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BLACK-SCHOLES EQUATION IN LAPLACE TRANSFORM DOMAIN

Igor Skachkov

Abstract

Laplace transformation is one of the most popular methods of solution of diffusion equations in many areas of science and technology. It is much less used in financial engineering. One reason is obvious: it is not supposed to be a way to solve a Nobel Prize winning problem. Another one is technical: not many people know that all that they need to do is to make simple calculations in the Laplace domain. Three years before Black-Scholes formula, the famous (in other areas) Stehfest algorithm of numerical inversion of Laplace transforms was published (Stehfest, 1970). The performance of the numerical procedure is comparable with evaluation of cumulative density functions: your solution in Laplace domain will be calculated in 10-14 predefined points on a real axes to achieve 5-6 digits accuracy.

It took more then 10 years to “discover” Stehfest algorithm in Hydrodynamics of porous media. In short time employing of the algorithm dramatically improved the quality of well testing analysis in the 80s.

Black-Scholes Equation in Laplace Transform Domain

Igor Skachkov, iskachkov@yahoo.com, January 2002

Laplace transformation is one of the most popular methods of solution of diffusion equations in many areas of science and technology. It is much less used in financial engineering. One reason is obvious: it is not supposed to be a way to solve a Nobel Prize winning problem. Another one is technical: not many people know that all that they need to do is to make simple calculations in the Laplace domain. Three years before Black-Scholes formula, the famous (in other areas) Stehfest algorithm of numerical inversion of Laplace transforms was published (Stehfest, 1970). The performance of the numerical procedure is comparable with evaluation of cumulative density functions: your solution in Laplace domain will be calculated in 10-14 predefined points on a real axes to achieve 5-6 digits accuracy.

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Double-Barrier Option Pricing as a General First Boundary Problem

Laplace transform has an advantage over other analytical approaches e.g. Fourier or Green's function methods in solving of many practical problems in finance. The most obvious one is a barrier options pricing.

For Call Options (Laplace space, Black-Scholes world) the system of equation and boundary conditions is:

$$\frac{1}{2}s^2\frac{\partial^2\overline{V}}{\partial x^2}+(r-\frac{1}{2}s^2)\frac{\partial\overline{V}}{\partial x}-(r+p)\overline{V}=-\max(0,e^x-1)$$

$$\overline{V}(x_d)=\bar{f}_d$$
$$\overline{V}(x_u)=\bar{f}_u$$

where prices are normalized by the strike.

Let $\overline{V}_v\ (x)$ be the Laplace transform of a value of plain vanilla call option (unbounded solution of a non-homogeneous diffusion equation). Then a complete solution can be presented as:

$$\overline{V}=\overline{V}_v+\overline{V}_b$$

$$\overline{V}_v=C_1e^{k_1x} \qquad x\le 0$$
$$\overline{V}_v=\frac{1}{p}e^x-\frac{1}{r+p}+C_2e^{k_2x} \qquad x>0$$

$$k_{1,2}=\frac{1}{s}(-a\pm l),\ l=\sqrt{a^2-b},\ a=\frac{1}{s}(r-\frac{1}{2}s^2),\quad b=-2(r+p),$$

To make a solution smooth at strike price we need

$$C_{1,2}=\frac{1}{k_1-k_2}\left(\frac{1-k_{2,1}}{p}+\frac{k_{2,1}}{r+p}\right).$$

$\overline{V}_b(x)$ is a solution of homogeneous equation

$$\overline{V}_b(x)=C_3e^{k_1x}+C_4e^{k_2x}$$

with the boundary conditions

$$\bar{V}_b(x_d) = j_d = \bar{f}_d - \bar{V}_v(x_d)$$
$$\bar{V}_b(x_u) = j_u = \bar{f}_u - \bar{V}_v(x_u)$$

Substituting the general solution in boundary conditions we finally find constants C_3 and C_4

$$C_{3,4} = \pm \frac{j_u e^{(k_{2,1}-k_2)x_d - k_1 x_u} - j_d e^{(k_{2,1}-k_1)x_u - k_2 x_d}}{(1-e^{-(k_1-k_2)(x_u-x_d)})}$$

That is it for the general first (Direchlet) boundary problem. The formulae above plus the put-call symmetry and reflection principle give us a number of different exotic options values in Laplace domain. Then, in a split of a second they could be converted into real prices.

Another advantage of Laplace transform method is in straightforward constructing of replications of complex options. Thus the results for double barrier options can be expanded in power series of small parameter

$$e = e^{-(k_1-k_2)(x_u-x_d)} < 1$$

and represented as a series of one side barrier options.

The second (Neumann) and the third (mixed) boundary problems are not harder to solve. Unfortunately, it is much easier to solve these problems than to find proper examples in financial practice. One and yet not pure application is an option on the underlying with the supported price due to the stock issuer repurchasing policy or governmental currency regulations. It means that we have an impenetrable wall at this level.

$$q = -((r - \frac{1}{2}s^2)\bar{V} + \frac{1}{2}s^2 \frac{\partial}{\partial x}\bar{V}) = 0, \quad x = x_d$$

where q is the full (convection and diffusion) flow rate.
If currency is kept in a corridor, then we have zero flow rates at both walls.

