Application of the Black-Derman-Toy Model (1990)
-Valuation of an Interest Rate CAP and a Call Option on a Bond-

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Synopsis

In this paper we aim at presenting the implementation of the Black, Derman and Toy[1] (1990) interest rate term structure model for the pricing of financial products using a generic and flexible MSExcel spreadsheet. Using the implemented tool we will compute the results for the valuation[2] of an interest rate cap and the determination of a strike price for an European call option on a bond that would make it "at the money".

These results will be, later in the paper, compared to the results obtained for the pricing of those financial instruments using the Black (1976) market model for the pricing of interest rate product derivatives.

Given this framework, the paper is structured as follows: Section I presents the summary results and methodology followed for each of the referred instruments using the BDT model; Section II presents the same information of the previous section but using the Visual Basic for Applications tool; Section III presents a comparison between the results obtained in the previous sections and those reached through the Black (1976) market model; and in Section IV we make some final considerations.

[1] BDT hereafter
[2] Please notice that all values are expressed in Euros (€).

I. Application of the BDT model

Section I is divided into each of the first three assignments we were requested, namely: (1) construction of the BDT model, (2) valuation of a cap; and the (3) calculation of the strike price that would make an European call option on a bond become at the money.

Construction of the BDT model

The first step in the construction of the BDT interest rate tree is the computation of the discount factors for each maturity according to the yield curve structure.

Secondly, we build a zero-coupon bond tree that pays a face value of 100 in 10 years. Each node is the average of the two resulting prices (up and down) in the subsequent period discounted back at the corresponding interest rate (that would be retrieved from the BDT interest rate tree) adjusted for the time interval. This is done starting in the 10 years moment and computing backwards until the present moment. At this point the entire tree shows the value "100" at each node (except for the first), as the interest rate tree was not yet built[3].

Thirdly we build the interest rate tree. We determine all our unknowns using the lowest interest rates for each period and the corresponding volatilities. After, we introduce a column denominated as "Destination cells". This column gives for each period the difference between the product of 100 by the discount factor determined through the yield curve, and the present value of the zero coupon bond calculated in the bond tree. Finally, we run "Solver" each period at a time starting from the first moment, only changing the location of the destination cell and of our unknown. For each period we set as our destination cell the value of our "Destination cells" column. This destination cell should be set equal to zero[4], by changing the cell of the respective lowest interest rate, using the methodology above. We do not need to impose any constraints.

The BDT interest rate tree is then achieved, with the determination of all interest rates[5]. However, using this methodology hampers the flexibility of the solution. Hence we devised an alternative solution based on Visual Basic for Applications (VBA) – see section II. Both solutions for the tree are presented in Appendix I.

Cap Valuation

A cap is a portfolio of call options on *forward-rate agreements* (FRAs)[6]. Each of these options – *caplets* – can be exercised at each moment when the borrowing rate is set. This option is of great

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[3] Which is equivalent to an interest rate of zero.
[4] As a result of its construction.
[5] The extreme values achieved in the tree are acceptable since the probabilities associated are very low. The results assume discrete discounting.
[6] Unlike FRAs these options are paid in arrears.

importance since it gives the holder the right, but not the obligation, to exercise it, making caps an effective protection against rises in interest rates while taking advantage of decreases in borrowing rates.

In the present case we have 11 caplets, which equal the number of interest payment minus one (the first), since it is already known by the time the cap is settled. To value the cap, first we have to value each individual caplet at the present moment, and then add them up.

Using the BDT model, we have to build a binomial tree for each of the caplets. Starting from the end (maturity of the option) we must compute the value for each caplet. The value of the caplet is the maximum of zero and the difference between: the expected[7] rate at maturity (taken from the corresponding nodes of the BDT tree) and the strike price[8]. Since these options are paid in arrears we must discount these values back to maturity, using the corresponding expected rate between the option's maturity and the payment date. Next, we must discount these values using the risk-neutral probability of 0.5 and the corresponding interest rate taken from the BDT tree until we reach the present moment. This procedure is undertaken for each of the caplets, and the sum of these present values is the present value of the whole cap.

The characteristics of the valued cap and the results of its valuation according to the term structure observed in 13th of June 2001 are summarized in Table 1. The results were obtained through the application of the BDT model previously described. The procedures undertaken to achieve these results can be found in Appendix II.

Table 1 - *Summary characteristics and results*

Strike	5,125%
Underlying	3 month LIBOR
Maturity	3 years
Notional Principal	1.000.000
Cap Value	7.179

Call Option on Bond

A call option on a bond is a contingent claim that gives the holder the right, but not the obligation, to buy a bond for a certain strike price at a given date. Since the option is European that date corresponds, in this case, to the option's maturity date.

