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NOTE 1 – DERIVING BLACK-SCHOLES FROM LOGNORMAL ASSET RETURNS

Mike Staunton

Author details:

Mike Staunton

Email: mstaunton@london.edu

Abstract

The Black-Scholes formula assumes that log share prices follow a continuous normal distribution. All options are valued in a risk-neutral environment, mirroring the insight behind the BS formula that a risk-free hedge portfolio can be created. The option value is then estimated as the discounted expectation of the option payoff.

The original Black-Scholes paper is littered with stochastic calculus and partial differential equations and, to my mind, this obscures the assumptions made about asset returns and, to a lesser extent, risk-neutrality. We're going to provide an alternative derivation free from any mention of Ito's lemma, integral signs and mathematical gobbledygook (such as add the exponents by using the trick known as completing the square). This alternative derivation highlights the key role played by the assumption that asset returns are lognormal and, in so doing, leads on to models that use higher moments such as skewness and kurtosis to reflect the volatility smile.

NOTE 1 – DERIVING BLACK-SCHOLES FROM LOGNORMAL ASSET RETURNS

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Risk-neutral pricing

Let us start by assuming that the correct way to value a call option is by discounting the expected option payoff at expiry, where the expectation is calculated in a risk-neutral world. We will also assume that log share prices follow a normal distribution with mean M and standard deviation $\sqrt{V}$; hence $\ln S_T \sim N(M,V)$. Along the way we will define d_2 to equal $(M-\ln(X)) / \sqrt{V}$ and use a standard normal random variable $Z_T \sim N(0,1)$.

The call has a value equal to the discounted expectation of it's payoff, under the risk-neutral probability measure Q:

$$c = \exp(-rT)\, E^Q [\, \max (S_T - X, 0)\,]$$

This can be divided into two terms:

$$c = \exp(-rT)\, E^Q [S_T \mid S_T > X] - \exp(-rT)\, E^Q [X \mid S_T > X]$$

We will first consider the expectation in the second term:

$$\begin{aligned} E^Q [X \mid S_T > X] &= X \text{ Prob } [\ln S_T > \ln X] \\ &= X \text{ Prob } [(\ln S_T - M) / \sqrt{V} > (\ln X - M) / \sqrt{V}\,] \\ &= X \text{ Prob } [Z_T > (\ln X - M) / \sqrt{V}\,] \\ &= X\, N(d_2) \end{aligned}$$

The variable $\ln S_T$ has been converted into a standard normal variable by subtracting its mean and dividing by its standard deviation. Then the left-hand tail probability has been converted into the right-hand tail probability, here the cumulative distribution function of the standard normal distribution (using the symmetry of the normal distribution function).

The expectation in the first term is more difficult since S_T is both the subject of the expectation and the conditional statement within the expectation. Instead we rely on two standard results, one obscure and one familiar, for lognormal and normal variables.

The obscure result says that if $\ln S_T \sim N(M,V)$ then $E[S_T \mid S_T > X] = E[S_T]\, N(\sqrt{V} - \{\ln(X)-M\}/\sqrt{V})$, where X is a number and N() is the cumulative normal distribution function. This result breaks the conditional expectation into an ordinary expectation times a normal distribution function.

The familiar result says that if $\ln S_T \sim N(M,V)$ then $E[S_T] = \exp(M+0.5V)$. This allows us to calculate the ordinary expectation and thus complete the simplification of the conditional expectation in the first term:

$$
\begin{aligned}
E^Q[S_T \mid S_T > X] &= E^Q[S_T]\, N(\sqrt{V} - \{\ln(X)-M\}/\sqrt{V}) \\
&= E^Q[S_T]\, N(d_2 + \sqrt{V}) \\
&= E^Q[S_T]\, N(d_1) \\
&= \exp(M+0.5V)\, N(d_1)
\end{aligned}
$$

Combining the two terms from the risk-neutral value now gives us the lognormal call option valuation formula:

$$c = \exp(-rT)\,[\exp(M+0.5V)\, N(d_1) - X\, N(d_2)]$$

where we have defined d_1 as $d_2 + \sqrt{V}$.

We see that $N(d_2)$ is the probability in the risk-neutral world that the option will be exercised, whereas $N(d_1)$ is the ratio of two risk-neutral expectations, the expectation of S_T conditional on option exercise divided by the unconditional expectation of S_T.

