by Fischer Black, Emanuel Derman and William Toy

A One-Factor Model of Interest Rates and Its Application to Treasury Bond Options

In one simple and versatile model of interest rates, all security prices and rates depend on only one factor—the short rate. The current structure of long rates and their estimated volatilities are used to construct a tree of possible future short rates. This tree can then be used to value interest-rate-sensitive securities.

For example, a two-year, zero-coupon bond has a known price at the end of the second year, no matter what short rate prevails. Its possible prices after one year can be obtained by discounting the expected two-year price by the possible short rates one year out. An iterative process is used to find the rates that will be consistent with the current market term structure. The price today is then determined by discounting the one-year price (in a binomial tree, the average of the two possible one-year prices) by the current short rate.

Given a market term structure and resulting tree of short rates, the model can be used to value a bond option. First the future prices of a Treasury bond at various points in time are found. These prices are used to determine the option's value at expiration. Given the values of a call or put at expiration, their possible values before expiration can be found by the same discounting procedure used to value the bond. The model can also be used to determine option hedge ratios.

THIS ARTICLE DESCRIBES a model of interest rates that can be used to value any interest-rate-sensitive security. In explaining how it works, we concentrate on valuing options on Treasury bonds.

The model has three key features.

1. Its fundamental variable is the short rate—the annualized one-period interest rate. The short rate is the one factor of the model; its changes drive all security prices.
2. The model takes as inputs an array of long rates (yields on zero-coupon Treasury bonds) for various maturities and an array of yield volatilities for the same bonds. We call the first array the *yield curve* and the second the *volatility curve*. Together these curves form the *term structure*.
3. The model varies an array of means and an array of volatilities for the future short rate to match the inputs. As the future volatility changes, the future mean reversion changes.

We examine how the model works in an imaginary world in which changes in all bond yields are perfectly correlated; expected returns on all securities over one period are equal; short rates at any time are lognormally distributed; and there are no taxes or trading costs.

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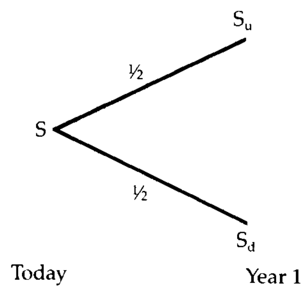
Figure A A One-Step Tree



Valuing Securities

Suppose we own an interest-rate-sensitive security worth S today. We assume that its price can move up to S_u or down to S_d with equal probability over the next time period. Figure A shows the possible changes in S for a one-year time step, starting from a state where the short rate is r.

The expected price of S one year from now is $1/2\ (S_u + S_d)$. The expected return is $1/2\ (S_u + S_d)/S$. Because we assume that all expected returns are equal, and because we can lend money at r, we deduce:

$$S = \frac{\frac{1}{2} S_u + \frac{1}{2} S_d}{1 + r}, \tag{1}$$

where r is today's short rate.

Getting Today's Prices from Future Prices

We can use the one-step tree to relate today's price to the prices one step away. Similarly, we can derive prices one step in the future from prices two steps in the future. In this way, we can relate today's prices to prices two steps away.

Figure B shows two-step trees for rates and prices. The short rate starts out at 10 per cent. We expect it to rise to 11 per cent or drop to 9 per cent with equal probability.

The second tree shows prices for a two-year, zero-coupon Treasury. In two years, the zero's price will be \$100. Its price one year from now may be \$91.74 (\$100 discounted by 9 per cent) or \$90.09 (\$100 discounted by 11 per cent). The expected price one year from now is the average of \$90.09 and \$91.74, or \$90.92. Our valuation formula, Equation (1), finds today's price by discounting this average by 10 per cent to give \$82.65.

We can in this way value a zero of any maturity, provided our tree of future short rates goes out far enough. We simply start with the security's face value at maturity and find the price at each earlier node by discounting future prices using the valuation formula and the short rate at that node. Eventually we work back to the root of the tree and find the price today.

