Barriers, Lookbacks and other Exotica

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Part I – Summary

In preparing our approach to pricing barrier and lookback options in the Black-Scholes framework, we shall consider the following tools:

T1. Binary Options (digitals)

T2. Gaussian Shift Theorem (GST)

T3. Static Replication

T4. Parity Relations

T5. Image Options and the Method of Images

T6. Equivalent European Payoffs

Not all of these tools are restricted to the Black-Scholes world.

Basic Pricing Methodology

It is well-known that there are two distinct, but equivalent approaches to arbitrage free pricing in the Black-Scholes framework.

1. The PDE approach

2. The EMM approach

Depending on the type of option, it is sometimes better to use the PDE method and at other times the EMM method. The trick is to use both methods in an integrated way that captures their most expedient features.

We briefly review the two methods.

Black-Scholes PDE

If $V(x,t)$ denotes the arbitrage-free present value of a European-style derivative with expiry payoff function $f(x)$. Then $V(x,t)$ satisfies the BS-pde:

$$\boxed{V_t = rV - (r-q)xV_x - \tfrac{1}{2}\sigma^2 x^2 V_{xx}}$$

in $t < T,\ x > 0$ and subject to the terminal value, $V(x,T) = f(x)$.

Examples:
European call option
$f(x) = (x-k)^+ = (x-k)\mathbb{I}(x>k)$

European put option
$f(x) = (k-x)^+ = (k-x)\mathbb{I}(x<k)$

We shall write the BS-pde in the operational form:

$$\boxed{\mathcal{L}V = 0}$$

Equivalent Martingale Measure

Theorem: *(Feynman-Kac)* Subject to regularity conditions on $f(x)$, the unique solution of the terminal value BS-pde $\mathcal{L}V = 0$, $V(x, T) = f(x)$ is given by

$$\boxed{V(x,t) = e^{-r\tau}\mathbb{E}_Q\{f(X_T)|\mathcal{F}_t\}}$$

where, under Q,

$$\boxed{X_T \stackrel{d}{=} xe^{(r-q-\frac{1}{2}\sigma^2)\tau+\sigma\sqrt{\tau}Z}}$$

and $\tau = T - t, \quad Z \sim N(0, 1)$.

The FK-formula is identical to the *First Fundamental Theorem on Asset Pricing* (Harrison & Pliska 1981) which states:

If the market is arbitrage-free, the discounted price of any derivative is a martingale wrt the measure Q.

Further, if the market is complete, then the EMM Q is unique.

T1. Binary Options

Contract	*Expiry T* *Payoff* $f(x)$	*Present Value Notation*
Zero Coupon Bond	1	$B(x,t)$
Unit asset	x	$A(x,t)$
Up-Bond Binary	$\mathbb{I}(x>\xi)$	$B_\xi^+(x,t)$
Down-Bond Binary	$\mathbb{I}(x<\xi)$	$B_\xi^-(x,t)$
Up-Asset Binary	$x\mathbb{I}(x>\xi)$	$A_\xi^+(x,t)$
Down-Asset Binary	$x\mathbb{I}(x<\xi)$	$A_\xi^-(x,t)$

It is easy to show:

$$\begin{aligned} A(x,t) &= xe^{-q\tau} \\ B(x,t) &= e^{-r\tau}; \qquad \tau = T - t \end{aligned}$$

. . . but how shall we price the binaries (digitals) ?

Answer: EMM + GST

T2. Gaussian Shift Theorem

Theorem: (*Uni-variate GST*)
Let $Z \sim N(0,1)$, c a real constant and $F(Z)$ a measurable function. Then

$$\boxed{\mathbb{E}\{e^{cZ}F(Z)\} = e^{\frac{1}{2}c^2}\,\mathbb{E}\{F(Z+c)\}}$$

Proof: With $\phi(y) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2}$,

$$\begin{aligned}
\text{RHS} &= e^{\frac{1}{2}c^2}\int_{-\infty}^{\infty} F(y+c)\phi(y)\,dy \\
&= e^{\frac{1}{2}c^2}\int_{-\infty}^{\infty} F(z)\phi(z-c)\,dz; \quad (z = y+c) \\
&= \int_{-\infty}^{\infty} e^{cz}F(z)\phi(z)\,dz \\
&= \mathbb{E}\{e^{cZ}F(Z)\} = \text{LHS}
\end{aligned}$$

The third line above comes from the identity

$$\phi(z-c) = \frac{e^{-\frac{1}{2}(z-c)^2}}{\sqrt{2\pi}} = e^{cz-\frac{1}{2}c^2}\phi(z) \qquad \square$$

Multi-Variate Gaussian Shift Theorem

Theorem: (*Multi-variate GST*)
Let $\boldsymbol{Z} \sim N(0, 1; R)$ be a standard Gaussian random vector with correlation matrix R, $\boldsymbol{c}$ any real constant vector and $F(\boldsymbol{Z})$ any measurable function of $\boldsymbol{Z}$ with finite expectation. Then

$$\boxed{\mathbb{E}\{e^{\boldsymbol{c}'\boldsymbol{Z}}\, F(\boldsymbol{Z})\} = e^{\frac{1}{2}\boldsymbol{c}'R\boldsymbol{c}}\,\mathbb{E}\{F(\boldsymbol{Z} + R\boldsymbol{c})\}}$$

Note that the multi-variate GST does reduce to the uni-variate GST, since in the uni-variate case, $R = 1$ and $\boldsymbol{c}$ is a scalar.

The multi-variate GST is used to price elementary contracts for the dual-expiry options, rainbow options and also for multi-period, multi-asset exotics in general.

One should not under-estimate the importance for option pricing of this simple yet powerful result. It obviates the need for Girsanov's Theorem, the Esscher Transform and Change of Numeraire.

Pricing the Asset Binary

For a down-type asset binary, we calculate, using

$$X_T = xe^{(r-q-\frac{1}{2}\sigma^2)\tau+\sigma\sqrt{\tau}Z}$$

$$\begin{aligned}
A_\xi^-(x,t) &= e^{-r\tau}\,\mathbb{E}_Q\{X_T\,\mathbb{I}(X_T<\xi)\} \\
&= xe^{-(q+\frac{1}{2}\sigma^2)\tau}\,\mathbb{E}_Q\{e^{\sigma\sqrt{\tau}Z}\,\mathbb{I}(Z<-d'_\xi)\}; \\
&= xe^{-q\tau}\,\mathbb{E}_Q\{\mathbb{I}(Z+\sigma\sqrt{\tau}<-d'_\xi)\}; \quad \text{by GST} \\
&= xe^{-q\tau}\,\mathbb{E}_Q\{\mathbb{I}(Z<-d'_\xi-\sigma\sqrt{\tau})\} \\
&= xe^{-q\tau}\,\mathbb{E}_Q\{\mathbb{I}(Z<-d_\xi)\}; \quad d_\xi = d'_\xi+\sigma\sqrt{\tau} \\
&= xe^{-q\tau}\,\mathcal{N}(-d_\xi) \qquad \square
\end{aligned}$$

The BS formula for the general asset binary can be written as:

$$\boxed{A_\xi^s(x,t) = xe^{-q\tau}\,\mathcal{N}(sd_\xi)}$$

where $s=+$ for up-type, and $s=-$ for a down-type.

Pricing the Asset/Bond Binary

In the previous slide, observe that:

$$d_\xi, d'_\xi = \frac{\log(x/\xi) + (r - q \pm \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}$$

and that the condition $X_T \gtrless \xi$ is equivalent to:

$$Z \gtrless -d'_\xi(x,\tau); \qquad \tau = T - t$$

A similar calculation yields the result for the bond-binary as:

$$\boxed{B^s_\xi(x,t) = e^{-r\tau}\mathcal{N}(sd'_\xi)}$$

This is where the hard work is done. The rest is really nothing but algebraic manipulation.

T3. Static Replication

Theorem: (*Principle of Static Replication*)
Let $V_i(x,t)$ for $i = 1, \ldots, n$ denote the present value of a set of elementary derivatives with given expiry T payoffs $V_i(x,T) = f_i(x)$. Then the arbitrage-free present value of a derivative contract with payoff

$$V(x,T) = f(x) = \sum_{i=1}^{n} \alpha_i \, f_i(x)$$

is given by

$$V(x,t) = \sum_{i=1}^{n} \alpha_i \, V_i(x,t).$$

This principle is equivalent to saying that: 'if a derivative contract has a payoff expressible as a portfolio of elementary contracts, then its fair price is the present value of the replicating portfolio'. □

Proof: Use linearity of BS-pde *(Duhamel's Principle)* or apply simple arbitrage arguments.

