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Schweizerische Vereinigung für Finanzanalyse
und Vermögensverwaltung

Formulae

Foundation Examination

Derivatives Analysis and Valuation

♦

Portfolio Management

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1. Derivatives Analysis and Valuation

1.1 Forwards and Futures

1.1.1 General cost of carry relationship

$$F_{t,T} = S_t(1+R_{t,T})^{T-t} + k(t,S) - FV(revenues)$$

where

$F_{t,T}$	forward or futures price at date t of a contract for delivery at date T
S_t	spot price of the underlying at date t
$R_{t,T}$	risk-free interest rate for the period t to T
FV(revenues)	future value of the revenues paid by the spot
$k(t,S)$	carrying costs, such as insurance costs, storage costs, etc.

1.1.2 Continuous time cost of carry relationship

$$F_{t,T} = S_t e^{(r_{t,T}-y)\cdot(T-t)}$$

where

$F_{t,T}$	futures price at date t of a contract for delivery at date T
S_t	spot price of the underlying at date t
y	continuous net yield (revenues minus carrying costs) of the underlying asset or commodity
$r_{t,T}$	continuously compounded risk-free interest rate

1.1.3 Stock Index Futures

$$F_{t,T} = I_t \cdot (1+R_{t,T}) - \sum_{i=1}^{I}\sum_{t_j=1}^{T} w_i \cdot D_{i,t_j} \cdot (1+R_{t_j,T})$$

where:

I_t	current spot price of the index
D_{i,t_j}	dividend paid by firm i at date t_j
w_i	weight of firm i in the index
$R_{t_j,T}$	interest rate for the time period t_j until T

1.1.4 Interest rates future cost of carry relationship

$$F_{t,T} = \frac{(S_t + A_t)\cdot(1+R_{t,T})^{T-t} - C_{t,T} - A_T}{Conversion\ Factor}$$

where

$F_{t,T}$ quoted futures "fair" price at date t of a contract for delivery at date T

$C_{t,T}$ future value of all coupons paid and reinvested between t and T

S_t spot value of the underlying bond

A_t accrued interest of the underlying at time t

A_T accrued interest of the delivered bond at time T

Theoretical futures at the delivery date

$$F_{T,T} = \frac{\text{spot price of cheapest to deliver}}{\text{conversion factor}}$$

1.1.5 Forward exchange rates

The forward exchange rate is given by

$$F_{t,T} = S_t \left(\frac{1 + R_{dom}}{1 + R_{for}} \right)^{T-t}$$

with continuous compounding

$$F_{t,T} = S_t e^{\left(r_{dom} - r_{for}\right)(T-t)}$$

where

$F_{t,T}$ forward exchange rate (domestic per foreign currency)

S_t spot exchange rate (domestic per foreign currency)

R_{dom} domestic risk-free rate of interest for the period t to T

R_{for} foreign risk-free rate of interest for the period t to T

r_{dom} continuous domestic risk-free rate of interest for t to T

r_{for} continuous foreign risk-free rate of interest for t to T

1.1.6 Commodity Futures

$$F_{t,T} = S_t \cdot (1 + R_{t,T}) + k(t,T) - Y_{t,T}$$

where

$F_{t,T}$ futures price at date t of a contract for delivery at date T

S_t spot price of the underlying at date t

$R_{t,T}$ risk-free interest rate for the period $(T - t)$

$k(t,T)$ carrying costs, such as insurance costs, storage costs, etc.

$Y_{t,T}$ convenience yield

1.2 Options

1.2.1 Binomial model in one period

The option price at the beginning of the period is equal to the expected value of the option price at the end of the period under the probability measure π, discounted with the risk-free rate.

$$O = \frac{O_u \cdot \pi + O_d \cdot (1-\pi)}{1+R}$$

$$\pi = \frac{1+R-d}{u-d}, \quad u = e^{\sigma\sqrt{\tau/n}}, \quad d = \frac{1}{u}, \quad d < 1+R < u$$

where

R	simple risk-free rate of interest for one period
O	value of the option at the beginning of the period
O_u	value of the option in the up-state at the end of the period
O_d	value of the option in the down-state at the end of the period
σ	volatility of the underlying
τ	time until expiry of the option
n	number of periods τ is divided in
u	upward factor of the underlying
d	downward factor of the underlying
π	risk neutral probability