Finding Short Rates from the Term Structure

The term structure of interest rates is quoted in yields, rather than prices. Today's annual yield, y, of the N-year zero in terms of its price, S, is given by the y that satisfies:

$$S = \frac{100}{(1 + y)^N}. \tag{2}$$

Similarly, the yields y_u and y_d one year from now corresponding to prices S_u and S_d are given by:

$$S_{u,\,d} = \frac{100}{(1 + y_{u,\,d})^{N-1}}. \tag{3}$$

We want to find the short rates that assure that the model's term structure matches today's

Figure B Two-Step Trees of Short Rates and Prices

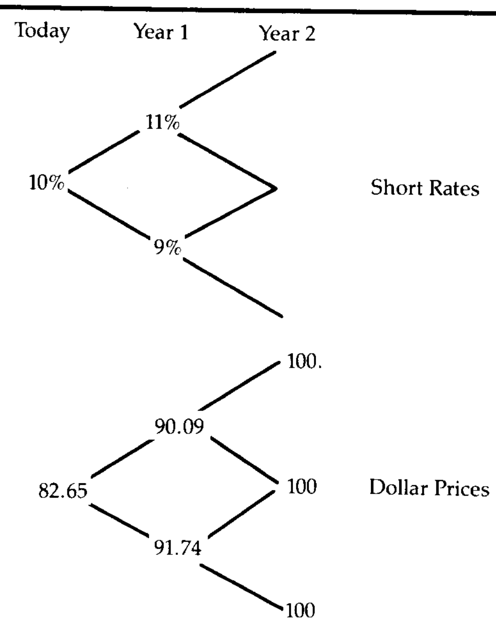
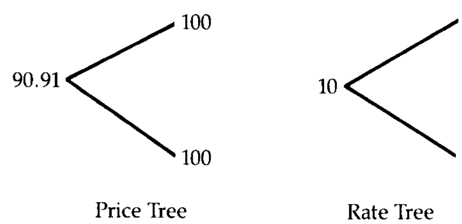


Table I A Sample Term Structure

Maturity (years)	*Yield (%)*	*Yield Volatility (%)*
1	10	20
2	11	19
3	12	18
4	12.5	17
5	13	16

Figure C Finding the Initial Short Rate Using a One-Year Zero



market term structure. Table I gives the assumed market term structure.

The price of a zero today is the expected price one period in the future discounted to today using the short rate. The short rate, r, is 10 per cent. Using the price tree of Figure C, we see that $S_u = S_d = 100$, and $S = 90.91$:

$$90.91 = \frac{\frac{1}{2}\,100 + \frac{1}{2}\,100}{1 + r} = \frac{100}{1 + r}.$$

Short Rates One Period in the Future

We can now find the short rates one year from now by looking at the yield and volatility for a two-year zero using the term structure of Table I.

Look at the two-year short-rate tree in Figure D. Let's call the unknown future short rates r_u and r_d. We want their values to be such that the price and volatility of the two-year zero match the price and volatility in Table I.

We know today's short rate is 10 per cent. Suppose we guess that $r_u = 14.32$ and $r_d = 9.79$.

Now look at the price and yield trees in Figure D. A two-year zero has a price of \$100 at all nodes at the end of the second period, no matter what short rate prevails. Using the valuation formula—Equation (1)—we can find the one-year prices by discounting the expected two-year price by r_u and r_d; we get prices of \$87.47 and \$91.08. Using Equation (3), we find that yields of 14.32 and 9.79 per cent correspond to these prices. These are shown on the yield tree in Figure D.

Now that we have the two-year prices and yields one year out, we can use the valuation formula to get today's price and yield for the two-year zero. Today's price is given by Equation (1) by discounting the expected one-year-out price by today's short rate:

$$\frac{\frac{1}{2}(87.47) + \frac{1}{2}(91.08)}{1.1} = 81.16.$$

We can get today's yield for the two-year zero, y_2, by using Equation (2) with today's price as S. As the yield tree in Figure D shows, y_2 is 11 per cent.