Pricing a Gap (Threshold) Call Option

The expiry T payoff is:

$$\begin{aligned} C_{h,k}(x,T) &= (x-k)\mathbb{I}(x>h); \quad h \geq k \\ &= x\mathbb{I}(x>h) - k\mathbb{I}(x>h) \\ &= A_h^+(x,T) - kB_h^+(x,T) \end{aligned}$$

The option is seen to be equivalent to a portfolio containing a long position in an up-type asset binary of exercise price h, and a short position in k up-type bond binaries of exercise price h. So by Static Replication

$$\boxed{C_{h,k}(x,t) = xe^{-q\tau}\mathcal{N}(d_h) - ke^{-r\tau}\mathcal{N}(d'_h)}$$

When $h = k$, this is recognised as the celebrated BS formula for a European call option.

T4. Parity Relations

Let $V_0(x,t)$ denote the present value of a general European style derivative with expiry payoff $f(x)$. Define two related binary options, $V_\xi^+(x,t)$ and $V_\xi^-(x,t)$ with corresponding payoffs:

$V_\xi^+(x,T) = f(x)\mathbb{I}(x>\xi)$ *i.e.* up-type
$V_\xi^-(x,T) = f(x)\mathbb{I}(x<\xi)$ *i.e.* down-type

Theorem: (*Up-Down Parity Relation*)
For all $x > 0$ and all $t \le T$,

$$\boxed{V_\xi^+(x,t) + V_\xi^-(x,t) = V_0(x,t)}$$

Proof: (*By static replication*). At expiry T,

$$\begin{aligned} V_\xi^+(x,T) + V_\xi^-(x,T) &= f(x)\mathbb{I}(x>\xi) + f(x)\mathbb{I}(x<\xi) \\ &= f(x)[\mathbb{I}(x>\xi) + \mathbb{I}(x<\xi)] \\ &= f(x) \\ &= V_0(x,T) \qquad \square \end{aligned}$$

Some Well-Known Parity Relations

Parity relations can be applied to the elementary contracts considered previously to give the following:

$$\begin{aligned} A^+_\xi(x,t) + A^-_\xi(x,t) &= xe^{-q\tau} \\ B^+_\xi(x,t) + B^-_\xi(x,t) &= e^{-r\tau} \\ Q^+_k(x,t) + Q^-_k(x,t) &= xe^{-q\tau} - ke^{-r\tau} \end{aligned}$$

where $\tau = T - t$ and

$$Q^s_k(x,t) = A^s_k(x,t) - kB^s_k(x,t)$$

Hence:

$$Q^+_k(x,t) = C_k(x,t) \qquad \text{and} \qquad Q^-_k(x,t) = -P_k(x,t)$$

The third equation above is of course the *put-call parity relation*.

Second Order Binaries

Second-order Binary Payoff Table	
Payoff at T_1	*Payoff at* T_2
$B^{s_2}_{\xi_2}(x_1, T_1)\mathbb{I}(s_1 x_1 > s_1 \xi_1)$	$\mathbb{I}(s_1 x_1 > s_1 \xi_1)\mathbb{I}(s_2 x_2 > s_2 \xi_2)$
$A^{s_2}_{\xi_2}(x_1, T_1)\mathbb{I}(s_1 x_1 > s_1 \xi_1)$	$x_2\mathbb{I}(s_1 x_1 > s_1 \xi_1)\mathbb{I}(s_2 x_2 > s_2 \xi_2)$

x_1, x_2 are the asset prices at times T_1 and T_2. The first order binaries above expire at time T_2 and since $s_{1,2} = \pm$, there are four different types of second-order binaries:

up-up
up-down
down-up
down-down

Second-order binaries are 'binaries of first-order binary options' ($i.e.$ compound binaries).

Pricing the Second-Order Binaries

Theorem: (*Second-order Asset and Bond Binaries*) Second-order asset and bond binaries have pv (in terms of the bi-variate normal):

$$\boxed{A^{s_1 s_2}_{\xi_1 \xi_2}(x,t) = x e^{-q\tau_2} \mathcal{N}(s_1 d_1, s_2 d_2;\ s_1 s_2 \rho)}$$

and

$$\boxed{B^{s_1 s_2}_{\xi_1 \xi_2}(x,t) = e^{-r\tau_2} \mathcal{N}(s_1 d'_1, s_2 d'_2;\ s_1 s_2 \rho)}$$

where for $i = 1, 2$,

$$[d_i, d'_i] = \frac{\log(x/\xi_i) + (r - q \pm \frac{1}{2}\sigma^2)\tau_i}{\sigma\sqrt{\tau_i}}; \quad \rho = \sqrt{\frac{\tau_1}{\tau_2}}$$

and $\tau_i = T_i - t$ denote the time intervals from the present time t to T_1 and T_2 respectively.

Proof: EMM + GST-2D

2nd-order Q-Options

Suppose the payoff at T_1 is a binary Q-option with expiry T_2. That is:

$$V(x_1, T_1) = Q^{s_2}_{k_2}(x_1, T_1; T_2)\mathbb{I}(s_1 x_1 > s_1 k_1)$$

Then the present value at time $t < T_1 < T_2$ is

$$\begin{aligned} V(x,t) &= Q^{s_1 s_2}_{k_1 k_2}(x, ; \tau_1, \tau_2); \qquad \tau_i = T_i - t \\ &= A^{s_1 s_2}_{k_1 k_2}(x;t) - k_2 B^{s_1 s_2}_{k_1 k_2}(x;t) \end{aligned}$$

Barrier Options

The pde for a down-and-out barrier option is:

$$\left.\begin{array}{rcl} \mathcal{L}V_{do}(x,t) & = & 0; \quad x > b,\ t < T \\ V_{do}(x,T) & = & f(x); \quad \forall\ x > b \\ V_{do}(b,t) & = & 0; \quad \forall\ t < T \end{array}\right\}$$

The presence of the BC at $x = b$ makes a significant difference.

Parity Relations

There are 3 parity relations for barrier options:

$$\boxed{\begin{array}{rcl} V_{do}(x,t) + V_{di}(x,t) & = & V_0(x,t) \\ V_{uo}(x,t) + V_{ui}(x,t) & = & V_0(x,t) \\ V_{ui}(x,t) & = & \overset{*}{V}_{di}(x,t) \end{array}}$$

V_0 is the pv of a standard option with the same payoff.

So knowing one (*e.g.* the D/O barrier option), we immediately get the remaining three.

T5(i). Image Options

Let $V(x,t)$ be an option price. Then the associated *image option* relative to $x = b$, denoted by $\overset{*}{V}(x,t)$, satisfies the properties:

Image Properties

1. $\overset{**}{V}(x,t) = V(x,t)$
2. $\mathcal{L}V(x,t) = 0$ with $V(x,T) = f(x)$ implies $\mathcal{L}\overset{*}{V}(x,t) = 0$ with $\overset{*}{V}(x,T) = \overset{*}{f}(x)$
3. $V = \overset{*}{V}$ when $x = b$
4. If $x > b$ (or $x < b$) is the active domain of $V(x,t)$, $x < b$ (or $x > b$) is the active domain of $\overset{*}{V}(x,t)$.

Theorem: For the BS-pde:

$$\boxed{\overset{*}{V}(x,t) = (b/x)^{\alpha} V(b^2/x, t)}$$

where $\alpha = 2(r-q)/\sigma^2 - 1$.

T5(ii). Method of Images

To solve the D/O pde, consider first the related terminal value pde

$$\left.\begin{array}{rcl} \mathcal{L}V_b(x,t) & = & 0; \qquad x>0,\ t<T \\ V_b(x,T) & = & f(x)\mathbb{I}(x>b) \end{array}\right\}$$

This is simply a European up-binary option where the payoff at expiry is adjusted so that it pays nothing if the asset price finishes below the barrier level, and the standard payoff $f(x)$ if above. It is usually easy to obtain by EMM or static replication.

