Implementation of the Black, Derman and Toy Model

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1. Introduction to Term Structure Models

Interest rate derivatives are instruments that are in some way contingent on interest rates (bonds, swaps or just simple loans that start at a future point in time). Such securities are extremely important because almost every financial transaction is exposed to interest rate risk – and interest rate derivatives provide the means for controlling this risk. In addition, same as with other derivative securities, interest rate derivatives may also be used to enhance the performance of investment portfolios. The interesting and crucial question is what are these instruments worth on the market and how can they be priced.

In analogy to stock options, interest rate derivatives depend on their underlying, i.e. the interest rate. Bond prices depend on the movement of interest rates, so do bond options. The question of pricing contingent claims on interest rates comes down to the question of how the underlying can be modeled. Future values are uncertain, but with the help of stochastic models it is possible to get information about possible interest rates. The tool needed is the *term structure* – the evolution of spot rates over time. One has to consider both, the term structure of interest rates and the term structure of interest rate volatilities. Term structure models, also known as yield curve models, describe the probabilistic behavior of all rates. They are more complicated than models used to describe a stock price or an exchange rate. This is because they are concerned with movements in an entire yield curve – not with changes to a single variable. As time passes, the individual interest rates in the term structure change. In addition, the shape of the curve itself is liable to change.

We have to distinguish between *equilibrium models* and *no-arbitrage models*. In an equilibrium model the initial term structure is an output from the model, in a no-arbitrage model it is an input to the model. Equilibrium models usually start with the assumption about economic variables and derive a process for the short-term risk-free rate *r*.[1] They then explore what the process implies for bond prices and option prices. The disadvantage of equilibrium models is that they do not automatically fit today's term structure. No-arbitrage models are designed to be exactly consistent with today's term structure. The idea is based on the risk-neutral pricing formula when a bond is valued over a single period of time with binominal lattice.

2. Term Structure Equation for Continuous Time

In our paper we prefer to use discrete time models, because the data available is always in discrete form and easier to compute. The continues time models do not provide us with useful solutions in the BDT world[2], nevertheless they are the fundamental and older part of interest rate derivatives. Continuous time models are more transparent and give a better understanding of the BDT model, therefore we here give a short general introduction into this material.

We assume that the following term structure equation[3] is given:

$$P_t + P_r \cdot \mu^{\Pi}(t,r) + \frac{1}{2} \cdot P_{rr} \cdot \sigma(t,r)^2 - P \cdot r(t) = 0$$

with

$$P_t = \frac{\partial P}{\partial t}\ ; \quad P_r = \frac{\partial P}{\partial r}\ ; \quad P_{rr} = \frac{\partial^2 P}{\partial r^2}$$

We have a second order partial differential equation. It is of course of great importance that the partial differential

[1] When *r(t)* is the only source of uncertainty for the term structure, the short rate is modeled by *One-Factor Models*.

[2] The continuous time models can lead to closed form mathematical solutions. But in the BDT case, we cannot find such a solution.

[3] Rudolf (2000) pp.38-40

equation above leads to solutions for interest derivatives. Not every Ito process can solve the term structure above.
It turns out that some Ito processes which are affine term structure models lead to a solution.
The zero bond price in time t to maturity T has the following form:

$$P(t,T) = e^{A(t,T)-B(t,T)\cdot r(t)} \qquad \text{where} \quad t < T$$

The drift and volatility have the general form:

$$\mu^{\Pi}(t,r) \equiv g(t) \cdot r(t) + h(t)$$

$$\sigma(t,r) \equiv \sqrt{c(t) \cdot r(t) + d(t)}$$

If the price P(t,T) above is given, we can derive P_r, P_{rr} and P_t as the following:

$$P_r = \frac{\partial P}{\partial t} = -B(t,T) \cdot e^{A(t,T)-B(t,T)\cdot r(t)} = -B \cdot P$$

$$P_{rr} = \frac{\partial^2 P}{\partial r^2} = B(t,T)^2 \cdot e^{A(t,T)-B(t,T)\cdot r(t)} = B^2 \cdot P$$

$$P_t = \frac{\partial P}{\partial t} = \left(A_t - B_t \cdot r\right) \cdot e^{A(t,T)-B(t,T)\cdot r(t)} = \left(A_t - B_t \cdot r\right) \cdot P$$

After substituting these three variables into the term structure equation we get

$$A_t(t,T) - \left[1 + B_t(t,T)\right] \cdot r(t) - B_t(t,T) \cdot \mu^{\Pi} + \frac{B_t(t,T)^2}{2} \cdot \sigma^2 = 0$$