Unlike the previous case, here we have only one option. The request is not to value the option (considering it is at-the-money), but to determine the strike price. As the option is at-the-money, there is only one price. The only common aspect with the previous exercise is the use of the BDT model.

[7] In moment 0, when we are valuing, these rates are expected.
[8] As they are call options.

We approached this assignment by dividing it in a three-step procedure:

1. Computation of the value of the bond at each node of the tree

We strip the bond, making all the payments (coupons and principal) equivalent to zero-coupon bonds. In this case, we have six zero-coupon bonds corresponding to the number of coupon payments. Subsequently, we build a tree for each of these bonds to implement the BDT model. In each tree we know that – independently of the state of nature – the bond will pay a certain value (coupon, or coupon plus face value at its maturity[9]). Then, we discount these values to the present moment by applying the risk-neutral probability of 0.5 and the corresponding interest rate. If we add all the corresponding nodes of these trees we obtain the tree of "dirty prices". By removing the respective period coupon (included in the "dirty price"), we get to the "clean prices" tree, which will be the basis for valuing the option.

2. Calculation of the call option's value for a randomly selected strike price

A call option's value at maturity is the maximum of zero and the difference between: the value of the bond (clean price) and the strike price. We discount these values (using the risk-neutral probability of 0.5 and the corresponding interest rate taken from the BDT tree) until we reach the present moment, and hence the option's value.

3. Determination of the strike price through the use of the "Goal Seek" function of MSExcel

This procedure ensures that the resulting strike price is the one that makes the option be at-the-money. Implementation: our destination cell is the present value of the option, which we set to zero, by changing the strike price (our unknown variable).

The characteristics of the option and the results attained according to the term structure observed in 13th of June 2001 are summarized in Table 2. The results were obtained through the application of the BDT model. The procedures undertaken to achieve these results can be found in Appendix III.

Table 2 - Summary characteristics and results

Underlying	3 year 5% coupon noncallable bond
Coupon	paid every 6 months
Option Type	European call
Option's Maturity	1year
Strike Price	103,02

We did not make any additional assumptions to those of the BDT model.

[9] Different for each zero-coupon bond and corresponding to the coupon payment moments.

II. Visual Basic for Applications

The use of VBA allows us to create an easy to use function, “BDTTree”, that requires only five inputs. This function, which works for any period of time and updates automatically with new data, proves to be very flexible.

The user of this function should first go to the “Insert Function” option from the Excel menu, and then choose “BDTTree” under the “user defined” category. The function prompts the user to specify the discount method (banking or finance), the location of the zero coupon bond values, their respective volatilities and maturity value (normally “1”), and the steps’ time interval on the tree (in this case 0,25). As it is a matrix function, the user should be careful when selecting the adequate area for the tree (in this case a 40x40 matrix), not forgetting to press Shift+Ctrl+Enter after introducing the function.

In broad terms, this function has one unknown in each step of the tree - which we set to be the maximum rate - and determines the other rates for that period through the volatility constraint. For each period, the function starts with one hypothetical value for the unknown – which we set to be the maximum rate of the previous period – and iterates until matching the calculated zero coupon bond value with the one observed in the market. Each iteration process stops when the error is less than the one we specified[10]. The code developed[11] uses the Newton-Raphson optimisation algorithm[12] in order to achieve the desired result with only a few iterations[13].

We also created a function for the calculation of the caplet using the potential provided by VBA, “CAPLET”[14]. Using a similar procedure to the one described above to the BDTree function, and choosing CAPLET in the Menu, the user, after choosing one of two types of instruments (“1” for caplets, and “2” for floorlets), must introduce the following inputs: (1) the strike price; (2) the first period interest rate given by the BDT tree[15]; (3) maturity of the caplet[16]; (4) delta t. The function automatically delivers us the value of the caplet (or floorlet).

In order to aid the calculation of the call option on the bond, we developed two additional functions: “BOND” and “BONDOPTION”. As in the previous cases, these functions are included in the “user defined” category. The first function gives us the binomial tree[17] for each zero-coupon bond stripped from the original one, while the second computes the option’s value. An intermediate

[10] Very close to zero.
[11] Based on Racicot (2001) and Jackson (2001).
[12] The same used by the “Solver” function.
[13] See Racicot (2001).
[14] This function assumes discrete discounting. With modifications on the VBA code this function can also be applied to continuous discounting.
[15] This rate must be the one achieved using the built “BDTTree” MSExcel function, and must be fixed (using F4 key).
[16] In number of periods.
[17] In a matrix format.

calculation is required in order to use the second function, and that is the determination of the dirty and clean prices. Both functions assume discrete discounting and require the same kind of inputs as the previous ones. The only difference concerns one input of the "BONDOPTION" function, which is the clean prices matrix[18]. This latter function can be applied to both call and put European style options[19].