Theorem: (*Method of Images*)
The unique solution of the D/O pde is given by

$$\boxed{V_{do}(x,t) = V_b(x,t) - \overset{*}{V}_b(x,t); \qquad x > b}$$

Proof: To see that this is indeed the required solution, observe that the representation

$$V_{do}(x,t) = V_b(x,t) - \overset{*}{V}_b(x,t); \qquad x > b$$

certainly satisfies the BS-pde, because both V_b and $\overset{*}{V}_b$ do and the BS-operator $\mathcal{L}$ is linear. Furthermore, $V_{do}(b,t) = 0$ since $V_b = \overset{*}{V}_b$ at $x = b$ and at expiry:

$$\begin{aligned} V_{do}(x,T) &= f(x)\mathbb{I}(x>b) - [f(x)\mathbb{I}(x>b)]^* \\ &= f(x)\mathbb{I}(x>b) - \overset{*}{f}(x)\mathbb{I}(x<b) \\ &= f(x); \qquad \text{for } \; x > b \end{aligned}$$

Note that the solution is valid only in the active domain $x > b$; obviously for $x < b$ we must have $V_{do}(x,t) = 0$.
$\square$

All the Barrier Options

Using the parity relations, the following representation for all barrier options in terms of the underlying standard option $V_0(x,t)$, its corresponding up-binary $V_b(x,t)$ and their images, is obtained:

$$\begin{aligned} V_{do}(x,t) &= V_b - \overset{*}{V}_b \\ V_{di}(x,t) &= V_0 - (V_b - \overset{*}{V}_b) \\ V_{ui}(x,t) &= \overset{*}{V}_0 + (V_b - \overset{*}{V}_b) \\ V_{uo}(x,t) &= (V_0 - \overset{*}{V}_0) - (V_b - \overset{*}{V}_b) \end{aligned}$$

It is remarkable that such a set of formulae exists, considering the complexity of barrier options. We need only price two European options: $V_0(x,t)$ and its related binary $V_b(x,t)$, and all four barrier option prices can then be determined.

T6. Equivalent European Payoffs

Barrier options are examples of *path-dependent* options. European options have payoffs which depend only on the asset price at T and are therefore path-independent options. It is possible, using the Method of Images, to write the price of any barrier option as an equivalent European option. All that is required is to find an equivalent payoff at T for an otherwise path-independent option, that replicates the price of the barrier option.

Let $f(x)$ be the payoff of any barrier option with barrier price at $x = b$. Then recall

$$\begin{aligned} V_0(x,T) &= f(x) \\ V_b(x,T) &= f(x)\mathbb{I}(x > b) \\ \overset{*}{V}_b(x,T) &= \overset{*}{f}(x)\mathbb{I}(x < b) \end{aligned}$$

So applying the barrier parity relations at $t = T$ we get:

$$\begin{aligned}
V_{do}^{eq}(x,T) &= f(x)\mathbb{I}(x>b) - \overset{*}{f}(x)\mathbb{I}(x<b) \\
V_{di}^{eq}(x,T) &= [f(x) + \overset{*}{f}(x)]\,\mathbb{I}(x<b) \\
V_{ui}^{eq}(x,T) &= [f(x) + \overset{*}{f}(x)]\,\mathbb{I}(x>b) \\
V_{uo}^{eq}(x,T) &= f(x)\mathbb{I}(x<b) - \overset{*}{f}(x)\mathbb{I}(x>b)
\end{aligned}$$

where, recall,

$$\overset{*}{f}(x) = (b/x)^{\alpha} f(b^2/x); \quad \alpha = \frac{2(r-q)}{\sigma^2} - 1$$

While barrier options exist only in restricted domains (either $x > b$ or $x < b$), it is very important to realise that their equivalent payoffs are defined for all values of $x > 0$.

D/O Call Barrier Option Price

Use the EEP (Equiv Euro Payoff):
For a call option of strike price a, barrier price b:

$$\begin{aligned} C_{do}^{eq}(x,T) &= (x-a)^+\mathbb{I}(x>b) - \text{its image} \\ &= \begin{cases} (x-a)\mathbb{I}(x>a) - \text{its image} & \text{if } a>b \\ (x-a)\mathbb{I}(x>b) - \text{its image} & \text{if } a<b \end{cases} \end{aligned}$$

Hence, for all $t<T$, by static replication,

$$C_{do}(x,t) = \begin{cases} C_a(x,t) - \overset{*}{C}_a(x,t) & \text{if } a>b \\ C_{b,a}(x,t) - \overset{*}{C}_{b,a}(x,t) & \text{if } a<b \end{cases}$$

where

$$C_a(x,t) = \text{pv(strike } a \text{ call option)}$$

and

$$C_{b,a}(x,t) = \text{pv(strike } a\text{, exercise } b \text{ gap-call option)}$$

European Call Barrier Prices

Option	*Case* $a > b$	*Case* $a < b$
D/O	$C_a - \overset{*}{C}_a$	$C_{b,a} - \overset{*}{C}_{b,a}$
D/I	$\overset{*}{C}_a$	$C_a - (C_{b,a} - \overset{*}{C}_{b,a})$
U/I	C_a	$\overset{*}{C}_a + (C_{b,a} - \overset{*}{C}_{b,a})$
U/O	0	$(C_a - \overset{*}{C}_a) - (C_{b,a} - \overset{*}{C}_{b,a})$

where $C_a(x,t)$ is the pv of a strike a call option and $C_{b,a}(x,t)$ is the pv of a strike a, exercise b threshold call option.

Note that it makes perfect sense that the up-and-out call barrier option is worthless if $a > b$; such an option can never finish in-the-money if the strike price is above the barrier price.

A similar analysis for the put barrier options yields the table:

European Call and Put Barrier Prices

Option	*Case* $a > b$	*Case* $a < b$
D/O	$(P_a - \overset{*}{P}_a) - (P_{b,a} - \overset{*}{P}_{b,a})$	0
D/I	$\overset{*}{P}_a + (P_{b,a} - \overset{*}{P}_{b,a})$	P_a
U/I	$P_a - (P_{b,a} - \overset{*}{P}_{b,a})$	$\overset{*}{P}_a$
U/O	$P_{b,a} - \overset{*}{P}_{b,a}$	$P_a - \overset{*}{P}_a$

where $P_a(x,t)$ is the pv of a strike a put option and $P_{b,a}(x,t)$ is the pv of a strike a, exercise b threshold put option.

Do not fail to see that these 16 different prices (including all calls and puts) can be written down virtually by inspection, once the tools of barrier option pricing (*i.e.* images, parity and static replication) are at ones disposal.

Lookback Options

Define

$$Y_t = \min_{0 \le s \le t} \{X_s\} \qquad \text{and} \qquad Z_t = \max_{0 \le s \le t} \{X_s\}$$

where X_s denotes the underlying asset price process (gBm). Thus Y_t represents the running minimum asset price up to time t, and Z_t the running maximum.

The payoffs for standard floating strike and fixed strike lookback calls and puts are shown in the following table.

Standard Lookback Payoffs

	Floating Strike	*Normal Fixed Strike*	*Reverse Fixed Strike*
Call	$(X_T - Y_T)^+$	$(Z_T - k)^+$	$(Y_T - k)^+$
Put	$(Z_T - X_T)^+$	$(k - Y_T)^+$	$(k - Z_T)^+$

Floating strike LB's are always exercised at expiry.

BS-pde's for Lookback Options

For a min-type LB option with running min y:

$$\boxed{\mathcal{L}U = 0; \quad U(x,y,T) = f(x,y); \quad U'(y,y,t) = 0}$$

in $x > y;\ t < T$, where $U'(x,y,t) = U_y(x,y,t)$.

For a max-type LB option with running max z:

$$\boxed{\mathcal{L}V = 0; \quad V(x,z,T) = F(x,z); \quad V'(z,z,t) = 0}$$

in the domain $x < z;\ t < T$ and $V'(x,z,t) = V_z(x,z,t)$.

Key Observation:
These can be transformed into the pde's for KO barrier options through:

$$\bar{U}(x,y,t) = U'(x,y,t); \qquad \bar{V}(x,z,t) = V'(x,z,t)$$

Equivalent Payoffs for Looback Options

These can be obtained by simply integrating the equivalent payoffs for D/O and U/O barrier options.

(a) The equivalent European payoff for a min-type LB option with expiry payoff $U(x,y,T) = f(x,y)$ is given by

$$\begin{aligned} U_{eq}(x,y,T) &= f(x,y)\mathbb{I}(x>y) + g(x,y)\mathbb{I}(x<y) \\ g(x,y) &= f(x,x) - \int_x^y \overset{*}{f}_\xi(x,\xi)\,d\xi \end{aligned}$$

(b) The equivalent European payoff for a max-type LB option with expiry payoff $V(x,z,T) = F(x,z)$ is given by

$$\begin{aligned} V_{eq}(x,z,T) &= F(x,z)\mathbb{I}(x<z) + G(x,z)\mathbb{I}(x>z) \\ G(x,z) &= F(x,x) + \int_z^x \overset{*}{F}_\xi(x,\xi)\,d\xi \end{aligned}$$

Generic Min/Max Lookbacks

These are lookback options which simply pay the min and max asset price over the lookback window and have expiry payoffs:

$$f(x,y) = y \qquad \text{and} \qquad F(x,z) = z$$

The equivalent payoff formulation then quickly leads to

$$\boxed{m(x,y,t) = (1+\beta)A_y^- + y[B_y^+ - \beta \overset{*}{B}{}_y^+]}$$

and

$$\boxed{M(x,z,t) = (1+\beta)A_z^+ + z[B_z - \beta \overset{*}{B}{}_z^-]}$$

where $\beta = \frac{\sigma^2}{2(r-q)}$.