The volatility of this two-year yield is defined as the natural logarithm of the ratio of the one-year yields:

$$\sigma_2 = \frac{\ln \frac{14.33}{9.79}}{2} = 19\%.$$

Figure D Finding the One-Year Short Rates Using a Two-Year Zero

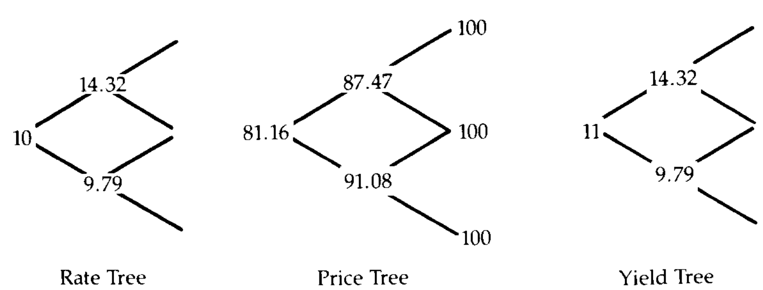
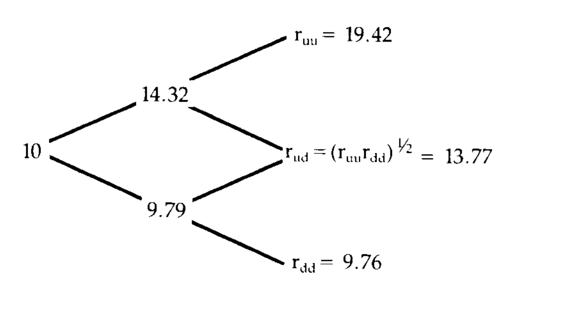


Figure E Finding the Two-Year Short Rates



With the one-year short rates we have chosen, the two-year zero's yield and yield volatility match those in the term structure of Table I. This means that our guesses for r_u and r_d were right. Had they been wrong, we would have found the correct ones by trial and error.

So an initial short rate of 10 per cent followed by equally probable one-year short rates of 14.32 and 9.79 per cent guarantee that our model matches the first two years of the term structure.

More Distant Short Rates

We found today's single short rate by matching the one-year yield. We found the two one-year short rates by matching the two-year yield and volatility. Now we find the short rates two years out.

Figure E shows the short rates out to two years. We already know the short out to one year. The three unknown short rates at the end of the second year are r_{uu}, r_{ud} and r_{dd}.

The values for these three short rates should let our model match the yield and yield volatility of a three-year zero. We must therefore match two quantities by guessing at three short rates. This contrasts with finding the one-year short rates, where we had to match two quantities with two short rates. As a rule, matching two quantities with two short rates is unique; there is only one set of values for the short rates that produces the right match. Matching two quantities with three short rates is not unique; many sets of three short rates produce the correct yield and volatility.

Remember, however, that our model assumes that the short rate is lognormal with a volatility (of the log of the short rate) that depends only on time. One year in the future, when the short rate is 14.32 per cent, the volatility is $1/2 \ln(r_{uu}/r_{ud})$; when the short rate is 9.79 per cent, the volatility is $1/2 \ln(r_{ud}/r_{dd})$. Because these volatilities must be the same, we know that $r_{uu}/r_{ud} = r_{ud}/r_{dd}$, or $r^2_{ud} = r_{uu}r_{dd}$.

So we don't really make three independent guesses for the rates; the middle one, r_{ud}, can be found from the other two. This means we have to match only two short rates—r_{uu} and r_{dd}—with two quantities—the three-year yield and volatility in the model. This typically has a unique solution.

In this case, Figure E shows that values for r_{uu}, r_{dd} and r_{ud} of 19.42, 9.76 and 13.77 per cent, respectively, produce a three-year yield of 12 per cent and volatility of 18 per cent, as Table I calls for.

We now know the short rates for one and two years in the future. Using a similar process, we can find the short rates on tree nodes farther in the future. Figure F displays the full tree of short rates at one-year intervals that matches the term structure of Table I.

Valuing Options on Treasury Bonds

Given the term structure of Table I and the resulting tree of short rates shown in Figure F, we can use the model to value a bond option.

Coupon Bonds as Collections of Zeroes

Before we can value Treasury bond options, we need to find the future prices of a Treasury bond at various nodes on the tree. Consider a Treasury with a 10 per cent coupon, a face value of \$100 and three years left to maturity. For convenience, consider this 10 per cent Treasury as a portfolio of three zero-coupon bonds—a one-year zero with a \$10 face value; a two-year zero with a \$10 face value; and a three-year zero with a \$110 face value.

