Savings and Portfolio Choice in a Two-Period Two-Asset Model

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Abstract

We extend Arrow's analysis of portfolio choice in a one-period model to savings and portfolio choice in a two-period model.

Arrow (1971) analyzed portfolio choice in a one-period model with one safe and one risky asset. He showed that with decreasing absolute risk aversion (DARA), the demand for the risky asset is increasing in wealth. He also showed that with increasing relative risk aversion (IRRA), the elasticity of demand for the safe asset with respect to wealth is greater than one. Thus, with both DARA and IRRA, both asset demands are normal. Furthermore, in any two-good model, the goods must be Hicksian substitutes. Following Arrow, Sandmo (1968) analyzed a two-period model with a safe and risky asset, and with intertemporally additive utility for consumption in both periods. He showed that with DARA and IRRA, both asset demands are normal, as is demand for first-period consumption.

This note shows that in Sandmo's two-period economy, DARA and IRRA also guarantee that each of the goods is a Hicksian substitute for each of the others (Proposition 6). Moreover, if the safe interest rate goes up and the price of stocks goes up, expected utility constant, then first-period consumption decreases (savings increase) (Proposition 7).[1] In passing, we also rederive Sandmo's original result on normal demands (Proposition 5).

Similar properties are derived for more general preferences of the form $U(x, y, z) = f(x) + g(y, z)$. We proceed by first considering this general case. In Proposition 1 we show that x and y are Hicksian substitutes if and only if y is a normal good. In Proposition 2 we show that y and z are Hicksian substitutes if increasing y decreases the marginal utility of z ($g_{yz} \leq 0$). In Proposition 3 we show that y is a normal good if and only if y is a normal good for the utility function $V(x, y, z) = g(y, z)$. In

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[1] We have used such a model to study savings and portfolio choice in a two-asset economy in the context of social security trust fund investment in risky securities (Diamond and Geanakoplos, 1999).

Proposition 4, we show that an increase in the price of y together with a decrease in the price of z that leaves utility constant increases demand for x if and only if the income elasticity of demand for good y is larger than the income elasticity of demand for good z. After proving these propositions, we apply the results to the savings-and-portfolio-choice problem.

In Proposition 8, and its discussion, we partially extend Propositions 1–3 to multiple consumption goods, multiple assets, and multiple periods.

1 Preferences of the Form $U(x, y, z) = f(x) + g(y, z)$

We begin by relating the Hicksian cross-price derivative to the Marshallian income derivative.

Let $U : \mathbb{R}^3_+ \to \mathbb{R}$ be a strictly increasing, strictly concave, twice differentiable utility function. Let p_x, p_y, p_z, W all be strictly positive and define

$$V(p_x, p_y, p_z, W) = \max_{x,y,z} U(x, y, z) \text{ subject to } p_x x + p_y y + p_z z \leq W.$$

The solution $x(p_x, p_y, p_z, W) = x(p, W)$, $y(p, W)$, $z(p, W)$ to the utility maximization problem is called the Marshallian demand, and V is called the indirect utility function. When all three of x, y, z are strictly positive, then it is well-known that Marshallian demand is differentiable. Furthermore, Roy's identity holds that $\partial V(p, W)/\partial p_\alpha = -\alpha(p, W)[\partial V(p, W)/\partial W]$ for any good $\alpha \in \{x, y, z\}$. Good α is called normal if $\partial \alpha(p, W)/\partial W > 0$. The income elasticity of good α is defined by $(W/\alpha)(\partial \alpha(p, W)/\partial W)$.

Similarly, define the expenditure function

$$e(p, U) = \min_{x,y,z} p_x x + p_y y + p_z z \text{ subject to } U(x, y, z) \geq U.$$

The solution $x(p, U)$, $y(p, U)$, $z(p, U)$ to the expenditure minimization problem is called the Hicksian or compensated demand. When all three of x, y, z are strictly positive, it is well-known that Hicksian demand is differentiable. Furthermore, $\partial e(p, U)/\partial p_\alpha = \alpha(p, U)$ for any good $\alpha \in \{x, y, z\}$. Goods α and β are called Hicksian substitutes if

$$\partial \alpha(p, U)/\partial p_\beta = \partial^2 e(p, U)/\partial p_\alpha \partial p_\beta = \partial \beta(p, U)/\partial p_\alpha > 0.$$