Pricing the Standard Lookbacks[1]

1. **The Floating Strike LB's**

$$\begin{aligned} V_c(x,y,t) &= x - m(x,y,t) \\ V_p(x,z,t) &= M(x,z,t) - x \end{aligned}$$

2. **The Normal Fixed Strike LB's**

$$\begin{aligned} V_c(x,z,t) &= M(x,z \vee k,t) - ke^{-r\tau} \\ V_p(x,y,t) &= ke^{-r\tau} - m(x,y \wedge k,t) \end{aligned}$$

3. **Reverse Fixed Strike LB's**

$$\begin{aligned} V_c(x,y,t) &= m(x,y,t) - m(x,y \wedge k,t) \\ V_p(x,z,t) &= M(x,z \vee k,t) - M(x,z,t) \end{aligned}$$

[1] PB/OK 2005, A new method of pricing lookback options, *Mathematical Finance*, **15(2)**, 245-259.

Conclusions

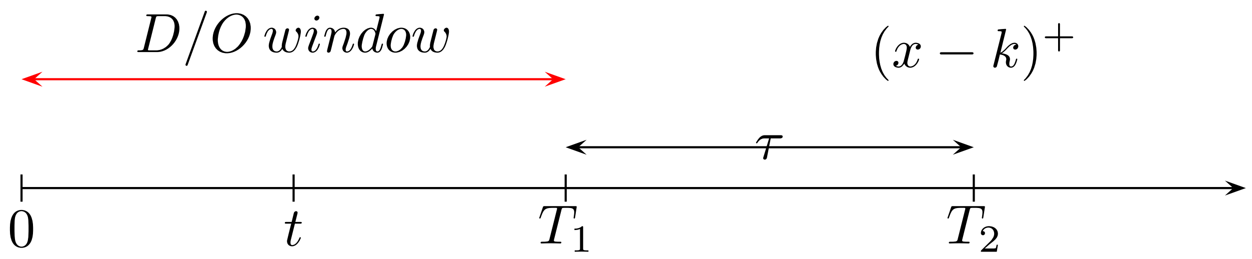
- The only prices formally calculated are the elementary building block contracts, such as first and higher-order asset and bond binaries.
- These calculations are greatly facilitated using the Gaussian Shift Theorem in the EMM method.
- Exotic option prices are then computed using static replication, parity relations, the image option and for barrier and lookback options, the notion of Equivalent European Payoffs.
- Once these tools are in place, many exotic options can be priced virtually by inspection.

—oOo—

Part II: Partial Barrier Options

- First considered in Heynen and Kat (1994)
- Also known as *early-ending* barrier options
- Barrier monitoring window a subset of the full lifetime of the option, $t \in [0, T_1]$
- Two expiry times T_1 and T_2, with $T_2 > T_1$
- At T_1, reverts to a standard option over $t \in [T_1, T_2]$
  - provided the barrier has not been breached in the knock-out case,
  - or alternatively is knocked-in to a standard option in the knock-in barrier case.
- There are four partial barrier call options, D/O, D/I, U/O & U/I as well as four corresponding puts.
- We can price the D/O barrier, get the other three (D/I, U/I, U/O) by parity.

D/O Partial Barrier Call Option



We derive the price of the down-and-out partial barrier call option.

At $t = T_2$, the holder of a *partial barrier call* option receives a standard call payoff with strike k:

$$V_{do}(x, T_2) = (x - k)^+$$

Provided that the stock price remains above the barrier level $x = a$ over $t \in [0, T_1]$.

Arbitrage arguments require the price at time $t = T_1$ is that of a standard call option with $\tau = T_2 - T_1$ to expiry:

$$V_{do}(x, T_1) = C_k(x, \tau)$$

Over $t \in [0, T_1]$ we therefore have

- a down-and-out barrier option with non-standard payoff $C_k(x, \tau)$ at $t = T_1$.

The partial barrier call therefore satisfies the PDE:

$$
\begin{aligned}
\mathcal{L}\, V_{do}(x,t) &= 0 \quad \text{for} \quad x > a \quad \text{and} \quad t < T_1 \\
V_{do}(x, T_1) &= C_k(x,\tau) = Q_k^+(x,\tau) \\
V_{do}(a,t) &= 0 \text{ for } t < T_1
\end{aligned}
$$

With related terminal-value problem for $V_a(x,t)$:

$$
\begin{aligned}
\mathcal{L}\, V_a(x,t) &= 0 \quad \text{for} \quad x > 0 \quad \text{and} \quad t < T_1 \\
V_a(x, T_1) &= Q_k^+(x,\tau)\mathbb{I}(x > a)
\end{aligned}
$$

The payoff of V_a at time $t = T_1$ is recognised as that of a second order Q option, so that by static replication we can write down the solution for $t < T_1$:

$$V_a(x,t) = Q_{ak}^{++}(x, \tau_1, \tau_2)$$

The solution for the required *down-and-out partial barrier call option* price is therefore given by:

$$V_{do}(x,t) = Q_{ak}^{++}(x,\tau_1,\tau_2) - \overset{*}{Q}{}_{ak}^{++}(x,\tau_1,\tau_2) \qquad (1)$$

where $\tau_i = T_i - t,\ i = 1,2$ and * denotes the image with respect to $x = a$.

The standard option associated with this down-and-out barrier option over $t \in [0, T_1]$ is simply a call option with time τ_2 to expiry:

$$V_s(x,t) = C_k(x,\tau_2)$$

By parity, we can write down the solutions for the three remaining related barriers over the window $t \in [0, T_1]$.

These are:

The *down-and-in partial barrier call option*:

$$V_{di}(x,t) = C_k(x,\tau_2) - \left[Q^{++}_{ak}(x,\tau_1,\tau_2) - \overset{*}{Q}{}^{++}_{ak}(x,\tau_1,\tau_2)\right]$$

The *up-and-in partial barrier call option*:

$$V_{ui}(x,t) = \overset{*}{C}_k(x,\tau_2) + \left[Q^{++}_{ak}(x,\tau_1,\tau_2) - \overset{*}{Q}{}^{++}_{ak}(x,\tau_1,\tau_2)\right]$$

The *up-and-out partial barrier call option*:

$$\begin{aligned} V_{uo}(x,t) &= C_k(x,\tau_2) - \overset{*}{C}_k(x,\tau_2) \\ &\qquad - \left[Q^{++}_{ak}(x,\tau_1,\tau_2) - \overset{*}{Q}{}^{++}_{ak}(x,\tau_1,\tau_2)\right] \end{aligned}$$

where $\tau_i = T_i - t$, $i = 1, 2$ and * denotes the image with respect to $x = a$.

Partial Barrier Put Options

The *down-and-out*, *down-and-in*, *up-and-in* and *up-and-out partial barrier put* options have respective prices given by:

$$V_{do}(x,t) = -\left[Q_{ak}^{+-}(x,\tau_1,\tau_2) - \overset{*}{Q}{}_{ak}^{+-}(x,\tau_1,\tau_2)\right]$$

$$V_{di}(x,t) = P_k(x,\tau_2) + Q_{ak}^{+-}(x,\tau_1,\tau_2) - \overset{*}{Q}{}_{ak}^{+-}(x,\tau_1,\tau_2)$$

$$V_{ui}(x,t) = \overset{*}{P}_k(x,\tau_2) - \left[Q_{ak}^{+-}(x,\tau_1,\tau_2) - \overset{*}{Q}{}_{ak}^{+-}(x,\tau_1,\tau_2)\right]$$

$$\begin{aligned} V_{uo}(x,t) &= P_k(x,\tau_2) - \overset{*}{P}_k(x,\tau_2) \\ &\quad + Q_{ak}^{+-}(x,\tau_1,\tau_2) - \overset{*}{Q}{}_{ak}^{+-}(x,\tau_1,\tau_2) \end{aligned}$$

where $\tau_i = T_i - t$, $i = 1,2$ and * denotes the image with respect to $x = a$.