1.2.2 Black and Scholes formula

The prices of European options on non dividend paying stocks are given by

$$C_E = S \cdot N(d_1) - Ke^{-r\tau} \cdot N(d_2)$$

$$P_E = Ke^{-r\tau} \cdot N(-d_2) - S \cdot N(-d_1)$$

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}, \quad d_2 = d_1 - \sigma\sqrt{\tau}$$

where

C_E	value of Euroean call
P_E	value of Euroean put
S	current stock price
τ	time in years until expiry of the option
K	strike price
σ	volatility p.a. of the underlying stock
r	continuously compounded risk-free rate p.a.
$N(x)$	cumulative distribution function for a standardised normal random variable (see Table 1.4), and

$$N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2}} ds$$

1.2.3 Put call parity for European and American options

$$P_E = C_E - S + D + Ke^{-r\tau}$$
$$C_{US} - S + Ke^{-r\tau} \le P_{US} \le C_{US} - S + K + D$$

where

τ	time until expiry of the option
K	strike or exercise price of the option
r	continuously compounded risk-free rate of interest
S	spot price of the underlying
C_E	value of European call option
P_E	value of European put option
C_{US}	value of American call option
P_{US}	value of American put option
D	present value of expected cash-dividends during the life of the option

1.2.4 Sensitivities of option prices

Sensitivity with respect to the stock price S (delta, Δ)

$$\Delta_c = \frac{\partial C}{\partial S} = N(d_1) \qquad (0 \le \Delta_c \le 1)$$
$$\Delta_P = \frac{\partial P}{\partial S} = N(d_1) - 1 \quad (-1 \le \Delta_P \le 0)$$

Sensitivity with respect to time τ (theta, θ)

$$\theta_C = \frac{\partial C}{\partial t} = -\frac{\partial C}{\partial \tau} = -\frac{S \cdot \sigma}{2 \cdot \sqrt{\tau}} \cdot n(d_1) - K \cdot r \cdot e^{-r\tau} N(d_2) \quad (\theta_C \le 0)$$
$$\theta_P = \frac{\partial P}{\partial t} = -\frac{\partial P}{\partial \tau} = -\frac{S \cdot \sigma}{2 \cdot \sqrt{\tau}} \cdot n(d_1) - K \cdot r \cdot e^{-r\tau} \left[N(d_2) - 1 \right]$$

where

N(x)	cumulative distribution function
n(x)	probability density function, and

$$n(x) = N'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

Sensitivity with respect to the volatility σ (kappa, κ)

$$\kappa_C = \frac{\partial C}{\partial \sigma} = S \cdot \sqrt{\tau} \cdot n(d_1) \qquad (\kappa_C \geq 0)$$

$$\kappa_P = \frac{\partial P}{\partial \sigma} = S \cdot \sqrt{\tau} \cdot n(d_1) = \kappa_C \qquad (\kappa_P \geq 0)$$

Sensitivity with respect to the interest rate r (rho, ρ)

$$\rho_C = \frac{\partial C}{\partial r} = K \cdot \tau \cdot e^{-r \cdot \tau} \cdot N(d_2) \qquad (\rho_C \geq 0)$$

$$\rho_P = \frac{\partial P}{\partial r} = K \cdot \tau \cdot e^{-r \cdot \tau} \cdot (N(d_2) - 1) \qquad (\rho_P \leq 0)$$

The second derivative with respect to the stock price S (gamma, Γ)

$$\Gamma_C = \frac{\partial^2 C}{\partial S^2} = \frac{n(d_1)}{S \cdot \sigma \cdot \sqrt{\tau}} \qquad (\Gamma_C \geq 0)$$

$$\Gamma_P = \frac{\partial^2 P}{\partial S^2} = \frac{n(d_1)}{S \cdot \sigma \cdot \sqrt{\tau}} = \Gamma_C \qquad (\Gamma_P \geq 0)$$

The leverage or elasticity of the option with respect to S (omega, Ω)

$$\Omega_C = \frac{\partial C}{\partial S} \cdot \frac{S}{C} \qquad \Omega_P = \frac{\partial P}{\partial S} \cdot \frac{S}{P}$$

1.2.5 Option pricing on stocks paying known dividends

The prices of European options on dividend paying stocks are given by

$$C_E = S^* \cdot N(d_1^*) - Ke^{-r\tau} \cdot N(d_2^*)$$

$$P_E = Ke^{-r\tau} \cdot N(-d_2^*) - S^* \cdot N(-d_1^*)$$

$$d_1^* = \frac{\ln\left(S^* / K\right) + (r + \sigma^2 / 2)\tau}{\sigma\sqrt{\tau}}, \quad d_2^* = d_1^* - \sigma\sqrt{\tau}, \quad S^* = S - \sum_{i=1}^{I} D_i \cdot e^{-r\tau_i}$$

where

τ_i	time in years until i^{th} dividend payment
D_i	dividend i
S	current stock price
τ	time in years until expiry of the option
K	strike price
σ	volatility p.a. of the underlying stock
r	continuously compounded risk-free rate p.a.