Proposition 1 *Let the utility function $U : \mathbb{R}^3_+ \to \mathbb{R}$ be additively separable into the form $U(x, y, z) = f(x) + g(y, z)$, where both f and g are twice-differentiable, strictly increasing and strictly concave. Then when demands x, y, z are strictly positive, $\partial x(p, U)/\partial p_y = -(f'/f'')[\partial y(p, W)/\partial W]$, where the first term is the compensated cross derivative and the last term is the income derivative.*

Proof Define the expenditure function as

$$e(p, U) = \min_{x,y,z} p_x x + p_y y + p_z y \text{ subject to } f(x) + g(y, z) \geq U.$$

The first order condition for expenditure minimization and the envelope theorem give:

$$\frac{\partial e(p,U)}{\partial U} = \frac{p_x}{f'(x)}. \tag{1}$$

Differentiation of (1) with respect to p_y yields:

$$\frac{\partial y(p,U)}{\partial U} = \frac{\partial^2 e(p,U)}{\partial U \partial p_y} = -\frac{p_x f''(x)}{(f'(x))^2}\frac{\partial x(p,U)}{\partial p_y}. \tag{2}$$

Differentiating the identity between compensated and ordinary demands

$$y(p,U) = y(p, e(p,U)) \tag{3}$$

with respect to U and using (1) gives

$$\frac{\partial y(p,U)}{\partial U} = \frac{\partial y(p,W)}{\partial W}\frac{\partial e(p,U)}{\partial U} = \frac{\partial y(p,W)}{\partial W}\frac{p_x}{f'(x)}. \tag{4}$$

Substituting (2) into (4) gives the result. ■

Proposition 1 shows that x and y are Hicksian substitutes if and only if y is normal. Note that the proof would hold with z being a vector.

We now consider the Hicksian cross-price derivative between goods y and z.

Proposition 2 *Suppose $U(x,y,z) = f(x) + g(y,z)$, where both f and g are twice-differentiable, strictly increasing and strictly concave. If $\partial^2 g(y,z)/\partial y \partial z \leq 0$, then goods y and z are Hicksian substitutes, whenever x, y, z are positive.*

Proof Consider again the expenditure minimization problem defined in the proof of Proposition 1. Suppose a solution is achieved at the Hicksian or compensated demands $(x,y,z) \gg 0$. Let p_y increase, and suppose, contrary to what we would like to prove, that the compensated demand for z stays the same or declines. Since Hicksian own effects are negative, y must decline. In order to maintain the same utility with y decreasing and z nonincreasing, x must rise, since utility is increasing in each variable. So the marginal utility of x falls (since f is strictly concave), and the marginal utility of z does not fall (since y and z fall or stay the same, and $\partial^2 g/\partial y \partial z \leq 0$). This is a contradiction, since p_x and p_z are the same. ■

Next, we consider the relationship among income derivatives in the full maximization and the submaximization over just y and z. Consider together the utility maximization problem

$$\max_{x,y,z} f(x) + g(y,z) \text{ subject to } p_x x + p_y + p_z z \leq W \tag{5}$$

and the sub-maximization problem

$$\max_{y,z} g(y,z) \text{ subject to } p_y y + p_z z \leq I. \tag{6}$$

Denote by $v(I)$ the maximum value of (6) for a fixed price vector. Problem (5) can be separated into first maximizing (6) and then choosing I optimally from

$$\max_{x,I} f(X) + v(I) \text{ subject to } p_x x + I \leq W. \tag{7}$$

Proposition 3 *Let $f(x)$ and $g(y,z)$ satisfy the same restrictions as in Proposition 1, twice-differentiable, strictly increasing and strictly concave. Demand for good y (or z) is increasing in wealth in sub-problem (6) if and only if it is increasing in wealth in problem (5). In particular, at least one of y or z must be increasing in wealth in problem (5).*

Proof From the strict concavity of $g(y,z)$ it follows that $v(I)$ is a strictly concave function of I. From (7) it is easy to see that given strict concavity of $f(x)$ and $v(I)$, both x and I are increasing in W. Hence y is increasing in W in problem (5) if and only if it is increasing in I in problem (6). ■

Note that the proof would hold with z being a vector.[2]

We turn next to simultaneous changes in the prices of y and z that leave utility constant.