These all agree with the published results of Heynen and Kat (1994), after notational differences are accounted for.

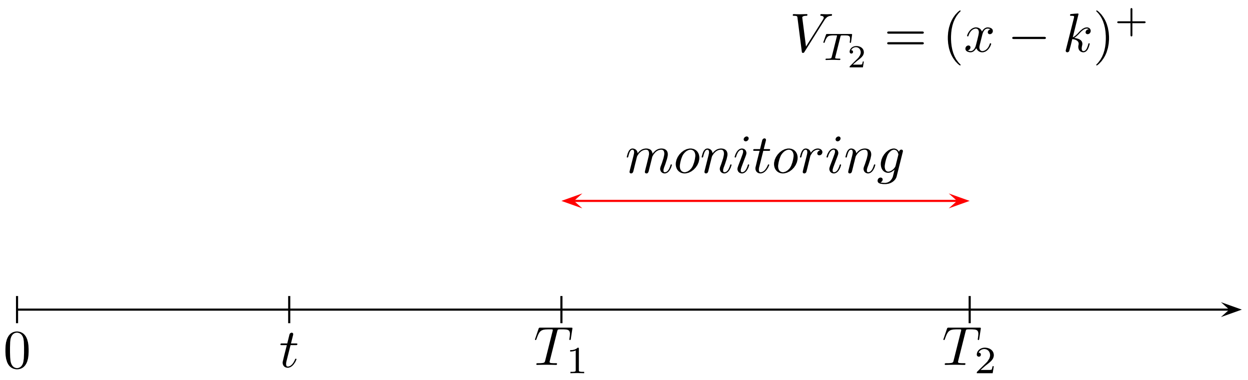
Partial Time End Out Options

- These are the B1-type partial barrier options of Heynen and Kat (1994).
- Have one barrier monitoring window.
- Monitoring doesn't begin at the start of the option, but rather at some time T_1 *after* initiation.
- The option is converted at time $t = T_1$ into a down-and-out barrier option over $t \in [T_1, T_2]$ if $X_{T_1} > b$.
- Or converted into an up-and-out barrier option over $t \in [T_1, T_2]$ if $X_{T_1} < b$.

where

- X_{T_1} is the stock price at time T_1, and
- $x = b$ is the barrier monitoring level.

Partial Time End Out Call Barrier



The payoff at time T_2 is $V_c^{eo}(x, T_2) = (x-k)^+$

At time $t = T_1$ the partial time end-out call option will be converted to either

- a down-and-out call barrier option over $[T_1, T_2]$, if the stock price x at T_1 is above b,
- into an up-and-out call barrier option if $x < b$.

Thus we can write the $t = T_1$ price of the partial time end-out call barrier option as:

$$V_c^{eo}(x, T_1) = V_{do}(x, T_1)\mathbb{I}(x > b) + V_{uo}(x, t)\mathbb{I}(x < b)$$

where

- $V_{do}(x, T_1)$ is the price of a *down-and-out call barrier option* with time $\tau = T_2 - T_1$ to expiry.
- $V_{uo}(x, T_1)$ is the price of an *up-and-out call barrier option* with time $\tau = T_2 - T_1$ to expiry.

We have expressions for both of these, as obtained by the MOI in the first half of the seminar.

It follows that the $t = T_1$ price of the partial time end-out call barrier option is given by:

$$V_c^{eo}(x, T_1) =$$

$$\begin{cases} \left[Q_k^+(x,\tau) - \overset{*}{Q}{}_k^+(x,\tau)\right] \mathbb{I}(x>b) & \text{if} \quad k > b \\ \left[A_b^+(x,\tau) - kB_b^+(x,\tau)\right] \mathbb{I}(x>b) \\ - \left[\overset{*}{A}{}_b^+(x,\tau) - k\overset{*}{B}{}_b^+(x,\tau)\right] \mathbb{I}(x>b) \\ + \left[Q_k^+(x,\tau) - \overset{*}{Q}{}_k^+(x,\tau)\right] \mathbb{I}(x<b) \\ - \left[A_b^+(x,\tau) - kB_b^+(x,\tau)\right] \mathbb{I}(x<b) \\ + \left[\overset{*}{A}{}_b^+(x,\tau) - k\overset{*}{B}{}_b^+(x,\tau)\right] \mathbb{I}(x<b) & \text{if} \quad k < b \end{cases}$$

where * is the image with respect to $x = b$ and $\tau = T_2 - T_1$.

We recognise these as the payoffs of second order Asset, Bond and Q options, so that we can then apply the principle of static replication:

The price of the *Partial Time End Out Call Barrier Option* for time $t < T_1$ is thus given by:

$$V_c^{eo}(x,t) =$$

$$\begin{cases} \quad \left[Q_{bk}^{++}(x,\tau_1,\tau_2) - \overset{*}{Q}{}_{bk}^{-+}(x,\tau_1,\tau_2)\right] & \text{if} \quad k > b \\ \quad [A_{bb}^{++}(x,\tau_1,\tau_2) - kB_{bb}^{++}(x,\tau_1,\tau_2)] & \\ - \left[\overset{*}{A}{}_{bb}^{-+}(x,\tau_1,\tau_2) - k\overset{*}{B}{}_{bb}^{-+}(x,\tau_1,\tau_2)\right] & \\ + \quad Q_{bk}^{-+}(x,\tau_1,\tau_2) - \overset{*}{Q}{}_{bk}^{++}(x,\tau_1,\tau_2) & \\ - [A_{bb}^{-+}(x,\tau_1,\tau_2) - kB_{bb}^{-+}(x,\tau_1,\tau_2)] & \\ + \quad \overset{*}{A}{}_{bb}^{++}(x,\tau_1,\tau_2) - k\overset{*}{B}{}_{bb}^{++}(x,\tau_1,\tau_2) & \text{if} \quad k < b \end{cases}$$

where $\tau_i = T_i - t$ for $i = 1, 2$ and * is the image with respect to $x = b$.

Similarly, we get the price of the *Partial Time End Out Put Barrier Option*:

$$V_p^{eo}(x,t) =$$

$$\begin{cases} - \quad Q_{bk}^{+-}(x,\tau_1,\tau_2) + \overset{*}{Q}{}_{bk}^{--}(x,\tau_1,\tau_2) \\ - \left[kB_{bb}^{+-}(x,\tau_1,\tau_2) - A_{bb}^{+-}(x,\tau_1,\tau_2)\right] \\ + \left[k\overset{*}{B}{}_{bb}^{--}(x,\tau_1,\tau_2) - \overset{*}{A}{}_{bb}^{--}(x,\tau_1,\tau_2)\right] \\ + \left[kB_{bb}^{--}(x,\tau_1,\tau_2) - A_{bb}^{--}(x,\tau_1,\tau_2)\right] \\ - \left[k\overset{*}{B}{}_{bb}^{+-}(x,\tau_1,\tau_2) - \overset{*}{A}{}_{bb}^{+-}(x,\tau_1,\tau_2)\right] & \text{if} \quad k > b \\ \quad - Q_{bk}^{--}(x,\tau_1,\tau_2) + \overset{*}{Q}{}_{bk}^{+-}(x,\tau_1,\tau_2) & \text{if} \quad k < b \end{cases}$$

where $\tau_i = T_i - t$ for $i = 1, 2$ and where * is the image with respect to $x = b$.

Allowing for notation, the above representations agree with the published solutions of Heynen and Kat (1994).

Theorem: *MOI For Double Barriers*

Given lower and upper barrier levels $x = (a, b)$ and arbitrary payoff $f(x)$, the unique solution to:

$$\begin{aligned} \mathcal{L}V(x,t) &= 0 \quad \text{for} \quad a < x < b,\, t < T \\ V(x,T) &= f(x) \\ V(a,t) &= V(b,t) = 0 \text{ for } t < T \end{aligned}$$

is given by:

$$V(x,t) = \sum_{n=-\infty}^{\infty} \lambda^{\alpha n} \left[U(\lambda^{2n}x,t) - \overset{*}{U}(\lambda^{2n}x,t) \right]$$

with $\lambda = b/a$, $\alpha = 2(r-q)/\sigma^2 - 1$,

$$\overset{*}{U}(\lambda^{2n}x,t) = \left(\frac{a}{x}\right)^{\alpha} U\left(\lambda^{2n}\frac{a^2}{x}, t\right)$$

denotes the image function wrt $x = a$.