Now we can also substitute μ^{Π} and σ into the equation above and get

$$A_t(t,T) - h(t) \cdot B(t,T) + \frac{B(t,T)^2}{2} \cdot d(t)$$

$$-r(t) \cdot \left[1 + B_t(t,T) + g(t,T) \cdot B(t,T) - \frac{B(t,T)^2}{2} \cdot c(t)\right] = 0$$

The equation above holds for all t, T and r under the following condition:

$$A_t(t,T) - h(t)\cdot B(t,T) + \frac{B(t,T)^2}{2}\cdot d(t) = 0$$

$$1 + B_t(t,T) + g(t)\cdot B(t,T) - \frac{B(t,T)^2}{2}\cdot c(t) = 0$$

The boundary conditions for solving this zero bond are:

$P(T,T) = 1$

$A(T,T) = 0$

$B(T,T) = 0$

The reason is that we know the price of the zero bond at maturity, so A(T,T) and B(T,T) must be zero. These two equations above lead to the Riccati-Problem for solving two non stochastic partial differential equations.

A general solution method for solving the term structure equation with an affine interest model could be to

1. Compare the coefficients $\mu^{\Pi}(t,r)$ and σ with the stochastic short rate process to identify the coefficients g(t), h(t), c(t) and d(t),
2. Substitute the coefficients into the two Riccati equations
3. Solve the two second order partial differential equations to get A(t,T) and B(t,T)

3. Overview - Basic Processes of One-Factor Models

Table I: Some basic single-factor models in continuous time[4]

Model	Process
Vasicek(1977) [Equilibrium model, Short rate model]	$dr_t = \left[\Phi - a \cdot r(t)\right] \cdot dt + \sigma \cdot dz^{\Pi}(t)$ dr = $a(b-r)$ dt + σ dz
Cox Ingersoll Ross (1985) [Equilibrium model, Short rate model]	$dr_t = \left[\Phi - a \cdot r(t)\right] \cdot dt + \sigma \cdot \sqrt{r} \cdot dz^{\Pi}(t)$ dr = $a(b-r)$ dt + c $\sqrt{r}$ dz
Ho Lee (1986) [First No-Arbitrage model]	$dr_t = \left[F_t(0,T) + \sigma^2 \cdot t\right] \cdot dt + \sigma \cdot \sqrt{r} \cdot dz^{\Pi}(t)$ dr = $\theta(t)$ dt + σ dz
Black Derman Toy (1990) [No-Arbitrage model, lognormal short rate model]	$\frac{dr_t}{r} = a(t) \cdot dt + \sigma(t) \cdot dz^{\Pi}(t)$ d ln r = $\theta(t)$ dt + σ dz
Hull White (1990) [No-Arbitrage model]	$dr_t = \left[\Phi(t) - a(t) \cdot r(t)\right] \cdot dt + \sigma(t) \cdot dz^{\Pi}(t)$ dr = [$\theta(t)$ - ar] dt + σ dz
Black Karasinski (1991)	$d \ln r_t = \left[\Phi(t) - a(t) \cdot \ln r(t)\right] \cdot dt + \sigma(t) \cdot dz^{\Pi}(t)$ d ln r = $(\theta - a \ln r)$ dt + σ dz
Heath Jarrow Morton (1992) [renown as a bridge between all term structure models; Forward rate model]	$dF(t,T) = \mu_F^{\Pi}(t,T) \cdot dt + \sigma_F(t,T) \cdot dz^{\Pi}(t)$

[4] See Rudolf (2000), p.64 and Clewlow, Strickland (1998)

4. The Black Derman and Toy Model (BDT)

4.1. Characteristics

The term structure model developed in 1990 by Fischer Black, Emanuel Derman and William Toy is a yield-based model which has proved popular with practitioners for valuing interest rate derivatives such as caps and swaptions etc. The Black, Derman and Toy model (henceforth BDT model) is a one-factor short-rate (no-arbitrage) model – all security prices and rates depend only on a single factor, the short rate – the annualized one-period interest rate. The current structure of long rates (yields on zero-coupon Treasury bonds) for various maturities and their estimated volatilities are used to construct a tree of possible future short rates. This tree can then be used to value interest-rate-sensitive securities.

Several assumptions are made for the model to hold:

- Changes in all bond yields are perfectly correlated.
- Expected returns on all securities over one period are equal.
- The short rates are log-normally distributed
- There exists no taxes or transaction costs.