All we now need to derive the familiar Black-Scholes formula (extended to allow for continuous dividends) is to replace M and V. Let us assume that $M = \ln S + (r-q-0.5\sigma^2)T$ and $V = \sigma^2 T$.

$$
\begin{aligned}
c &= \exp(-rT)\,[\exp(M+0.5V)\, N(d_1) - X\, N(d_2)] \\
&= \exp(-rT)\,[S \exp((r-q)T)\, N(d_1) - X\, N(d_2)] \\
&= S \exp(-qT)\, N(d_1) - X \exp(-rT)\, N(d_2)
\end{aligned}
$$

Though this is not the route that Black and Scholes followed in their original derivation, you should now see how risk-neutral pricing is combined with the assumption of normal log share prices to give the lognormal option valuation formula. Adding the parameter values for the normal distribution then leads to the Black-Scholes formula. The next section shows how the parameters of the lognormal and normal distributions are linked.

Linking normal and lognormal distributions

In finance, a standard assumption is that share prices follow a lognormal distribution – as a consequence log share prices will follow a normal distribution. Equivalently we can say that log share returns follow a normal distribution – as a consequence share returns will follow a lognormal distribution. Hence it doesn't matter if we talk of lognormal share prices or normal log share returns, as these are just two sides of the same assumption.

Asset	Prices : S_T	Log Prices : ln S_T
	Returns : $S_T/S - 1$	Log Returns : ln (S_T/S)
Distribution	Lognormal	Normal
First moment	M1	M
Second moment	M2	V
First moment link	M1 = exp (M + 0.5V)	M = 2 ln (M1) - 0.5 ln (M2)
Second moment link	M2 = exp (2M + 2V)	V = -2 ln (M1) + ln (M2)

We can characterise a distribution through its moments about the mean (as for the normal distribution in the above table) or equivalently through its moments about zero (as for the lognormal distribution). The first moment of any distribution is its mean (denoted M1 or simply M), while the second moment (about the mean) is the variance, V as opposed to the second moment about zero, M2.

This equivalence allows us to translate parameters between the equivalent exact normal and lognormal distributions. Let us assume that the log share returns have a normal distribution with mean M and variance V. Then we can find the central moments (M1 and M2) of the equivalent lognormal distribution for returns using the link given at the bottom of the second column in the table. We can reverse this process by finding formulas for the mean and variance of the normal distribution (M and V) in terms of the central moments of the equivalent lognormal distribution using the link given at the bottom of the third column in the table.

The spreadsheet layout

	A	B	C	D	E	F	G	H
1	**MikeWF01.XLS**		**Deriving Black-Scholes from Lognormal Asset Returns**					
2								
3	Share price (S)	**100.00**		M	4.6188			
4	Exercise price (X)	**98.00**		V	0.0128			
5	Int rate-cont (r)	**5.00%**						
6	Dividend yield (q)	**1.00%**		d_2	0.2988			
7	Option life (T,years)	**0.50**		d_1	0.4119			
8	Volatility (σ)	**16.00%**						
9				N (d_2)	0.6174		$E^Q[S_T]$	102.02
10				N (d_1)	0.6598		$E^Q[S_T \mid S_T>X]$	67.31
11								
12				call (LN)	6.6352			
13								
14				via LN fn	6.6352			
15				via BS fn	6.6352			

Writing VBA functions

Once you are familiar with Excel, writing code for VBA functions is relatively straightforward – the following table shows how to translate mathematical expressions into Excel and then into VBA. The code for functions is then entered into a VBA module sheet (hit Alt+F11 from Excel) – you should find the code for the functions I have written by looking for the module sheet contained in the project named as the spreadsheet MikeWF01.xls. To create your own module sheet, use Insert / Module from the VBA menu in the Project Explorer window and Alt+F11 will return you to Excel.