And where $U(x,t)$ is the solution to the *related terminal-value problem*:

$$\begin{aligned} \mathcal{L}\,U(x,t) &= 0 \quad \text{for} \quad x > 0 \quad \text{and} \quad t < T \\ U(x,T) &= f(x)\,\mathbb{I}(a < x < b) \end{aligned}$$

Lemma:

The image can alternatively be taken wrt to $x = b$:

$$V(x,t) = \sum_{n=-\infty}^{\infty} \lambda^{\alpha n} \left[U(\lambda^{2n}x,t) - \overset{*}{U}(\lambda^{2n}x,t) \right]$$

$$\overset{*}{U}(\lambda^{2n}x,t) = \left(\frac{b}{x}\right)^{\alpha} U\left(\lambda^{2n}\frac{b^2}{x}, t\right)$$

Proof: Replace a with b/λ and $n-1$ with n in the second term of the double-infinite sum. □

Proof of the main result (MOI for Double Barriers):

$U(\lambda^{2n}x, t)$ and $\overset{*}{U}(\lambda^{2n}x, t)$ satisfy the Black Scholes PDE for any n, hence $V = U - \overset{*}{U}$ also does.

Since $U = \overset{*}{U}$ when $x = a$, it follows that $V(a, t) = 0$.

Replacing the image in $\overset{*}{U}$ wrt $x = b$, it follows that $V(b, t) = 0$ as well.

When $t = T$, we obtain:

$$\begin{aligned} V(x, T) &= \sum_{n=-\infty}^{n=\infty} \lambda^{\alpha n} \left[f_n(x) \mathbf{I}(a < \lambda^{2n} x < b) \right. \\ &\quad \left. - \left(\frac{a}{x}\right)^{\alpha} \overset{*}{f}_n(x) \mathbf{I}\left(a < \lambda^{2n} \frac{a^2}{x} < b\right) \right], \\ f_n(x) &= f(\lambda^{2n} x) \quad \text{and} \quad \overset{*}{f}_n(x) = f\left(\lambda^{2n} \frac{a^2}{x}\right) \end{aligned}$$

Note,

$$\begin{aligned}
\mathbf{I}\left(a<\lambda^{2n}x<b\right) &\equiv \mathbf{I}\left(\frac{a}{\lambda^{2n}}<x<\frac{b}{\lambda^{2n}}\right) \\
&\equiv \begin{cases} 1 & \text{if} \quad n=0 \\ 0 & \text{if} \quad n\neq 0 \end{cases} \\
\mathbf{I}\left(a<\lambda^{2n}\frac{a^2}{x}<b\right) &\equiv \mathbf{I}\left(a\lambda^{2n-1}<x<a\lambda^{2n}\right) \\
&\equiv 0
\end{aligned}$$

for all n whenever $a < x < b$.

Only the $n = 0$ term remains, giving payoff $f(x)$. □

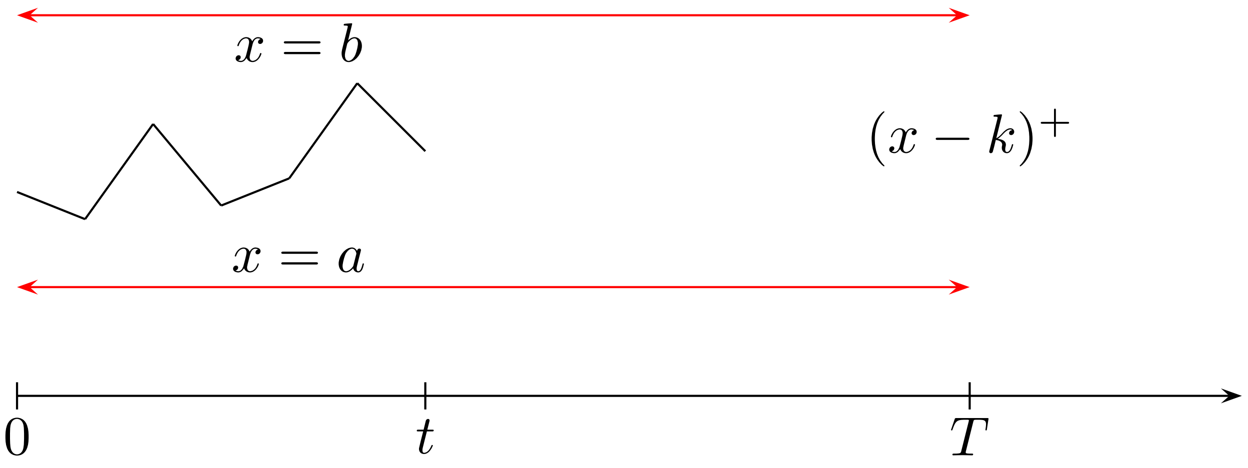
Knock-In Double Barrier Options

In the domain $a < x < b$, the knock-out and knock-in double barrier options satisfy:

$$\boxed{V^O(x,t) + V^I(x,t) = V_s(x,t)}$$

where $V_s(x,t)$ is the standard European option.

Knock-Out Double Barrier Call Option



- Lower barrier at $x = a$, upper at $x = b$
- Option knocked-out if either barrier breached for $t < T$.
- If option survives, get standard call payoff $(X_T - k)^+$

The price $V = V_{DB}^{OC}(x,t)$ satisfies the PDE:

$$\begin{aligned} \mathcal{L}V(x,t) &= 0 \quad \text{in } a < x < b,\ t < T \\ V(x,T) &= (x-k)^+ \\ V(a,t) &= V(b,t) = 0, \quad t < T \end{aligned}$$

Consider the related terminal-value problem for $V_{ab}(x,t)$:

$$\begin{aligned} \mathcal{L}\,V_{ab}(x,t) &= 0 \quad \text{in } x>0,\ t<T \\ V_{ab}(x,T) &= (x-k)^{+}\mathbb{I}(x>a)\mathbb{I}(x<b) \end{aligned}$$

Removing the plus superscript, the payoff of V_{ab} can be written as:

$$V_{ab}(x,T) = (x-k)\mathbb{I}(x>k)\mathbb{I}(x>a)\mathbb{I}(x<b)$$

- If $k>b$,

$$\mathbb{I}(x>k)\mathbb{I}(x>a)\mathbb{I}(x<b) \equiv 0$$

- If $a<k<b$,

$$\mathbb{I}(x>k)\mathbb{I}(x>a)\mathbb{I}(x<b) \equiv \mathbb{I}(x>k) - \mathbb{I}(x>b)$$

- If $k<a<b$,

$$\mathbb{I}(x>k)\mathbb{I}(x>a)\mathbb{I}(x<b) \equiv \mathbb{I}(x>a) - \mathbb{I}(x>b)$$

By defining

$$k' = k \vee a = \max(k, a)$$

when $k < b$, we can thus summarise the product of the three indicator functions as:

$$\mathbb{I}(x > k)\mathbb{I}(x > a)\mathbb{I}(x < b) \equiv \mathbb{I}(x > k') - \mathbb{I}(x > b)$$

It follows that

$$V_{ab}(x, T) = (x - k)\left[\mathbb{I}(x > k') - \mathbb{I}(x > b)\right]$$

For $t < T$, static replication therefore requires that

$$\begin{aligned} V_{ab}(x, t) &= A^+_{k'}(x, \tau) - A^+_b(x, \tau) \\ &\qquad -k\left[B^+_{k'}(x, \tau) - B^+_b(x, \tau)\right] \end{aligned}$$

By the MOI for double barriers, the price of the double barrier knock-out call option is:

$$
\begin{aligned}
&V_{DB}^{OC}(x,t) = \\
&\sum_{n=-\infty}^{\infty} \lambda^{\alpha n}\left[A_{k'}^{+}(\lambda^{2n}x,\tau) - A_{b}^{+}(\lambda^{2n}x,\tau)\right] \\
&- \sum_{n=-\infty}^{\infty} \lambda^{\alpha n}\left[\overset{*}{A}{}_{k'}^{+}(\lambda^{2n}x,\tau) - \overset{*}{A}{}_{b}^{+}(\lambda^{2n}x,\tau)\right] \\
&-k \sum_{n=-\infty}^{\infty} \lambda^{\alpha n}\left[B_{k'}^{+}(\lambda^{2n}x,\tau) - B_{b}^{+}(\lambda^{2n}x,\tau)\right] \\
&+k \sum_{n=-\infty}^{\infty} \lambda^{\alpha n}\left[\overset{*}{B}{}_{k'}^{+}(\lambda^{2n}x,\tau) - \overset{*}{B}{}_{b}^{+}(\lambda^{2n}x,\tau)\right]
\end{aligned}
$$

where $\lambda = b/a$, * denotes the image with respect to $x = a$ (or $x = b$), $k' = k \vee a$ and $\alpha = 2r/\sigma^2 - 1$.