As with the original *Ho and Lee* model, the model is developed algorithmically, describing the evolution of the term structure in a discrete-time binominal lattice framework. Although the algorithmic construction is rather opaque with regard to its assumptions about the evolution of the short rate, several authors have shown that the implied continuous time limit of the BDT model, as we take the limit of the size of the time step to zero, is given by the following stochastic differential equation[5] :

$$d \ln r(t) = \left[\theta(t) - \frac{\partial \sigma(t) / \partial t}{\sigma(t)} \ln r(t) \right] dt + \sigma(t) dz$$

[5] See Clewlow and Strickland (1998), p.221

This representation of the model allows to understand the assumption implicit in the model. The BDT model incorporates two independent functions of time, θ(t) and σ(t), chosen so that the model fits the term structure of spot interest rates and the term structure of spot rate volatilities. In contrast to the *Ho and Lee* and *Hull and White* model, in the BDT representation the short rates are log-normally distributed; with the resulting advantage that interest rates cannot become negative. An unfortunate consequence of the model is that for certain specifications of the volatility function σ(t) the short rate can be mean-fleeing rather than mean-reverting. It is popular among practitioners, partly for the simplicity of its calibration and partly because of its straightforward analytic results. The model furthermore has the advantage that the volatility unit is a percentage, confirming with the market conventions.

4.2. Modeling of an "artificial" Short-Rate Process

In this chapter we describe a model of interest rates that can be used to value any interest-rate-sensitive security. In explaining how it works, we concentrate on valuing options on Treasury bonds. We want to examine how the model works in an imaginary world in which changes in all bond yields are perfectly correlated, expected returns on all securities over one period are equal, short rates at any time are lognormally distributed and there are neither taxes nor trading costs.

We can value a zero bond of any maturity (providing our tree of future short rates goes out far enough) using backward induction. We simply start with the security's face value at maturity and find the price at each node earlier by discounting future prices using the short rate at that node. The term structure of interest rates is quoted in yields, rather than prices. Today's annual yield, y, of the N-zero in terms of its price, *S*, is given by the *y* that satisfies the following equation (1). Similarly the yields (up and down) one year from now corresponding to the prices S_u and S_d are given by equation (2).

$$S = \frac{100}{(1+y)^N} \quad (1), \qquad S_{u,d} = \frac{100}{(1+y_{u,d})^{N-1}} \quad (2)$$

We want to find the short rates that assure that the model's term structure matches today's market term structure. Using Table II we can illustrate how to find short rates one period in the future.[6]

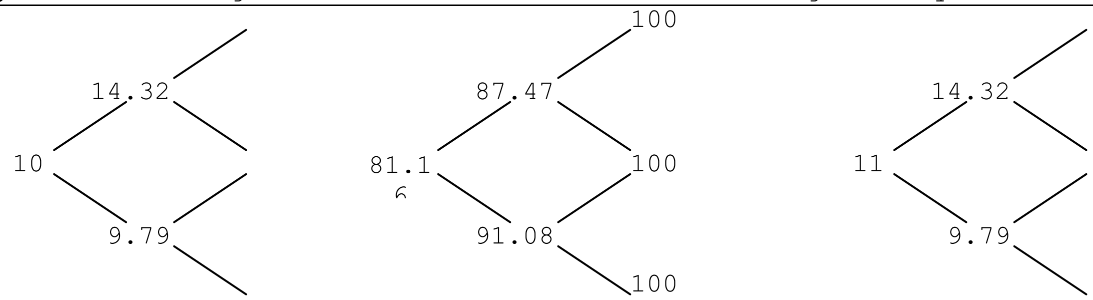
Maturity (yrs)	Yield (%)	Yield Volatility (%)
1	10	20
2	11	19
3	12	18
4	12.5	17
5	13	16

Table II – A Sample Term Structure

[6] Example is taken from Black, Derman and Toy (1990), pp.33-39

The unknown future short rates r_u and r_d should be such that the price and volatility of our two-year zero bond match the price and volatility given in Table II. We know that today's short rate is 10 per cent. Suppose we make a guess for r_u and r_d assuming that r_u=14.32 and r_d=9.79, see Figure I a. A two-year zero has a price of EUR 100 at all nodes at the end of the second period, no matter what short rate prevails. We can find the one-year prices by discounting the expected two year prices by r_u and r_d; we get prices of EUR 87.47 and EUR 91.08, see Figure I b. Using equation (2) we find that yields of 14.32 and 9.79 per cent indeed correspond to these prices. (e.g. 87.47=100/1+y_u, so y_u=14.32%). Today's price is given by equation (1) by discounting the expected one-year out price by today's short rate, i.e. (0.5(87.47)+0.5(91.08))/1.1=81.16. We can get today's yield for the two year zero, y_2, using equation (1) with today's price as *S*.

