Construction of interest rate binomial tree for Black-Derman-Toy model using Arrow-Debreu prices

We shall give a description on how to construct an interest rate binomial tree for Black-Derman-Toy model using Arrow-Debreu prices (see Appendix) . To start, lets define some notation. For $t = 1, 2, 3, \ldots$, let

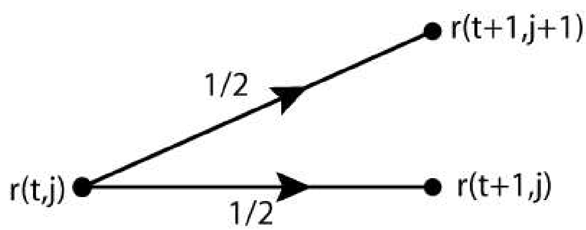
- $D(t)$ be the discount factor over time period $[0, t]$. $D(t)$ could be thought of as the value at $t = 0$ of a \$1 face value default free zero bond that matures at time t.
- $r(t)$ the interest rate over $[0, t]$. Note that

$$D(t) = \begin{cases} e^{-t \cdot r(t)} & \text{for continuously compounded interest} \\ \frac{1}{(1+r(t))^t} & \text{for simple interest} \end{cases}$$

- $\sigma(t)$ be the volatility, with respect to the risk neutral probability, of the interest rate at time t.
- $D(t, j)$ be the discount factor at time t and state j, at (t, j) for short, over the time period $[t, t+1]$.
- $r(t, j)$ be the spot interest rate at (t, j) over time period $[t, t+1]$. Note that

$$D(t,j) = \begin{cases} e^{-r(t,j)} & \text{for continuously compounded interest} \\ \frac{1}{1+r(t,j)} & \text{for simple interest} \end{cases}$$

Note that $r(0, 0) = r(1)$. At each time t, we may assume without loss of generality that $r(t, j)$ will go up to $r(t + 1, j + 1)$ with neutral probability $\frac{1}{2}$. Hence $r(t, j)$ will go down to $r(t + 1, j)$ with neutral probability $\frac{1}{2}$.



Suppose for $t \geq 0$, we have

$$r(t + 1, j + 1) = r(t + 1, j) \cdot e^{2\sigma(t+1)} \tag{1}$$

Then

$$D(t,j) = \begin{cases} e^{-r(t,0)e^{2j\sigma(t)}} & \text{for continuously compounded interest} \\ \frac{1}{1+r(t,0)e^{2j\sigma(t)}} & \text{for simple interest} \end{cases} \tag{2}$$

Given $D(1), D(2), \ldots, D(n)$ and $\sigma(1), \sigma(2), \ldots, \sigma(n-1)$, where $n \geq 2$, we now show how to find $r(t, j)$ inductively,where $1 \leq t \leq n-1$, $0 \leq j \leq t$, which satisfies (1) and there is no arbitrage opportunity. These $r(t, j)$'s are discretisation of the Black-Derman-Toy model

$$d \ln r = \theta(t)dt + \sigma(t)dW$$

Please consult [1, Chapter 8, Section 4], [2, Chapter 18] and [3, Chapter 15] for more details.

At time $t = 0$, consider

- portfolio A that consists of a zero bond which matures at time $t = 2$ with a face value of \$1.
- portfolio B that consists of a derivative which pays

$$\begin{cases} D(1,0) & \text{at } (1,0) \\ D(1,1) & \text{at } (1,1) \end{cases}$$

The value of portfolio A at time $t = 0$ is $D(2)$. The value of portfolio B at time $t = 0$ is $G(1,0)D(1,0) + G(1,1)D(1,1)$, where $G(t,j)$'s are the Arrow-Debreu prices and they are known (see Appendix). As both portfolios have the same payoff at $t = 1$, by the no arbitrage argument, their value at time $t = 0$ must be the same. Hence

$$D(2) = G(1,0)D(1,0) + G(1,1)D(1,1) \tag{3}$$

From (2), we can express $D(1,0), D(1,1)$ in terms of $r(1,0)$. Hence (3) would become an equation with one unknown $r(1,0)$. $r(1,0)$ could be solved using a numerical method. Once we know $r(1,0)$, $r(2,0)$ follows from (1).

Now that we have worked out the spot rates at time $t = 1$, we move on to time $t = 2$.

At time $t = 0$, consider (new portfolios)

- portfolio A that consists of a zero bond which matures at time $t = 3$ with a face value of \$1.
- portfolio B that consists of a derivative which pays

$$\begin{cases} D(2,0) & \text{at } (2,0) \\ D(2,1) & \text{at } (2,1) \\ D(2,2) & \text{at } (2,2) \end{cases}$$

Both portfolios A and B have the same payoff at time $t = 2$. By the no arbitrage argument they must have the same value at time $t = 0$. This gives

$$D(3) = G(2,0)D(2,0) + G(2,1)D(2,1) + G(2,2)D(2,2) \tag{4}$$

From (2), we can express $D(2,0), D(2,1), D(2,2)$ in terms of $r(2,0)$. Hence (4) would become an equation with one unknown $r(2,0)$. $r(2,0)$ could be solved using a numerical method. Once we know $r(2,0)$, $r(2,1), r(2,2)$ follows from (1).