Proposition 4 *Let the utility function be $U(x,y,z) = f(x) + g(y,z)$, where both f and g are twice-differentiable, strictly increasing and strictly concave. Let p_x* and W be fixed. *If p_y increases and p_z decreases so that the indirect utility V is constant, then the demand for good x increases (decreases) if the income elasticity of demand for good y is larger (smaller) than the income elasticity of demand for good z.*

Proof Define $p_z(p_y)$ so that utility is constant as a function of p_y, given p_x and income W. Then, by Roy's identity, we have

$$p_z'(p_y) = -(\partial V/\partial p_y)/(\partial V/\partial p_z) = -y/z.$$

Taking the total derivative of x with respect to p_y, with p_z varying according to $p_z(p_y)$, leaves both income and utility fixed. By Proposition 1, we have

$$\frac{dx}{dp_y} = \frac{\partial x(p,W)}{\partial p_y} - \frac{y}{z}\frac{\partial x(p,W)}{\partial p_z} = \frac{\partial x(p,U)}{\partial p_y} - \frac{y}{z}\frac{\partial x(p,U)}{\partial p_z} = -\frac{f'}{f''}\left(\frac{\partial y(p,W)}{\partial W} - \frac{y}{z}\frac{\partial z(p,W)}{\partial W}\right)$$

or

$$\frac{dx}{dp_y} = -\frac{yf'}{Wf''}\left(\frac{W\partial y(p,W)}{y\partial W} - \frac{W\partial z(p,W)}{z\partial W}\right).$$

[2] Given Proposition 3, intuition for the sign structure in Proposition 1 becomes clear. Consider the effect of a compensated price increase of x on the demand for I (the composite expenditure on y and z). Since this is a two-good problem the compensated effect on I must be positive, and the resulting effect on y and z depends on normality in the suboptimization. Given the additive separable structure of the problem this is the only channel for the compensated price change to affect the demand for y and z.

■

In Proposition 3 we saw the relationship between income derivatives in the full optimization and the suboptimization, $\partial y(p,W)/\partial W = [\partial y(p,I)/\partial I](\partial I/\partial W)$. From this relationship, we have a corollary using the income elasticities in the suboptimization.

Corollary *If p_y increases and p_z decreases so that expected utility is constant, then the demand for good x increases (decreases) if the income elasticity of demand for good y is larger (smaller) than the income elasticity of demand for good z in the suboptimization over y and z.*

2 Savings and Portfolio Choice

Next, consider a special case where $g(y,z) = E\{u(y+zR)\} = \int u(y+zR)dF(R)$, where u is strictly increasing, twice-differentiable and strictly concave. The utility maximization problem, (5), using this subutility function, is the savings-and-portfolio-choice problem described in the introduction. The goods x, y, z are, respectively, first-period consumption, the safe investment good, and the risky investment good. The price ratio p_x/p_y can then be interpreted as the safe interest rate plus one. R is the random payoff of the risky investment. (We suppose R only takes on nonnegative values.) We now present the applications of Propositions 1–4 to this problem.

Proposition 5 *In the savings-and-portfolio-choice problem (i) first-period consumption is a normal good. If u globally satisfies decreasing absolute risk-aversion (DARA), then (ii) risky investment is a normal good. Furthermore, if u satisfies increasing relative risk-aversion (IRRA) then (iii) safe investment is a normal good. In short, if u satisfies both DARA and IRRA, then both investment goods are normal goods.*

Proof The strict concavity of u implies the strict concavity $g(y,z)$. This gives (i) and the applicability of Proposition 3. Applying Arrow's original result in the second period sub-maximization and using Proposition 3 yields (ii) and (iii). ■

Proposition 6 *Let the second period utility function, u, globally satisfy decreasing absolute risk-aversion (DARA). Then the compensated demand for first-period consumption is increasing in the price of the risky investment good, as is the compensated demand for the safe investment good. Furthermore, if second period utility satisfies increasing relative risk-aversion (IRRA), then the compensated demand for first-period consumption is increasing in the price of the safe investment good, as is the compensated demand for the risky investment good. In short, if u satisfies both DARA and IRRA, then each good is a Hicksian substitute for each of the others.*