The code created for the functions and related comments are presented in Appendix IV.

III. Comparison with Black's market model

In this section we will start by briefly explaining the methodology employed in the valuation of the same derivative instruments described in Section I through the use of Black's market model. We conclude this section by comparing these results with those of the BDT.

Cap Valuation

The valuation of a cap using Black's model involves several inputs: (1) the discount factor[20]; (2) the accrual[21]; (3) the forward rate[22]; (4) the option's maturity[23]; and finally, (5) the forward rate volatility of the corresponding period[24]. These inputs are needed for each caplet. Having determined the value of each caplet, the cap's value is equal to the sum of the present value of all the caplets.

We did not make any assumption in addition to those embedded in the Black market model.

Call Option on Bond

In what refers to the call option on the bond, the inputs are basically the same as those of the cap valuation, but some adjustments must be made. Firstly, we do not need the accrual as the underlying asset is not a rate, but a bond's price. Therefore, we do not need to correct it for the period. Secondly, we estimate the underlying asset's forward price and volatility in a somewhat different way[25] explained in the following paragraphs.

The forward price of the bond was estimated by discounting all the subsequent certain cash flows (coupons and principal) back to the exercise date through the forward rates[26] (between each exercise date and the cash-flow's moment).

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[18] It is crucial that the clean prices be presented also in a matrix format.
[19] With modifications on the VBA code this function can also be applied to American options.
[20] Corresponding to the moment the option value is received
[21] Corresponding to the period of the effects of the option's exercise in order to correct the annual interest rates to the period.
[22] For the same period of the accrual. The forward rate is estimated using the spot-forward parity.
[23] Determined as difference between the moment the option can be exercised and the present moment.
[24] We assumed forward-forward volatilities.
[25] Notice that, the underlying reasoning is the same. The differences arise only because of the nature of the underlying asset – price – whereas before we had a rate.
[26] Calculated using the spot-forward parity.

The forward price volatility was calculated through the transformation of the forward yield volatility. For this purpose, we first determined the forward yield volatility as the average of the yield volatilities between the exercise date and the underlying asset's maturity. Next, we determined the duration of the bond[27] and the forward yield between the end of year 1 and the end of year 3. Given these three inputs, we estimated the forward price volatility.

Having all the inputs needed we were able to run Black's model and compute the option's value.

BDT versus Black model: comparison of results

The results attained through the application of the Black market model and BDT are summarized in Table 3. The procedures undertaken to achieve these results can be found in Appendix V.

As it can be observed, there is a difference in the cap's value, depending on the model used[28]. The reason for the difference may be due to:

Table 3 - Summary results and comparison

	Black	BDT
Cap	6.312	7.179
Call Option	0,017	0

(1) the different assumptions of volatility in the models – constant in Black's model and varying in time in the BDT model;

(2) the relatively high t in BDT compromising the accuracy of the estimate of the true value;

(3) whilst Black uses a central estimate of the discount rate, BDT uses a whole spectrum of rates.

The differences between the models' assumptions should have a higher impact in those caplets with longer maturities. In fact, if we compare the differences for each caplet, they grow as each caplet's maturity is longer. Furthermore as in this case the cap is made of eleven caplets the differences between the models are amplified.

Obviously, the reasons for the difference of valuation of a call option on a bond using the Black's model or the BDT are the same as above. However, as expected[29], in this case it is Black's model that gives a higher value for the option.

IV. Final Considerations

In conclusion we may declare that the choice between the two models depends on the assumptions we make about the volatilities and the simplicity of modelling.

[27] Only taken into account the values after year 1 and referring to that moment. The use of the discount factors referred to the present moment is innocuous in the calculation of the duration, since we just need to determine the weight of each cash-flow and to achieve that we must put them in the same moment in time, whatever this might be.
[28] The notional principal used was the same for both models (1.000.000).
[29] Inverse relationship between rates and bond prices.

The BDT model provides a way to incorporate the stochastic behaviour of interest rates. However this comes at a price. Since it is a discrete time model, the values given are just rough approximations. If we want a more rigorous value we need to diminish the time period between each step. This involves more time and computational capacity.

On the contrary, the Black model is a continuous time model that can give, in a quick way, a value for the option. However, this model assumes constant volatility during the option’s life and has other inconsistencies embedded.

A practical solution might be running both models in order to achieve an approximate interval for the option’s value. In our opinion, due to volatility considerations, and If accuracy is the prime factor, the BDT model with very small time interval steps is the preferred solution, when these two models are considered. More developed models have emerged in the last years, which seem to solute many of the critiques risen above by us.

References

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