Figure I – Finding out the One-Year Short Rates (using a two-year zero)



(a) Rate Tree (Guessed values)

(b) Price Tree (Discounting the price at maturity with ru and rd to get the prices in t=1; then discounting these using today's short rate in equation (1))

(c) Yield Tree Using equation 2 to calculate the yields in t=1, for today : 81.16=100/(1+y)^2

The volatility of this two year yield is defined as the natural logarithm of the ratio of the one-year yields : $\sigma_2 = \frac{\ln {}^{14.33}/_{9.79}}{2} = 19\%$.

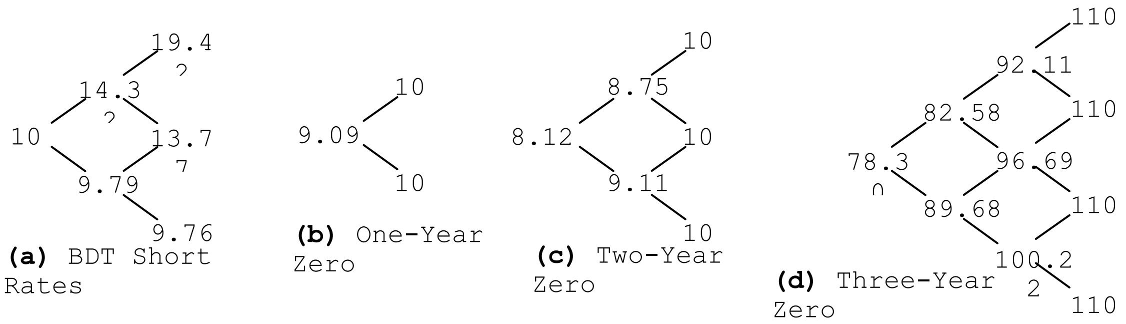
With the one-year short rates we have guessed, the two-year zero bond's yield and yield volatility match those in the observed term structure of Table II. This means that our guesses for r_u and r_d were right. Had they been wrong, we would have found the correct ones by trial and error.

So an initial short rate of 10 per cent followed by equally probable one-year short rates of 14.32 and 9.79 per cent guarantee that our model matches the first two years of the term structure.

Valuing Options on Treasury Bonds

We can use the model to value a bond option. But before we can value Treasury bond options, we need to find the future prices of the bond at various nodes on the tree. For our example shown in Figure II and III, we consider a Treasury with a 10 per cent coupon, a face value of EUR 100 and three years left to maturity. The application of the BDT model provided us with the short rates, which are shown in Figure II a.

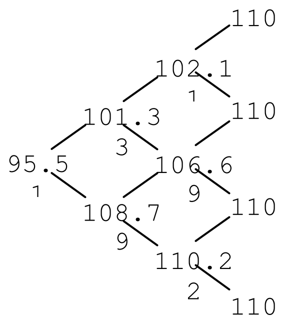
Figure II – Valuation of a portfolio of three zeroes



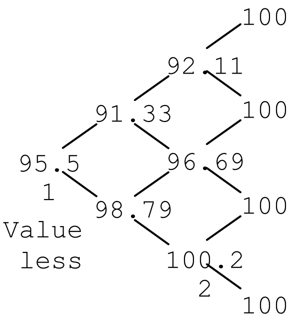
For convenience, we can consider this 10 per cent Three-year Treasury as a portfolio of three zero-coupon bonds[7]. See Figure II panels (b, c, d). The tree shown in panel (a) was built, the same way as explained in the previous section of this paper, to value all zero's according to today's yield curve. Figure III a below shows the price of the 10 per cent Treasury as the sum of the present values of the zeroes – EUR 95.51. The tree in Figure III b shows the Three-year Treasury prices obtained after subtracting EUR 10 of accrued interest on each coupon date.