This portfolio has exactly the same annual

Figure F Short Rates that Match the Term Structure of Table I

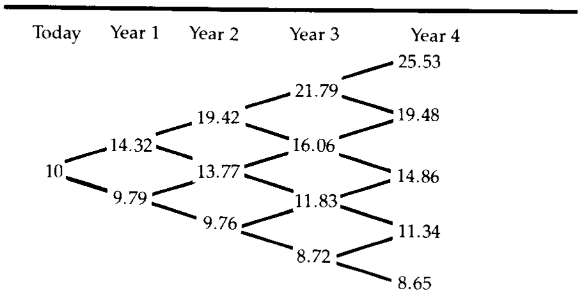
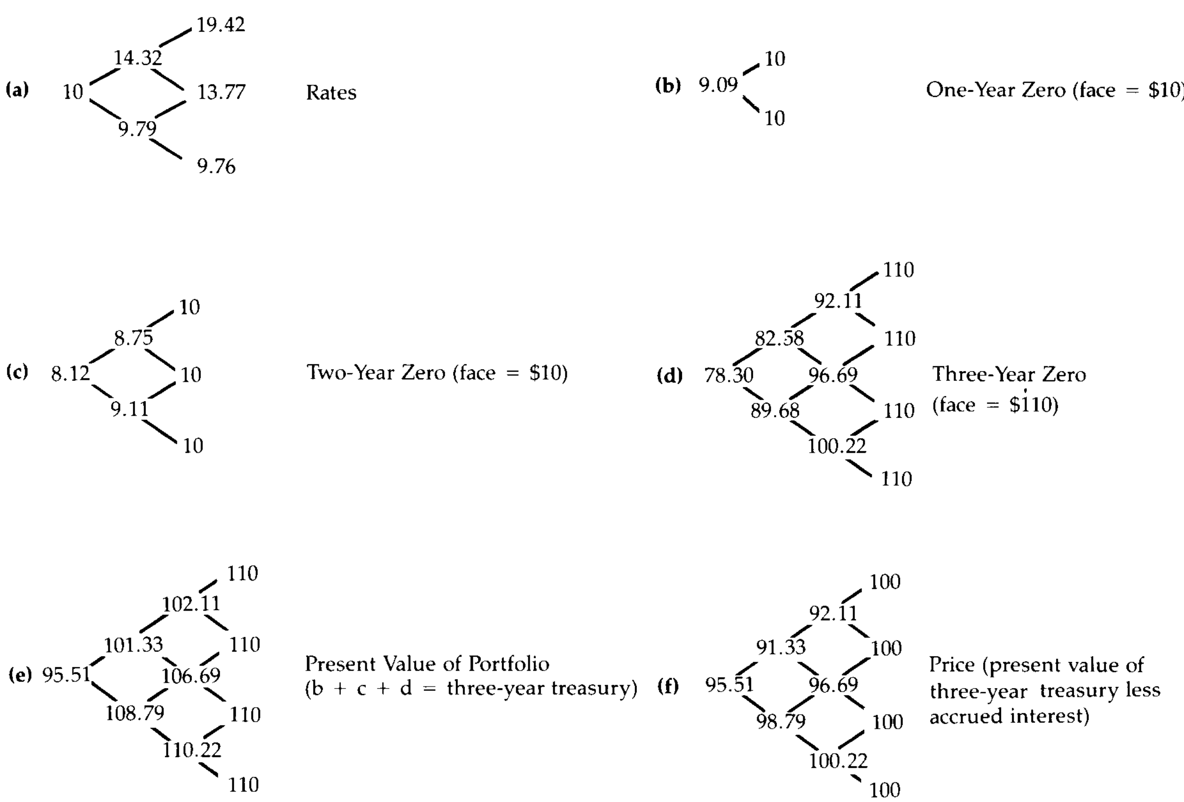


Figure G Three-Year Treasury Values Obtained by Valuing an Equivalent Portfolio of Zeroes



payoffs as the 10 per cent Treasury with three years to maturity. So the portfolio and the Treasury should have the same value. The tree in Figure F was built to value all zeroes according to today's yield curve, hence we can use it to value the three zeroes in the portfolio above.