Similarly, we can price the *double barrier knock-out put option*.

By parity, we get the knock-in barrier prices as well, for both calls and puts.

- We get agreement with Ikeda and Kunitomo (1992) in the case of flat barriers.

- We can apply the same procedure to obtain the prices of the **partial time double barrier options**, with early monitoring knock-out window $t \in [0, T_1]$:

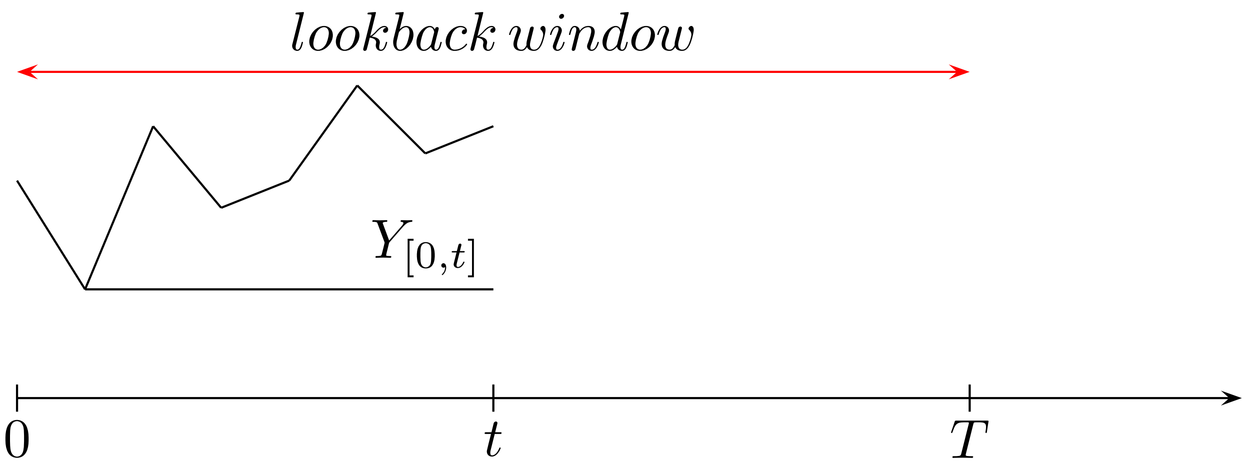
e.g. The price of the *partial time knock-out double barrier call option* is:

$$V_o(x,t) = \sum_{n=-\infty}^{\infty} \lambda^{\alpha n} \left[Q_{ak}^{++}(\lambda^{2n}x, \tau_1, \tau_2) - Q_{bk}^{++}(\lambda^{2n}x, \tau_1, \tau_2) \right] - \sum_{n=-\infty}^{\infty} \lambda^{\alpha n} \left[\overset{*}{Q}{}_{ak}^{++}(\lambda^{2n}x, \tau_1, \tau_2) - \overset{*}{Q}{}_{bk}^{++}(\lambda^{2n}x, \tau_1, \tau_2) \right]$$

where $\lambda = b/a$, $\tau_i = T_i - t$, $i = 1, 2$ and * denotes the image with respect to $x = a$ (or $x = b$).

New option not priced before by other methods, to our knowledge.

Partial Price Lookback Options



Recall,

$$y = \min_{[0,T]} X_s \quad \text{and} \quad z = \max_{[0,T]} X_s$$

- The standard methods of pricing lookbacks first explored in Goldman Sossin and Gatto (1979).
- 'Partial time' lookbacks priced in Heynen and Kat (1995)
- 'Partial price' lookbacks first priced in Conze and Viswanathan (1991)

Partial Price Lookback Call Option

At expiry time $t = T$, this option pays

$$V^c(x, y, T) = (x - \lambda y)^+ \quad \text{and } \lambda \geq 1$$

thus satisfying the following PDE in $t \in [0, T]$:

$$\begin{aligned} \mathcal{L}V^c(x, y, t) &= 0 \quad \text{in } x > y,\ t < T \\ V^c(x, y, T) &= (x - \lambda y)\mathbb{I}(x > \lambda y) \\ \frac{\partial V^c}{\partial y} &= 0 \quad \text{at } x = y \end{aligned}$$

We now make the transformation

$$U(x, y, t) = \frac{\partial V^c}{\partial y}$$

to obtain the following D/O barrier problem for U:

$$\begin{aligned} \mathcal{L}U(x, y, t) &= 0 \quad \text{in } x > y,\ t < T \\ U(x, y, T) &= -\lambda\,\mathbb{I}(x > \lambda y) \\ U &= 0 \quad \text{at } x = y \end{aligned}$$

We solve this using the MOI and Static Rep. for $t < T$:

$$U_y(x, y, t) = -\lambda B^+_{\lambda y}(x, \tau)$$

so that

$$U(x, y, t) = -\lambda \left[B^+_{\lambda y}(x, \tau) - \overset{*}{B}{}^+_{\lambda y}(x, \tau) \right]$$

where $*$ is the image with respect to $x = y$.

We thus recover the price by quadratures:

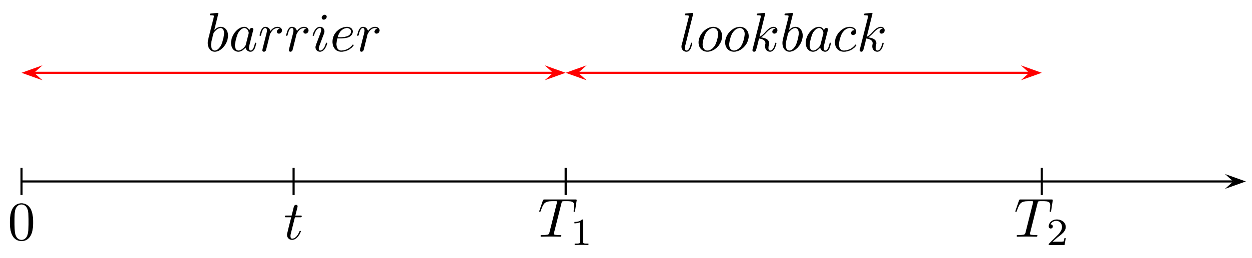
$$\begin{aligned} V^c(x, y, t) &= V^c(x, 0, t) \\ &\qquad -\lambda \int_0^y \left[B^+_{\lambda \xi}(x, \tau) - \overset{*}{B}{}^+_{\lambda \xi}(x, \tau) \right] d\xi \\ &= x - \lambda \int_0^y \left[B^+_{\lambda \xi}(x, \tau) - \overset{*}{B}{}^+_{\lambda \xi}(x, \tau) \right] d\xi \end{aligned}$$

Evaluating the integrals at $t = T$, we obtain:

$$V^c(x, y, t) = Q^+_{\lambda y}(x, \tau) - \lambda^{\alpha+2} D^-_{y/\lambda}(x, \tau) \qquad (2)$$

where $\tau = T - t$, and $D^-_{y/\lambda}(x, \tau)$ is a linear combination of Asset and Bond binaries and their images

Lookbarrier Options



- First defined in Bermin (1998), they're a whole family of options rather than one specific option.

- They combine a *partial time barrier* window, with a forward starting *lookback* window.

- Many combinations, *e.g.* U/O barrier over $[0, T_1]$ with Fixed Strike Lookback over $[T_1, T_2]$, D/O barrier with Floating Strike Lookback etc

- Rationale is to reduce the price from the standard and *expensive* lookback price, which has the full lookback window, $[0, T_2]$.

Up-and-Out Fixed Strike Lookbarrier Call Option

Over $t \in [0, T_1]$, we have an up-and-out barrier window with barrier level $x = h$.

The $t = T_2$ payoff is given by:

$$V_{uo}^c(x, T_2) = (Z_{[T_1,T_2]} - k)^+ \mathbb{I}(Z_{[0,T_1]} < h)$$

In $t \in [0, T_1]$, $V_{uo}^c(x, t)$ satisfies:

$$\begin{aligned} \mathcal{L} V_{uo}^c(x, t) &= 0 \quad \text{in } x < h, t < T_1 \\ V_{uo}^c(x, T_1) &= F^c(x, x, \tau) \\ V_{uo}^c(h, t) &= 0 \end{aligned}$$

where $\tau = T_2 - T_1$ and

$$F^c(x, z, \tau) = M(x, z \vee k, \tau) - ke^{-r\tau}$$

is the price of a fixed strike lookback call option over the time window $[T_1, T_2]$.