Figure III – Three-Year Treasury Values

[7] a one-year zero with face-value of EUR 10, a two-year zero with a EUR 10 face-value and a three-year zero with a EUR 110 face-value



(a) Present Value of Portfolio
(Fig. II b+c+d = 3-year-

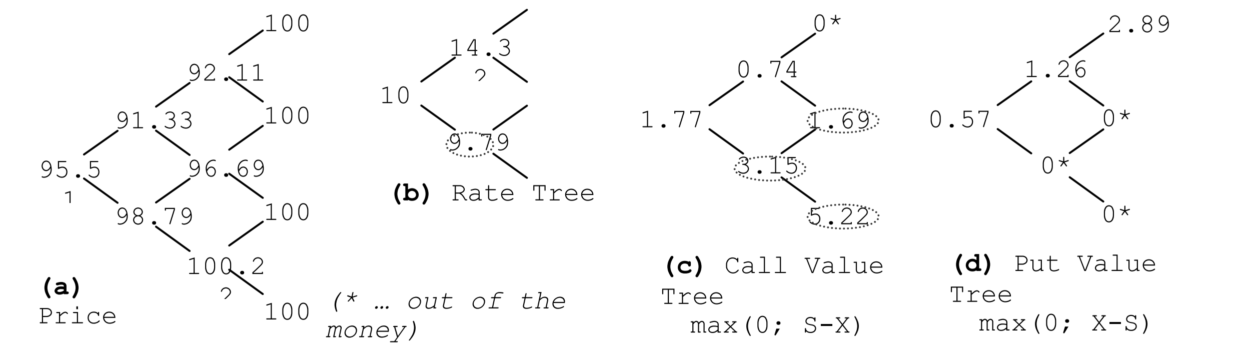


(b) Price (Present Value of 3-year-treasury less accrued interest)

Having found a price for the Treasury – EUR 95.51. We can easily value options on this security – e.g. a two-year European call and put struck at EUR 95. The process is shown in Figure IV. At expiration the difference between the bond's price and the strike price is calculated to get the calls (vice versa- the puts) value. Is the option out of the money, its value is zero. To get the possible values one-year before expiration we discount with the short rate for that period.

For example $\frac{0.5(1.69)+0.5(5.22)}{1+9.79\%}=3.15$

Figure IV – Two-Year European Options on a Three-Year Treasury



We have price European-style options by finding their values at any node as the discounted expected value one step in the future. American-style options can be valued in a similar manner with little extra effort.

In the beginning of this chapter we outlined the approach taken by Black, Derman and Toy in their original publication[8] of their model. They indicate a trial and error procedure, without providing any hints to facilitate the guessing game. Jamishidian[9] (1991) improved the implementation of the original BDT model by implementing state-contingent prices and forward induction to construct the short interest rate tree.

[8] Black, Derman and Toy "A One-Factor Model of Interest Rates and Its Application to Treasury Bond Options", Financial Analysts Journal, Jan-Feb 1990, pp.33-39

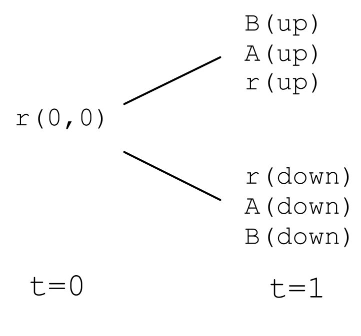
[9] Jamshidian, "Forward Induction and Construction of Yield Curve Diffusion Models", Journal of Fixed Income, June 1991, pp.62-74

Bjerksund and Stensland [10] developed an alternative to Jamshidian's forward induction method which proves to be even more efficient. With the help of two formulas they provide a closed form solution to the calibration problem. Given the initial yield and volatility curves a bond price tree is modeled that helps to approximate the short interest rate tree. The idea is to use information from the calculated tree to adjust input, which is used to generate a new tree by the two approximation formulas. Avoiding iteration procedures they assure shorter computation times and accurate results.

4.3. The BDT-Model and Reality

The BDT model offers in comparison to the Ho-Lee model more flexibility. In the case of constant volatility the expected yield of the Ho-Lee model moves exactly parallel, but the BDT model allows more complex changes in the yield-curve shape.

Figure V – short rate [11]



The short rate can be calculated by

$$\frac{A(up)-A(down)}{B(up)-B(down)}=\frac{\dfrac{A(up)-A(down)}{r(up)-r(down)}}{\dfrac{B(up)-B(down)}{r(up)-r(down)}}$$

[10] Bjerksund, Stensland "Implementation of the Black-Derman-Toy Model", Journal of Fixed Income, Vol.6(2), Sept 1996, pp.66-75

[11] See Rebonato (2002) p.265

From the equation above (right side) we can easily derive the sensitivity of Bond prices of different maturities to changes in the short rates.[12]

The sensitivity of the short rates is strongly dependent on the shape of the yield curve. Upward sloping term structure tend to produce an elasticity above 1.