Translation Table

Expression	Excel	VBA
xy	x*y	x*y
x^2	x^2	x^2
$\log_e x$	LN(x)	Log(x)
exp(x)	EXP(x)	Exp(x)
$\sqrt{x}$	SQRT(x)	Sqr(x)
3!	FACT(3)	Application.Fact(3)
$N(d_1)$	NORMSDIST(d1)	Application.NormSDist(d1)
$N'(d_1)$ or $n(d_1)$	NORMDIST(d1,0,1,FALSE)	Application.NormDist(d1,0,1,FALSE)
$\Phi[a:n,p]$	1-BINOMDIST(a-1,n,p,TRUE)	1-Application.BinomDist(a-1,n,p,TRUE)
Uniform (0,1)	RAND()	Rnd
Normal (0,1)	NORMSINV(RAND())	Application.NormSInv(Rnd)
Normal (M,V)	M+SQRT(V)* NORMSINV(RAND())	M+Sqr(V)* Application.NormSInv(Rnd)

The VBA Code

The Option Explicit line forces you to declare all variables (apart from input parameters) using Dim statements, while the Option Base 1 line ensures that VBA arrays are numbered starting from 1 (to conform with Excel). The Dim statement declares the variables with the default Variant type. The VBA functions, Sqr, Log and Exp, must be used in place of their Excel equivalents. Excel functions are used with the preface Application. (alternatively the newer WorksheetFunction. preface will do). Once written, the function can be called from the Function Wizard (in the User Defined category) just like any ordinary Excel functions.

```
Option Explicit
Option Base 1

Function LNCallValue(M, V, X, r, tyr)
'   Returns LogNormal Call Value
    Dim d1, d2, Nd1, Nd2
    d2 = (M - Log(X)) / Sqr(V)
    d1 = d2 + Sqr(V)
```

```
    Nd1 = Application.NormSDist(d1)
    Nd2 = Application.NormSDist(d2)
    LNCallValue = Exp(-r * tyr) * (Exp(M + 0.5 * V) * Nd1 - X * Nd2)
End Function
```

For completeness, I have written another function with the more traditional inputs – note how I have incorporated the LNCallValue function in the code.

```
Function BSCallValue(S, X, r, q, tyr, sigma)
'   Returns Black-Scholes Call Value
    Dim M, V
    M = Log(S) + (r - q - 0.5 * sigma ^ 2) * tyr
    V = (sigma ^ 2) * tyr
    BSCallValue = LNCallValue(M, V, X, r, tyr)
End Function
```

Next steps

There are two important directions that this alternative derivation of the Black-Scholes formula points towards. The first is using higher moments to allow for deviations from strict normality (this can be seen in the Gram-Charlier expansion for vanilla options below and also can be used to derive analytic approximations for Asian and Basket options). The second is that the valuation of European options simply involves the calculation of a mathematical expectation – so showing why standard mathematical techniques such as Monte Carlo simulation and numerical integration have found their way into option valuation.

Exercise

In recent years, more interest has been shown in allowing for deviations from strict lognormality (motivated by the existence of volatility smiles). One way is to incorporate third and fourth moments that differ from normality (skewco=0, kurtco=3) and Corrado & Su use the Gram-Charlier expansion to adjust the normal density function. Their call option formula adds two terms to the standard LN/Black-Scholes option value. You might also like to write a user-defined function for the Gram-Charlier call option value.

$C(GC) = C(BS) + \text{skewco}\ Q_3 + (\text{kurtco}-3)\ Q_4$

$Q_3 = (1/3!)\ S \exp(-qT)\ \sigma \sqrt{T}\ [\ (2\ \sigma \sqrt{T} - d_1)\ n(d_1) + (\sigma \sqrt{T})^2\ N(d_1)\ ]$

$Q_4 = (1/4!)\ S \exp(-qT)\ \sigma \sqrt{T}\ [\ (d_1^2 - 1 - 3\ \sigma \sqrt{T}\ d_2)\ n(d_1) + (\sigma \sqrt{T})^3\ N(d_1)\ ]$

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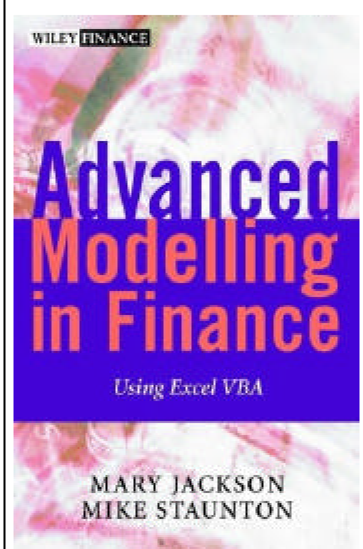

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About the Authors:

Mary Jackson and Mike Staunton have worked together teaching spreadsheet modelling to both graduate students and practitioners

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