A Simple Way to Find Prices, Greeks, and Static Hedges for Barrier Options*

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Abstract

We give simple proofs of some pricing and hedging results for barrier options.

Following the probabilistic approach in Andreasen (2001), we prove some pricing and hedging results for single barrier zero-rebate options. The results are all well-established, but the proofs are self-contained and simple; no reference to first-hitting-time densities or the joint distribution of Brownian motion & its maximum is needed, be this explicit (as for instance in Björk (1998)[Chapter 13]) or implicit (as in Carr & Chou (1997)[Appendix]). All that is needed is a suitable change of measure.

We look at the Black-Scholes model for a dividend-paying stock, i.e. an asset whose price dynamics are

$$dS(t) = (r-q)S(t)dt + \sigma S(t)dW^Q(t),$$

under the equivalent martingale measure Q (with the bank-account as numeraire). Put $p = 1 - 2(r-q)/\sigma^2$. Consider a simple claim with a payoff at time T specified by a (suitably nice) payoff function g (a "g-claim" for short). Its arbitrage-free time-t price

$$\pi^g(t) = e^{-r(T-t)}\mathbf{E}_t^Q(g(S(T))) = e^{-r(T-t)}f(S(t),t),$$

where of course $f(S(t),t) = \mathbf{E}_t^Q(g(S(T)))$, and the Markov property of S ensures that this is non-deceptive notation.

Let B be a constant, define a new function $\widehat{f}$

$$\widehat{f}(x,t) = (x/B)^p f\left(B^2/x,t\right),$$

and look at a simple claim with payoff-function $\widehat{g}(x) = \widehat{f}(x,T)$.

Define the process Z by

$$Z(t) = \left(\frac{S(t)}{B}\right)^p.$$

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Using the Ito formula we get

$$dZ(t) = p\sigma Z(t)dW^Q(t),$$

so $Z(t)/Z(0)$ is a positive, mean-1 Q-martingale. Here the exact form of p is needed, the result does not hold if σ isn't constant. (That is, unless it happens that $r = q$, as is the case in Andreasen (2001).) Hence

$$\frac{dQ^Z}{dQ} = \frac{Z(T)}{Z(0)} \text{ on } \mathcal{F}_T$$

defines a probability measure $Q^Z \sim Q$.

Now recall the "Abstract Bayes Formula" for conditonal means, see Øksendal (1995)[Lemma 8.24] for instance. To recap the theorem: Let μ and ν be equivalent probability meausures on a space $(\Omega, \mathcal{F})$, and let f be their Radon/Nikodym-derivative, ie.

$$\frac{d\nu}{d\mu} = f.$$

Let X be a random variable and $\mathcal{H}$ a σ-algebra such that $\mathcal{H} \subset \mathcal{F}$. Then

$$\mathbf{E}^\nu(X|\mathcal{H})\mathbf{E}^\mu(f|\mathcal{H}) = \mathbf{E}^\mu(fX|\mathcal{H}). \tag{1}$$

Now apply the formula with the Q^Z in the role of ν, Q as μ, $Z(T)/Z(0)$ as f, $\mathcal{F} = \mathcal{F}_T$ and $\mathcal{H} = \mathcal{F}_t$. Because $Z(t)/Z(0)$ is a martingale this gives us that the price of the $\widehat{g}$-claim can be written as

$$\begin{aligned} \pi^{\widehat{g}}(t) &= e^{-r(T-t)}\mathbf{E}_t^Q\left(\left(\frac{S(T)}{B}\right)^p g\left(\frac{B^2}{S(T)}\right)\right) \\ &= e^{-r(T-t)}\left(\frac{S(t)}{B}\right)^p \mathbf{E}_t^{Q^Z}\left(g\left(\frac{B^2}{S(T)}\right)\right). \end{aligned}$$

Girsanov's theorem (Øksendal (1995)[Theorem 8.26]) tells us that

$$dW^{Q^Z}(t) = dW^Q - p\sigma dt \tag{2}$$

defines a Q^Z-Brownian motion. Put $Y(t) = B^2/S(t)$. Then the Ito formula and (2) gives us that (again, the particular form of p is needed)

$$dY(t) = (r-q)Y(t)dt + \sigma Y(t)\left(-dW^{Q^Z}(t)\right),$$

which means the law of Y under Q^Z is the same as the law of S under Q. Therefore

$$\mathbf{E}_t^{Q^Z}(g(Y(T))) = f(Y(t),t) = f(B^2/S(t),t),$$

and the $\widehat{g}$-claim's price is

$$\pi^{\widehat{g}}(t) = e^{-r(T-t)}\widehat{f}(S(t),t) = e^{-r(T-t)}(S(t)/B)^p f(B^2/S(t),t).$$

And why is this useful?

Consider a 0-rebate knock-out (at a barrier B) version of a claim with payoff function g ("the barrier option" in the following). Without loss of generality we may assume that $g(x) = 0$ for $x \leq B$ (in the down-and-out case) or $g(x) = 0$ for $x \geq B$ (in the up-and-out case). Consider a simple claim with payoff function $h = g - \widehat{g}$ (the "adjusted payoff" in the language of Carr & Chou (1997)). Note that $h(x) = g(x)$ if $x \geq B$ ($x \leq B$ "for up-and-out"). The time-t price of the h-claim is

$$\pi^h(t) = e^{-r(T-t)} \left(f(S(t), t) - (S(t)/B)^p f(B^2/S(t), t) \right). \tag{3}$$

We see that if $S(t) = B$, then the h-claim has a price of 0. Suppose now that we buy the h-claim, sell it again if the stock price hits the barrier, and if this doesn't happen, simply hold it until expiry. If the stock price stays above (below) the barrier, we receive $g(S(T))$ at expiry, otherwise we get 0. In other words, exactly the same as the barrier option, so we can read off its price directly from (3). In this way we can establish all formulas from Björk (1998)[Chapter 13.2-3] (possibly trough in-out parities). We also easily get information about the various partial derivatives of the price wrt. input variables (often called "greeks"). For instance we immediately see that the Vega (i.e. ∂price$/\partial\sigma$) of a knock-out barrier call is lower than that of the plain vanilla call; it could even be negative. Further, since any simple claim can be statically hedged by a portfolio of plain vanilla puts & calls, we can also devise static hedges for the barrier option. Note, however, that for a general p, the h-function is not piecewise linear, so the static hedge portfolio involves a continuum of options. In fact, h could be discontinuous at the barrier level (this happens for up-and-out calls), in which case matching the pay-off becomes problematic in practice (unless some sort of digital options exist).
If we do assume that $r = q$ (this could be because we're really looking at forward prices, or because we're in an exchange rate setting with equal foreign and domestic interest rates) and consider a call option, $g(x) = (x - K)^+$, then we find that

$$\widehat{g}(x) = (x/B)(B^2/x - K)^+ = (K/B)(B^2/K - x)^+,$$

which is the payoff of K/B puts with strike B^2/K. So the knock-out call can be hedged by buying the plain vanilla call and shorting K/B strike-B^2/K puts.

References

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