It must be stressed that using BDT, which is a one-factor model, does not mean that the yield curve is forced to move parallel. The crucial point is that only one source of uncertainty is allowed to affect the different rates. In contrast to linearly independent rates, a one factor model implies that all rates are perfectly correlated. Of course, rates with different maturity are not perfectly correlated. One factor models are brutally simplifying real life. So two (or three) factor models would be a better choice to match the rates. [13]

[12] See Rebonato (2002) p.264

[13] Francke (2000) p.11-14

Mainly three advantages using one factor models rather than two or three factor models can be mentioned:

1. It is easier to implement
2. It takes much less computer time
3. It is much easier to calibrate

The ease of calibration to caps is one of the advantages in the case of the BDT model. It is considered by many practitioners to outperform all other one-factor models.

The BDT model suffers from two important disadvantages[14]:

- Substantial inability to handle conditions where the impact of a second factor could be of relevance because of the one-factor model
- Inability to specify the volatility of yields of different maturities independently of future volatility of the short rate

An exact match of the volatilities of yields of different maturities should not be expected and, even if actually observed, should be regarded as a little more than fortuitous.

Another disadvantage of the BDT model is that the fundamental idea of a short rate process, that follows a mean reversion, does not hold under the following circumstance. If the continuous time BDT risk neutral short rate process has the form:

$$d\ln r(t) = \{\theta(t) + f' \cdot [\psi(t) - \ln r(t)]\} \cdot dt + \sigma(t) \cdot dz(t)$$

where:

$u(t)$.. is the median of the short rate distribution at time t

$$\theta(t) = \frac{\partial \ln u(t)}{\partial t}$$

$$f' = -\frac{\partial \ln \sigma(t)}{\partial t}$$

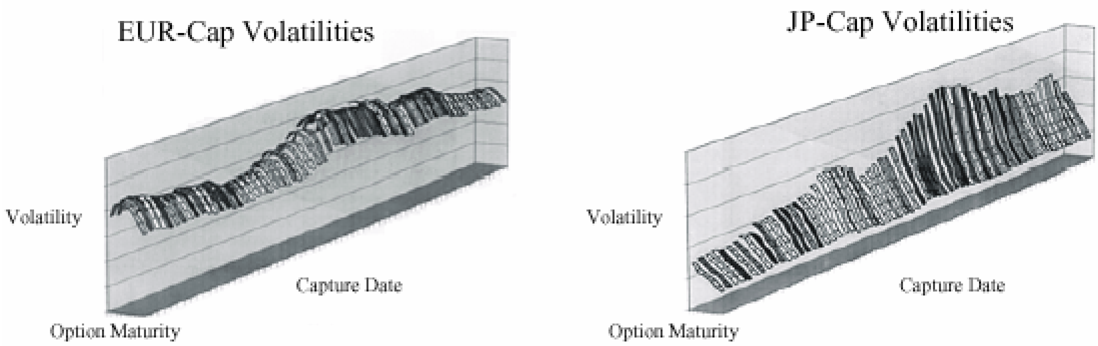
$$\psi(t) = \ln u(t)$$

14 See Rebonato (2002) p.268

For constant volatility ($\sigma = const$), $f' = 0$, the BDT model does not display any mean reversion.

Furthermore we have to note that volatility term structures are not necessary the same for different markets. The BDT model is for example used in the US- and the European market where cap volatilities are declining rather smoothly possibly after an initial hump. This is not the case for the Japanese cap volatilities, which decrease rapidly with option maturity. See Figure VI.

Figure VI - Different cap volatility term structure for the European and the Japanese markets



It would not be appropriate to use the BDT model in the Japanese market, because of the sharp decrease in volatility which would imply that we have more information about a future time period than about an earlier time period, which is not correct.[15]

5. Implementation and Application of the BDT-Model

In this chapter we want to outline the implementation of the BDT model using a spreadsheet. To match our model to the observed term structure we need a spreadsheet package that includes an equation-solving routine[16]. The spreadsheet

[15] For further details see Francke (2000), pp.13-14
[16] For our demonstration we used Microsoft Excel and its "Solver"

equation takes advantage of the forward equation and is an appropriate method when the number of periods is not large.
A simple way[17] of writing our model is to assume that the values in the short rate (r_{ks}) lattice are of the form

$$r_{ks} = a_k e^{b_k s}$$

We index the nodes of a short rate lattice according to the format (*k,s*), where *k* is the time (*k*=0,1,…,*n*) and *s* is the state (*s*=0,1,…,*k*). Here a_k is a measure of the aggregate drift, b_k represents the volatility of the logarithm of the short rate from time *k*-1 to *k*. Many practitioners choose to fit the rate structure only[18], holding the future short rate volatility constant[19]. So in the simplest version of the model, the values of b_k are all equal to one value *b*. The a_k's are then assigned so that the implied term structure matches the observed term structure.