Proof From Propositions 1 and 5, the compensated demand for first period consumption is increasing in the price of the risky good. Since increased holdings of either asset decreases the marginal utility of the other asset, Proposition 2 guarantees that the compensated demand for the safe good is increasing in the price of the risky good. This proves the first half of the proposition. The proof that the compensated demands for both first period consumption and risky investment are increasing in the price of the riskless investment good is handled in exactly the same way. ■

Proposition 7 *Let the second period utility function, u, globally satisfy decreasing absolute risk-aversion (DARA) and increasing relative risk-aversion (IRRA). An increase in the interest rate accompanied by an increase in the price of stocks that just leaves expected utility constant decreases first period consumption.*

Proof From Arrow, we know that the income elasticity of the demand for the safe asset in the portfolio choice problem exceeds one, and so must be larger than the income elasticity of the demand for the risky asset. Together with the Corollary to Proposition 4, this completes the proof. ■

3 Two Examples

While Proposition 6 refers to compensated demands, using normality, we can also sign the response of the ordinary demand for first-period consumption (and so savings) to a change in the price of an asset for some settings of initial endowments. Consider a budget constraint in which income is derived from the sale of initial endowments, so that utility maximization is:

$$\max_{x,y,z} f(x) + g(y,z) \text{ subject to } p_x x + p_y y + p_z z = p_x x_0 + p_y y_0 + p_z z_0, \qquad (8)$$

where 0-subscripted goods are initial endowments. From the Slutsky equation, we know that if good α is a normal good and the consumer is a net seller of good β, then the Marshallian demand derivative $\partial\alpha(p,W)/\partial p_\beta$ must be positive if the Hicksian demand derivative $\partial\alpha(p,U)/\partial p_\beta$ is positive. We consider two examples of such a setting.

Consider a consumer whose income is derived entirely from the sale of endowments, as in (8), and whose utility satisfies the assumptions of Proposition 6. We can think of the risky investment good z as a proxy for stocks, and the safe investment good y as a proxy for government bonds. Suppose the FED lowers the interest rate on bonds. Propositions 5 and 6 imply that if the consumer was a borrower to begin with, $p_y y < p_y y_0$, then the consumer will respond to the lower interest rate (stock prices held constant) by borrowing more, increasing both investment in the stock market and first period consumption. His savings (first-period income, x_0, less first-period consumption, x) must go down.[3] Notice that a drop in the interest rate is equivalent

[3] We define savings in the manner corresponding to national income accounting. We do not claim that the total value held of stocks and bonds would go down.

to a rise in the price of the safe investment good, and a borrower is made richer by the fall in the interest rate.

If instead the consumer had been a seller of stocks (say a middle aged investor winding down his accounts as old age approached), and the stock market suddenly ran up in price, with no change in the expected payoffs of stocks or the interest rate, then the consumer would respond by increasing his demand for immediate consumption and his investment in bonds.

4 Extensions to Multiple Assets and Multiple Periods

Consider now a multiperiod model with uncertainty, where at each date-event the agent consumes and adjusts his portfolio, in an effort to maximize lifetime expected utility. For the moment, we continue to assume that there are two assets available in the first period, but now suppose that in each state of nature in the second period there is an opportunity to consume and a new opportunity to trade. We can write lifetime expected utility as $f(x)+g(y,z)$ where $g(y,z) = E\{u_R(y+zR)\} = \int u_R(y+zR)dF(R)$, where u_R is the Bellman value function for income available starting in period 2 in the state of nature where the return on the risky asset was R. While u_R is strictly increasing, twice-differentiable and strictly concave, there is no guarantee that it satisfies DARA and IRRA even if the utility of consumption does. Thus Propositions 1–4 apply, but not Propositions 5–7.