In general, suppose $t \geq 0$ and we have worked out $r(t,j)$ and $G(t,j)$ for $j = 0, 1, \ldots, t$. (Note that $r(0,0) = r(1)$ and $G(0,0) = 1$.) Then (see (10)) for $j = -1, 0, \ldots, t$,

$$G(t+1, j+1) = \frac{1}{2}D(t,j)G(t,j) + \frac{1}{2}D(t,j+1)G(t,j+1) \tag{5}$$

The no arbitrage argument described above gives

$$D(t+2) = \sum_{j=0}^{t+1} G(t+1,j)D(t+1,j)$$

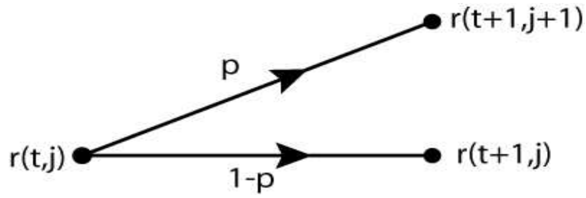
It follows from (2) that

$$D(t+2,j) = \begin{cases} \sum_{j=0}^{t+1} G(t+1,j)e^{-r(t+1,0)e^{2j\sigma(t+1)}} & \text{for continuously compounded interest} \\ \sum_{j=0}^{t+1} \frac{G(t+1,j)}{1+r(t+1,0)e^{2j\sigma(t+1)}} & \text{for simple interest} \end{cases} \tag{6}$$

Note that (6) is an equation with one unknown $r(t+1,0)$. ($r(t+1,0)$ could be solved using a numerical method (such as the Bisection method). Once we know $r(t+1,0)$, the $r(t+1,j)$'s, $j=1,2,\ldots,t+1$, could be deduced from (1).

Appendix Arrow-Debreu prices

Let $r(t,j)$ be the interest rate at time t and state j, at (t,j) for short, over time period $[t,t+1]$ on a binomial tree. Let p the risk neutral probability that the interest rate will go up from $r(t,j)$ to $r(t+1,j+1)$. (Hence $r(t,j)$ will go down to $r(t+1,j)$ with probability $1-p$.



For $0 \le t_0, 0 \le j_0$, let $G(t_0,j_0)$ be the value of a derivative at time 0 and the payoff at $t=t_0$ is given by

$$\delta_{j_0 j} \text{ where j is the state reached at time } t_0 \tag{7}$$

(We also use $G(t_0,j_0)$ to denote the above defined derivative.) Note that $G(0,0)$ is 1. The $G(t,j)$'s are known as the Arrow-Debreu prices.

Let $V(t,j)$ be the value (payoff) of an arbitrary derivative at (t,j). It can be easily verified that $V(0,0)$, the value of the derivative at time $t=0$ is given by

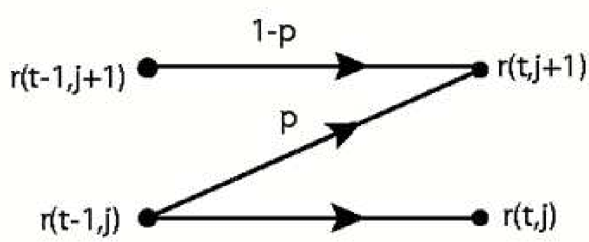
$$V(0,0) = \sum_{s=0}^{t} V(t,s)G(t,s) \tag{8}$$

Let t_0, j_0 be given. The value of $G(t_0, j_0+1)$ at time t_0-1 is

$$\begin{cases} (1-p)D(t_0-1,j+1) & \text{at state } j_0+1 \\ pD(t_0-1,j) & \text{at state } j_0 \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

where $D(t,j)$ is the discount factor at (t,j) over $[t,t+1]$. We have

$$D(t,j) = \begin{cases} e^{-r(t,j)} & \text{for continuously compounded interest} \\ \frac{1}{1+r(t,j)} & \text{for simple interest} \end{cases}$$



Let $t \geq 1, -1 \leq j \leq t-1$ be given. Let $V(t-1, j+1)$ be the payoff (value) of $G(t, j+1)$ at time $t-1$. Note that the time 0 value of $V(t-1, j+1)$ is $G(t, j+1)$. By (9) and (8), we have

$$G(t, j+1) = (1-p)D(t-1, j+1)G(t-1, j+1) + pD(t-1, j)G(t-1, j) \tag{10}$$

Note that we define $G(t, j) = 0$ if $t < 0$ or $j < 0$ or $j > t$. From (10), we see that $G(t, j)$ could be calculated inductively.

S H Man 30/3/03

References

[1] L Clewlow and C Strickland *Implementing Derivatives Models*, Wiley

[2] K Cuthbertson and D Nitzsche, *Financial Engineering-Derivatives and Risk Management*, Wiley

[3] R Jarrow and S Turnbull, *Derivative Securities*, South-Western College Publishing