(**Step 1**) We get the data of a yield curve and paste it into Excel.[20] To match it to our BDT model, we have to make assumptions concerning volatility. For our case we suppose to have measured the volatility to be 0.01 per year, which means that the short rate is likely to fluctuate about one percentage point during a year. (Inputs are shown with grey background in Figure VII a)
(**Step 2**) We introduce a row for the parameters a_k. These parameters are considered variable by the program. Based on these parameters the short rate lattice is constructed. We can enter some values close to the observed spot rate, so we can neatly see the development of the lattices. What we have done so far is shown in Figure VII a.

[17] See Luenberger (1998), p.400
[18] For a justification of this see Clewlow and Strickland (1998) Section 7.7. (pp.222-223)
[19] The convergent continuous time limit as shown on page 8 therefore reduces to the following equation : $d \ln r = \theta(t)\, dt + \sigma\, dz$. This process can be seen as a lognormal version of the Ho and Lee model.
[20] In our case we obtained the spot rate curve from the ÖKB website (http://www.profitweb.at/apps/ yieldcourse/index.jsp).

(**Step 3**) We use the BDT model ($r_{ks} = a_k e^{b\,s}$) to construct our short rate lattice.

(**Step 4**) Using the short rates (shown in Figure VII b) we can construct a new lattice for the elementary prices/state prices with the forward equations[21] shown below.

(Here $d_{k,s-1}$ and $d_{k,s}$ are the one period discount factors (determined from the short rates at those nodes).

Three equations for the forward recursion :

- For the middle branch : $P_0(k+1,s) = \frac{1}{2}\left[d_{k,s-1}P_0(k,s-1) + d_{k,s}P_0(k,s)\right]$
- For going down : $P_0(k+1,0) = \frac{1}{2}d_{k,0}P_0(k,0)$
- For going up : $P_0(k+1,k+1) = \frac{1}{2}d_{k,k}P_0(k,k)$

(**Step 5**) The sum of the elements in any column gives us the price of a zero-coupon bond with maturity at that date. From these prices the spot rates can be directly computed. (Figure VII d)

(**Step 6**) We introduce one more row for the (squared) difference between the observed spot rate and the one that we just computed.

(**Step 7**) Now we run the equation-solving routine, which adjusts the *a* values until the sum of errors is minimized, i.e. until the calculated spot rate equals the assumed spot rate in the second row.

(**Step 8**) We can now use the results for valuing interest rate contingent derivatives such as bond options, caps, floors, swaptions.

[21] The method of Forward induction was first introduced by Jamishidian (1991).

Figure VII – Using a spreadsheet program to implement the BDT model

Year	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
Spot rate I	2,83	2,9	3,22	3,62	4,01	4,35	4,64	4,88	5,08	5,12	
a	2,83	2,940	3,786	4,686	5,365	5,770	6,023	6,130	6,178	4,916	
b	0,020										

(a)

State										
9		*short rates*								5,89
8		*(Black, Derman and Toy)*							7,25	5,77
7								7,05	7,11	5,66
6							6,79	6,91	6,97	5,54
5						6,38	6,66	6,77	6,83	5,43
4					5,81	6,25	6,52	6,64	6,69	5,33
3				4,98	5,70	6,13	6,40	6,51	6,56	5,22
2			3,94	4,88	5,58	6,01	6,27	6,38	6,43	5,12
1		3,00	3,86	4,78	5,47	5,89	6,14	6,25	6,30	5,02
0	2,83	2,94	3,79	4,69	5,36	5,77	6,02	6,13	6,18	4,92

(b)

State											0,001
9		*state prices*								0,001	0,006
8		*(through forward induction)*							0,003	0,011	0,026
7								0,006	0,021	0,045	0,070
6							0,012	0,039	0,074	0,104	0,124
5						0,026	0,072	0,119	0,149	0,157	0,150
4					0,054	0,128	0,181	0,199	0,187	0,158	0,125
3				0,114	0,217	0,256	0,242	0,199	0,150	0,106	0,072
2			0,236	0,341	0,325	0,257	0,182	0,120	0,075	0,046	0,027
1		0,486	0,472	0,341	0,217	0,129	0,073	0,040	0,022	0,011	0,006
0	1,000	0,486	0,236	0,114	0,054	0,026	0,012	0,006	0,003	0,001	0,001