Consider the situation where an agent has chosen an optimal lifetime plan, and then period 1 prices change, but the spot prices for period 2 and onward stay the same in all states of nature, as do subjective probabilities. Thus, the Bellman value functions u_R do not change. Suppose the agent is a short-term borrower at date 1. Then by Proposition 2, if the FED lowered the interest rate at date 1 (leaving it unchanged at every other date-event), he would invest more in the stock market. We could not be sure if he would consume more in date 1, or borrow more, even if his utility u satisfied DARA and IRRA, because there is no guarantee that the expectation of u_R satisfies DARA and IRRA.

However, we know by Proposition 1 that if the safe asset were a normal good, then the increase in its price (which is equivalent to the drop in the interest rate) would indeed raise period 1 consumption, and so reduce savings, and increase borrowing. Finally, we know from Proposition 3 that either the safe asset or the risky asset must be normal. Thus we can say that in a multiperiod, two asset economy, a rise in either asset price in period 1 (all else equal) will raise the Hicksian demand for the other asset. Furthermore, if the consumer is a net seller of both assets, then for at least one of the assets, a rise in its price will increase consumption and so reduce savings.

We turn next to settings with explicit attention to more consumption goods and more assets. As noted above, the proofs of Propositions 1 and 3 did not use the property that z was a scalar, and so extend to the case where z is a vector. Proposition 2 also admits extensions. Moreover, a version of the results carries over when x is a vector.

Proposition 8 *Let $U : R^k \times R^\ell \to R$ be a utility function which is additively separable into the form $U(x,y) = f(x)+g(y)$ where both f and g are twice-differentiable, strictly increasing and strictly concave, and x and y are vectors. If some x_i is a normal good for the utility function $F(x,y) = f(x)$, and some y_j is a normal good for the utility function $G(x,y) = g(y)$, then (i) x_i and y_j are normal goods for the utility function U, and (ii) x_i and y_j are Hicksian substitutes. Moreover, if y_j is a normal good, then (iii) increasing p_{y_j} increases the compensated aggregate expenditure on goods x. Whether or not y_j is normal, if all the mixed partials of g are less than or equal to zero, then (iv) y_j and some y_k are Hicksian substitutes.*

Proof Consider the subproblem:

$$\max_x f(x) \text{ subject to } p_x x \leq I. \tag{9}$$

Denote by $v(I)$ the maximum value of (9) for a fixed price vector. The full maximization can be separated into first maximizing (9) and then choosing I optimally from

$$\max_{y,I} g(y) + v(I) \text{ subject to } p_y y + I \leq W. \tag{10}$$

Since v is strictly concave, the proof of Proposition 3 applies to (10) and to the problem with the roles of x and y reversed. This proves (i).

The proof of Proposition 1 also applies to (10). Hence an increase in the price of any y_j increases the compensated demand for I. This proves (iii).

Moreover, if x_i is normal in (9), this increase in the compensated demand for I increases the compensated demand for x_i in the original problem, proving (ii).

Finally, the proof of Proposition 2 can be extended in a straightforward manner to prove that some y_k must be a Hicksian substitute for y_j in (10), which proves (iv). ■

In one interpretation, we can think of x as the vector of first period consumption goods, and y as a vector of assets in a two-period problem. If g is the expected utility of the aggregate asset payoffs, and if the return from each asset is nonnegative, then Proposition 8 applies. If any asset is a normal good, then it is a Hicksian substitute for first period aggregate consumption and a Hicksian substitute for at least one other asset. This can be extended to Marshallian demands in some settings. If we knew, say, that investments in mortgage derivatives increased with income, then an increase in the price of mortgage derivatives, all else equal, would raise aggregate consumption in the first period and increase investment in at least one other asset, for any investor who maximized expected utility and was a net seller of mortgage derivatives.

In another interpretation, we can think of y as two assets paying off in some states in period 2, x_i as first-period consumption, and the rest of the x variables as assets paying off only in period 2 states disjoint from those in which the y assets pay off, or paying off in later periods (provided intertemporal preferences are additive and there is no additional trading at the time of consumption of the proceeds of assets y). Then again Proposition 8 applies. From separability of f, we know that

first period consumption is a normal good. Using the von Neumann expected utility interpretation of g, the two assets y are Hicksian substitutes and, if asset y_j is a normal good, then it is a Hicksian substitute for first-period consumption.

5 References

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