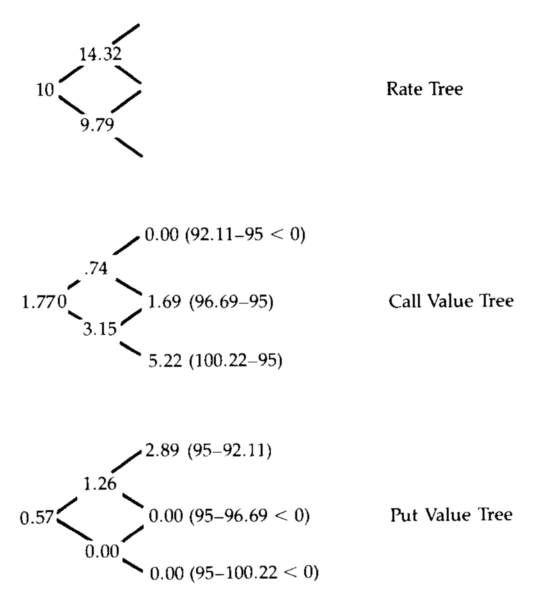
Panel (e) of Figure G shows the price of the 10 per cent Treasury as the sum of the present values of the zeroes—$95.51. The tree in panel (f) gives the three-year Treasury prices obtained after subtracting $10 of accrued interest on each coupon date.

Puts and Calls on Treasuries

We found a price of $95.51 for a three-year, 10 per cent Treasury. The security is below par today; it has a 10 per cent coupon, and yields in today's yield curve are generally higher than 10 per cent.

We want to value options on this security—a two-year European call and a two-year European put, both struck at $95. From Figure G(e) we see that in two years the three-year Treasury

Figure H Two-Year Options on a Three-Year Treasury



bond may have one of three prices—$110.22, $106.69 or $102.11. The corresponding prices without accrued interest are $100.22, $96.69 and $92.11.

At expiration, the $95 call is in the money if the bond is worth either $100.22 or $96.69. The call's value will be the difference between the bond's price and the strike price. The $95 call will be worth $5.22 if the bond is trading at $100.22 at expiration and $1.69 if the bond is trading at $96.69. The call is out of the money, and therefore worth zero, if the bond is trading at $92.11 at expiration. Figure H shows the short-rate tree over two years, as well as possible call values at expiration of the option in two years.

At expiration the put is in the money if the bond is worth $92.11 (without accrued interest). The put's value will be the difference between $92.11 and the $95 strike price—$2.89. The put is worthless if the bond's price is one of the two higher values, $100.22 or $96.69. Figure H gives the put values.

Knowing the call values at expiration we can find the possible values of the call one year before expiration, using the valuation formula given by Equation (1). If the short rate is 14.32 per cent one year from today, the call's value one year before expiration will be:

$$\frac{\frac{1}{2}(0.00) + \frac{1}{2}(1.69)}{1.1432} = 0.74.$$

If the short rate is 9.79 per cent one year from today, the call's value will be:

$$\frac{\frac{1}{2}(1.69) + \frac{1}{2}(5.22)}{1.0979} = 3.15.$$

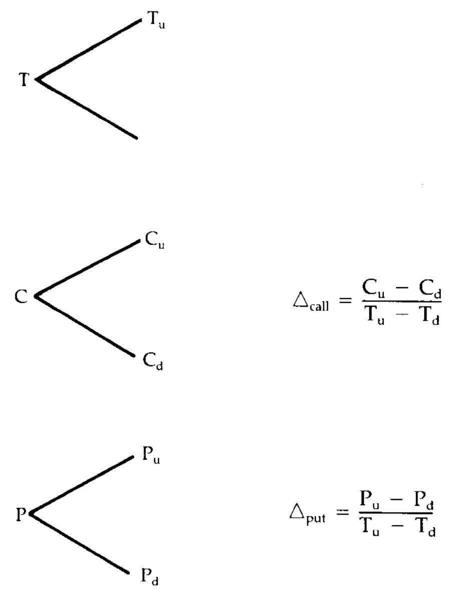
Given the call values one year out, we can find the value of the call today when the short rate is 10 per cent:

$$\frac{\frac{1}{2}(0.74) + \frac{1}{2}(3.15)}{1.1} = 1.77$$

Put values are derived in a similar manner. Figure H shows the full trees of call and put values.

