| < tol$, $|f(x_1)| \le |f(x_2)|$ or $2|P| \ge 3|mQ|$ a bisection is performed instead of the above. The tolerance is denoted tol.

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