To generalize the case we can put a membrane instead of a wall. The value of option has a break, proportional to delta and coefficient of “skin” effect.

$$\bar{V}_+ - \bar{V}_- = -sq, \quad x = x_d$$

where s is a “skin” coefficient. Passing over “sacred” round numbers could be another example of skin effect.

From Microbiology and Porous Hydrodynamics, we turn to Kinetic Theory of Gases with “jump” and “slip” conditions on surfaces as useful abstractions to obtain second order approximations for concentration and temperature distributions. Market manipulations with barrier options could make their prices behave as if

$$\bar{V} = -k_{jump} q, \quad x = x_d$$

while actually options have zero value at that barrier.

All these parameters might look very artificial, but they are not more artificial than implied volatility and much easier to integrate in flexible pricing models.

Beyond Barriers and a Nutcracker’s Smile

Double-barrier step options lose their prices proportionally to the time the underlying is beyond a prescribed barrier level. (I wonder why there are no American options amortizing their principal all the time or options with the payoff equal to *max(0,exp(-Dt)S-E)* which mimic options on a stock providing continuous dividend yield?) The pricing of double-barrier step options can be described as a 3-zone problem. We have standard initial condition and distributed sinks/sources outside of the corridor with the power density proportional to option value: $\bar{Q} = \{-r_d \bar{V}_d, 0, -r_u \bar{V}_d\}$. The system of equations and boundary conditions in Laplace domain can be obtained by subtracting a source term from the right side of Black-Scholes equation and applying continuity conditions for the function and its space derivative (delta).

We can apply the results of previous point to option pricing with the volatility smile. The simplest approach would be 3-zone model with three different but constant inside of each zone diffusion coefficients (Nutcracker's smile). As before, we need to obtain similar solutions in each zone and calculate arbitrary coefficients applying continuity conditions. Note, that continuity for the flow rate cannot be reduced to continuity for delta in this case. Generalization that makes a smile less ugly is obvious. A combination with numerical methods for ordinary differential equations is also a possibility.

Diffusion equation with time-dependent coefficients also would not be an obstacle for our method as far as we stay with the fixed boundary conditions (see P. Wilmott, 1999 for the transformations).

Numerical Algorithms

H. Stehfest's algorithm was the first fast and accurate method to convert **smooth** functions. Fortunately, it is almost always true for the diffusion equation solutions.

Recently other powerful algorithms of numerical inversion of Laplace transforms were introduced (J. Abate, W. Whitt, 1995). Abate and Whitt approaches are a little slower than the Stehfest one, but provide a better precision and give us an additional opportunity to control the accuracy of calculations (see *Appendix* for a short description of EULER algorithm).

Euler binomial summation and Stehfest linear combination accelerates convergence dramatically. Try to apply a brute force (Abate and Whitt EULER algorithm is reduced to trapezoidal-rule approximation of the Bromwich inversion integral by setting parameter *m* to zero) and feel the difference!

High precision (up to 10^{-11} with the 8 bytes float point arithmetic) and excellent performance seems a useless luxury if all we need is to solve Black-Scholes equation with the known coefficients, but I expect, that they would be greatly appreciated for multi-parameter fitting and optimization purposes, especially in real-time environment.

H. Stehfest provided ALGOL code and J. Abate and W. Whitt provided BASIC code in their articles.
Do not hesitate to ask me for C++ code.

References

1. J. Abate and W. Whitt, 1995. Numerical Inversion of Laplace Transforms of Probability Distributions, ORSA Journal on Computing, 7, 36-43
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3. P. Wilmott, 1999. Derivatives, John Wiley & Sons

Appendix

Abate and Whitt EULER algorithm is specified by

$$f(t;m,n) = \frac{e^{A/2}}{t} 2^{-m} \sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix} \sum_{l=0}^{n+k} (-1)^l a_l(t)$$

$$a_0 = \frac{1}{2} \mathrm{Re}\left(F\left(\frac{A}{2t} \right) \right), \quad a_k = \mathrm{Re}\left(F\left(\frac{A + 2kpi}{2t} \right) \right), \text{ k>0}$$

Euler summation was used as an acceleration technique that can be described as the weighted average of the last *m* partial sums by a binomial probability distribution with parameters *m* and *p*=1/2.
For better performance, we changed the order of summation and used pre-calculated sum of binomial coefficients.

$$f(t;m,n) = \frac{e^{A/2}}{t} \left(\sum_{l=0}^{n} (-1)^l a_l(t) + \sum_{l=1}^{m} (-1)^{n+l} w_{ml} a_{n+l}(t) \right)$$

where $w_{ml} = 2^{-m} \sum_{k=0}^{m-l} \begin{bmatrix} m \\ k \end{bmatrix}$

Parameter A defines discretization error: approximately, to produce 10^{-k} accuracy $A=k*log10$ should be taken. However, with A too high round-off errors increase. Authors of the algorithm used k=8, m=11, and n=15 to convert their functions. Our calculations (VC++ compiler, 8 bytes double and long double types) achieve the best precision with k=10, m is from 12 to 17 and n is in the range of 35 - 45. Therefore 10, 12 and 35 were chosen as the default values.

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