We have priced European-style options by finding their values at any node as the discounted expected value one step in the future. American-style options can be valued with little extra effort. Because an American option may be exercised at any time, its value at any node is the greater of its value if held or its value if exercised. We obtain its value if held by using the valuation formula to get any node's value in terms of values one step in the future. Its value if exercised is the difference between the bond price at the node and the strike price.

Figure I Hedge Ratios for a Call and a Put on a Treasury



Option Hedge Ratios

When interest rates change, so do the prices of bonds and bond options. Bond option investors are naturally interested in how much option prices change in response to changes in the price of the underlying bond. We measure this relation by the hedge ratio (or delta).

Figure I shows one-step trees for a Treasury, a call and a put. For a call worth C on a Treasury with price T, the hedge ratio is:

$$\Delta_{call} = \frac{C_u - C_d}{T_u - T_d}, \quad (4)$$

where C_u and C_d are the values of the call one period from today in the tree corresponding to possible short rates r_u and r_d. A similar formula holds for a put, P, on a Treasury; we simply replace C with P in Equation (4).

For the two-year put and call on the three-year Treasury considered above, we start by finding the differences between possible prices one year from today. Given the Treasury prices shown in Figure G and the option prices from Figure H:

$$T_u - T_d = 91.33 - 98.79$$
$$= -7.46$$
$$C_u - C_d = 0.74 - 3.15$$
$$= -2.41$$
$$P_u - P_d = 1.26$$
$$= 1.26.$$

We can now derive the hedge ratios, using Equation (4):

$$\Delta_{call} = \frac{-2.41}{-7.46} \quad (5)$$
$$= 0.32$$
$$\Delta_{put} = \frac{1.26}{-7.46} \quad (6)$$
$$= -0.17.$$

These hedge ratios give us the sensitivity of the option to changes in the underlying Treasury price by describing the change in the option's price per dollar change in the Treasury's price. They therefore tell us how to hedge the Treasury with the option, and vice versa. The call hedge ratio is positive because the call prices increase when the Treasury price increases. In contrast, the put hedge ratio is negative because put prices decrease as the Treasury price increases.

Reducing the Interval Size

In the examples above, the short-rate tree had coarse one-year steps, Treasuries paid annual coupons and options could only be exercised once a year.

To get accurate solutions for option values, we need a tree with finely spaced steps between today and the option's expiration. Ideally, we would like a tree with one-day steps and a 30-year horizon, so that coupon payments and option exercise dates would always fall exactly on a node. We would also like to have many steps to expiration, even for options on the verge of expiring.

In practice, our computer doesn't have enough memory to build a 30-year tree with daily steps. And even if it did, it would take us hours to value a security. Instead, we can build a sequence of short-rate trees, each with the same number of steps but compressed into shorter and shorter horizons. Thus each tree has finer spacing than the one before it. For example, we might use today's term structure to build short-rate trees that extend over 30 years, 15 years, 7-1/2 years and so on. In this way, no matter when the option expires, we will always have one tree with enough steps to value the option accurately.

To value an option on any Treasury, we use two trees—a coarse one with enough steps to value the Treasury accurately from its maturity back to today, and a fine one with enough steps to value the option accurately from its expiration until today. We find the Treasury values on the coarse tree by using the model's valuation formula from maturity to today. Then we interpolate these Treasury values onto the fine tree, which may often have as many as 60 periods. Maturity, expiration and coupon dates that fall between nodes are carefully interpolated to the nearest node in time. In this way, the option can be accurately valued.

Interpolating across trees gives us accurate, yet rapid, results. Once model values have been found to match the term structure, we can value options in a few seconds.

Improving the Model

We considered more complex models that use more than one factor to describe shifts in the yield curve. Increasing the number of factors improves the model results. But a multifactor model is much harder to think about and work with than a single-factor model. It also takes much more computer time. We therefore think it pays to work with different single-factor models before moving on to a multifactor model.

Along these lines, we examined the effects on our model of letting forward mean reversion and forward short-rate volatility vary independently. (They are tied together in the current model only by the geometry of a tree with equal time spacing throughout.) We found that varying forward mean reversion and varying forward short-rate volatility give very different results. We can use one or the other alone, or a mixture of both, in matching the term structure. ■