(c)

P(0,k)	1,000	0,972	0,944	0,909	0,867	0,822	0,775	0,728	0,683	0,640	0,607
Spot rate II		2,83	2,9	3,22	3,62	4,01	4,35	4,64	4,88	5,08	5,12
Errors		7E-14	1E-13	2E-12	1E-11	1E-10	5E-10	1E-09	9E-10	5E-10	6E-11
Sum of errors	3E-09										

(d)

The implementation in a Microsoft Excel spreadsheet can be found on a floppy disc attached to this seminar paper. (Please enable macros and the add-in "Solver".)

6. List of Symbols and Abbreviations

a	mean reversion factor
$P(t,T)$	value (price) of the zero bond at time t with maturity time T
σ	short rate volatility
μ	drift of the short rate
$\Phi(t)$	time various drift in Hull-White model
t	time
T	time at maturity
dr	infinitesimal increment in short rate
dt	infinitesimal increment of time
dz	infinitesimal increment in a standard Wiener Process
Π	risk neutral Martingale measure
Φ	forward measure
$F(t,T)$	forward rate at time t to time T
K	strike or exercise price of a contingent claim
S	asset price
r	continuous or simply compounded interest rate over one time step
$\theta(t)$	time dependent drift
r_{ks}	short rate in time k and state s in the lattice
a	aggregate drift
b	volatility of the logarithm of the short rate
k	time in the lattice
s	state in the lattice
$d_{k,s-1}$, $d_{k,s}$	one period discount factors of the forward equation

7. References

- Petter Bjerksund, Gunnar Stensland "Implementation of the Black-Derman-Toy Model", *Journal of Fixed Income*, Vol.6(2), Sept 1996, pp.67-75
- Fischer Black, Emanuel Derman and William Toy "A One-Factor Model of Interest Rates and Its Application to Treasury Bond Options", *Financial Analysts Journal*, Jan-Feb 1990, pp.33-39
- Les Clewlow and Chris Strickland, "Implementing Derivative Models: Numerical Methods", Wiley 1998
- Otto Francke, "The Impact of Default Risk when Pricing American Bond Options when using the Jarrow-Turnbull Approach", Master Thesis, Mathematical Statistics Institute, KTH Stockholm, 2000
- John C. Hull, "Options, Futures and other Derivatives", Prentice Hall 1997
- Farshid Jamshidian, "Forward Induction and Construction of Yield Curve Diffusion Models", *Journal of Fixed Income*, June 1991, pp.62-74
- David G. Luenberger, "Investment Science", Oxford Univ. Press 1998
- Riccardo Rebonato, "Interest Rate Option Models – Understanding, Analyzing and using Models for Exotic Interest Rate Options", Wiley 1997
- Markus Rudolf, "Zinsstrukturmodelle", Physica Verlag 2000
- Rémi Vrignaud, "Valuation of interest derivatives with lattice models", Master Thesis, Departement of Business Studies, University of Vienna, 2000

8. Suggestions for further reading

- T.G., Bali, Karagozoglu "Implementation of the BDT-Model with Different Volatility Estimators", *Journal of Fixed Income*, March 1999, pp.24-34
- Fischer Black, Piotr Karasinsky , "Bond and Option Pricing when Short Rates are Lognormal", *Financial Analysts Journal*, July-August 1991, pp.52-59
- John Hull, Alan White, "New Ways with the Yield Curve", *Risk*, October 1990a
- John Hull, Alan White, "One Factor Interest-Rate Models and the Valuation of Interest-Rate Derivative Securities", *Journal of Financial and Quantitative Analysis*, Vol. 28, June 1993, pp.235-254
- John Hull, Alan White, "Pricing Interest-Rate Derivative Securities", *Review of Financial Studies*, Vol. 4, 1990b, pp.573-591
- R. Jarrow and S. Turnbull "Pricing Derivatives on Financial Securities subject to Credit Risk". *The Journal of Finance*, March 1995, pp. 53-85
- R. Jarrow, D. Lando and Turnbull, "A Markov Model for the Term Structure of Credit Risk Spreads". *The review of financial studies*, 1997 Vol. 10, No.2.