We can write:

$$F^c(x,x,\tau) = \begin{cases} xg(\tau) - ke^{-r\tau} & \text{if} \quad x > k \\ C_k(x,\tau) + D_k^+(x,\tau) & \text{if} \quad x < k \end{cases}$$

for some function $g(\tau)$.

Hence over $[0, T_1]$, we have an up-and-out barrier option with non-standard payoff given above.

The related terminal-value problem is given by:

$$\begin{aligned} \mathcal{L}\, V_h(x,t) &= 0 \quad \text{in } x > 0, t < T_1 \\ V_h(x,T_1) &= F^c(x,x,\tau)\mathbb{I}(x<h) \end{aligned}$$

with $t = T_1$ payoff (after some simplification):

$$V_h(x,T_1) =$$

$$\begin{cases} \left[xg(\tau) - ke^{-r\tau}\right]\left[\mathbb{I}(x>k) - \mathbb{I}(x>h)\right] & \\ + \left[C_k(x,\tau) + D_k^+(x,\tau)\right]\mathbb{I}(x<k) & \text{if} \quad h > k \\ \left[C_k(x,\tau) + D_k^+(x,\tau)\right]\mathbb{I}(x<h) & \text{if} \quad h < k \end{cases}$$

We can thus solve for V_h when $t < T_1$ by Static Rep:

$$V_h(x,t) = \begin{cases} g(\tau)\left[A_k^+(x,\tau_1) - A_h^+(x,\tau_1)\right] \\ -ke^{-r\tau}\left[B_k^+(x,\tau_1) - B_h^+(x,\tau_1)\right] \\ +Q_{kk}^{-+}(x,\tau_1,\tau_2) + D_{kk}^{-+}(x,\tau_1,\tau_2) & \text{if} \quad h > k \\ \\ Q_{hk}^{-+}(x,\tau_1,\tau_2) + D_{hk}^{-+}(x,\tau_1,\tau_2) & \text{if} \quad h < k \end{cases}$$

The final expression for the *Up-and-Out Fixed Strike Lookbarrier Call Option* is:

$$V_{uo}^c(x,t) = \begin{cases} g(\tau)\left[A_k^+(x,\tau_1) - \overset{*}{A}{}_k^+(x,\tau_1)\right] \\ -g(\tau)\left[A_h^+(x,\tau_1) - \overset{*}{A}{}_h^+(x,\tau_1)\right] \\ -ke^{-r\tau}\left[B_k^+(x,\tau_1) - \overset{*}{B}{}_k^+(x,\tau_1)\right] \\ +ke^{-r\tau}\left[B_h^+(x,\tau_1) - \overset{*}{B}{}_h^+(x,\tau_1)\right] \\ +Q_{kk}^{-+}(x,\tau_1,\tau_2) - \overset{*}{Q}{}_{kk}^{-+}(x,\tau_1,\tau_2) \\ +D_{kk}^{-+}(x,\tau_1,\tau_2) - \overset{*}{D}{}_{kk}^{-+}(x,\tau_1,\tau_2) & \text{if} \quad h > k \\ \\ Q_{hk}^{-+}(x,\tau_1,\tau_2) - \overset{*}{Q}{}_{hk}^{-+}(x,\tau_1,\tau_2) \\ +D_{hk}^{-+}(x,\tau_1,\tau_2) - \overset{*}{D}{}_{hk}^{-+}(x,\tau_1,\tau_2) & \text{if} \quad h < k \end{cases}$$

where * is the image with respect to $x = h$, $\tau_i = T_i - t$, for $i = 1, 2$, $\tau = T_2 - T_1$ and

$$\begin{aligned} g(\tau) &= (1 + \tfrac{\sigma^2}{2r})\mathcal{N}(a) + e^{-r\tau}(1 - \tfrac{\sigma^2}{2r})\mathcal{N}(-a') \\ [a, a'] &= \frac{(r \pm \frac{1}{2}\sigma^2)\sqrt{\tau}}{\sigma} \end{aligned}$$

This agrees with Bermin (1998).

- Bermin (1998) only gave expressions for two of the lookbarriers.
- We can similarly price *all* of the lookbarriers, including the Knock-In versions.
- We can extend the analysis and readily price *Look Double Barrier* options, using the MOI for Double Barriers.

Bermin's Extreme Spread Options

Bermin (1998) originally defined four extreme spread options. Using the notation

$$Y_{[a,b]} = \min_{t \in [a,b]} X_t \quad \text{and} \quad Z_{[a,b]} = \max_{t \in [a,b]} X_t$$

they are:

	Normal Extreme Spread	*Reverse Extreme Spread*
Call	$(Z_{[T_1,T_2]} - Z_{[0,T_1]})^+$	$(Y_{[T_1,T_2]} - Y_{[0,T_1]})^+$
Put	$(Y_{[0,T_1]} - Y_{[T_1,T_2]})^+$	$(Z_{[0,T_1]} - Z_{[T_1,T_2]})^+$

T_2 payoffs of extreme spread lookback options

The time interval $[0, T_2]$ is divided into two contiguous lookback windows $[0, T_1]$ and $[T_1, T_2]$.

These options have payoffs determined by the minimum and maximum asset prices within these two windows.

We can price all four of Bermin's extreme spread options in terms of $m(x, y, \tau)$ and $M(x, z, \tau)$

Lemma

For $0 < T_1 < T_2$ the following identities hold:

$$\begin{aligned}
\left(Y_{[0,T_1]} - Y_{[T_1,T_2]}\right)^+ &= Y_{[0,T_1]} - Y_{[0,T_2]} \\
\left(Y_{[T_1,T_2]} - Y_{[0,T_1]}\right)^+ &= Y_{[T_1,T_2]} - Y_{[0,T_2]} \\
\left(Z_{[T_1,T_2]} - Z_{[0,T_1]}\right)^+ &= Z_{[0,T_2]} - Z_{[0,T_1]} \\
\left(Z_{[0,T_1]} - Z_{[T_1,T_2]}\right)^+ &= Z_{[0,T_2]} - Z_{[T_1,T_2]}.
\end{aligned}$$

Proof:
For the first identity:

$$\begin{aligned}
\left(Y_{[0,T_1]} - Y_{[T_1,T_2]}\right)^+ &= Y_{[0,T_1]} - \min\left(Y_{[0,T_1]}, Y_{[T_1,T_2]}\right) \\
&= Y_{[0,T_1]} - Y_{[0,T_2]}
\end{aligned}$$

Similar proofs hold for the rest.

We summarise the $t < T_1$ prices of the generic lookback options with the given payoffs at time $t = T_2$.

Denoting $\tau = T_2 - T_1$, we have:

Option	*Payoff at T_2*	*Price at $t < T_1$*
$V_1(x,t)$	$Y_{[0,T_1]}$	$e^{-r\tau}\, m(x,y,\tau_1)$
$V_2(x,t)$	$Y_{[T_1,T_2]}$	$m(x,x,\tau)$
$V_3(x,t)$	$Y_{[0,T_2]}$	$m(x,y,\tau_2)$
$V_4(x,t)$	$Z_{[0,T_1]}$	$e^{-r\tau}\, M(x,z,\tau_1)$
$V_5(x,t)$	$Z_{[T_1,T_2]}$	$M(x,x,\tau)$
$V_6(x,t)$	$Z_{[0,T_2]}$	$M(x,z,\tau_2)$

We now price *all* of Bermin's extreme spread options:

Option	*Payoff at T_2*	*Price at $t < T_1$*
Normal Put	$(Y_{01} - Y_{12})^+$	$e^{-r\tau}\, m(x,y,\tau_1) - m(x,y,\tau_2)$
Reverse Call	$(Y_{12} - Y_{01})^+$	$m(x,x,\tau) - m(x,y,\tau_2)$
Normal Call	$(Z_{12} - Z_{01})^+$	$M(x,z,\tau_2) - e^{-r\tau}\, M(x,z,\tau_1)$
Reverse Put	$(Z_{01} - Z_{12})^+$	$M(x,z,\tau_2) - M(x,x,\tau)$

Conclusions

- As emphasised before, the only prices we need to formally calculate are the elementary building block contracts, here being first and second-order asset and bond binaries.
- Exotic options are statically replicated in terms of these building blocks, and their *images*.
- With the *Method of Images for Double Barriers*, we can easily extend the framework to price complicated exotic options with Double Barrier features as well.
- Relatively simple integrals arise in the evaluation of lookback option prices.
- Many complicated options can be priced by these methods, including:
  - partial price maximal spread options
  - multi-dimensional barrier and lookback options.