1.2.6 Option pricing when the underlying stock pays a known dividend yield

The prices of European options on stocks paying a continuous dividend yield are given by

$$C_E = S \cdot e^{-y\tau} \cdot N(d_1) - Ke^{-r\tau} \cdot N(d_2)$$
$$P_E = Ke^{-r\tau} \cdot N(-d_2) - S \cdot e^{-y\tau} \cdot N(-d_1)$$
$$d_1 = \frac{\ln(S/K) + (r - y + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}, \quad d_2 = d_1 - \sigma\sqrt{\tau}$$

where

y	continuous dividend yield
S	current stock price
τ	time in years until expiry of the option
K	strike price
σ	volatility p.a. of the underlying stock
r	continuously compounded risk-free rate p.a.

1.2.7 Options on financial futures

The prices of European options on a financial future are given by

$$C_E = e^{-r\tau}\left[F \cdot N(d_1) - K \cdot N(d_2)\right]$$
$$P_E = e^{-r\tau}\left[K \cdot N(-d_2) - F \cdot N(-d_1)\right]$$
$$d_1 = \frac{\ln(F/K)}{\sigma\sqrt{\tau}} + \frac{1}{2}\sigma\sqrt{\tau}, \quad d_2 = d_1 - \sigma\sqrt{\tau}$$

where

F	current futures price
τ	time in years until expiry of the option
K	strike price
σ	volatility p.a. of the futures returns
r	continuously compounded risk-free rate p.a.

1.2.8 Options on foreign currencies

$$C_E = S \cdot e^{-r_{for}\tau} \cdot N(d_1) - Ke^{-r\tau} \cdot N(d_2)$$
$$P_E = Ke^{-r\tau} \cdot N(-d_2) - S \cdot e^{-r_{for}\tau} \cdot N(-d_1)$$
$$d_1 = \frac{\ln(S/K) + (r - r_{for} + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}, \quad d_2 = d_1 - \sigma\sqrt{\tau}$$

where

S	current exchange rate (domestic per foreign currency units)
τ	time in years until expiry of the option
K	strike price (domestic per foreign currency units)
σ	volatility p.a. of the underlying stock
r	continuously compounded risk-free rate p.a.
r_{for}	continuously compounded risk-free rate p.a. of the foreign currency

1.4 Normal Distribution Table

x	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
4.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Probability that a normal random variable is smaller than x.

2. Modern Portfolio Theory

2.1 Return

2.1.1 Holding Period Return

$$R_t = \frac{P_t - P_{t-1} + \sum_{j=1}^{J} D_{t_j} \cdot \left(1 + R^*_{t_j,t}\right)^{t-t_j}}{P_{t-1}}$$

where

R_t	simple (or discrete) return of the asset over period $t-1$ to t
P_t	price of the asset at date t
D_{t_j}	dividend or coupon paid at date t_j between $t-1$ and t
t_j	date of the j^{th} dividend or coupon payment
$R^*_{t_j,t}$	risk-free rate p.a. for the period t_j to t
J	number of intermediary payments

2.1.2 Compounded Returns

$$1 + R_{eff} = \left(1 + \frac{R_{nom}}{m}\right)^m$$

where

R_{eff}	effective rate of return over entire period
R_{nom}	nominal return
m	number of sub-periods

2.1.3 Continuously compounded versus simple (discrete) returns

$$r_t = \ln \frac{P_t}{P_{t-1}} = \ln(1 + R_t)$$

$$R_t = e^{r_t} - 1$$

where

r_t	continuously compounded return between time $t-1$ and t
R_t	simple (discrete) return between time $t-1$ and t

2.1.4 Average return

Geometric average return over a holding period using discrete compounding

$$R_A = \sqrt[N]{(1+R_1)\cdot(1+R_2)\cdot\ldots\cdot(1+R_N)} - 1$$

where

R_A geometric average return over N sequential periods
R_i discrete return for the period i

Arithmetic average return over a holding period using continuous compounding

$$r_A = \frac{1}{N}\sum_{i=1}^{N} r_i$$

where

r_A average continuously compounded return over N sequential periods
r_i continuously compounded return for the period i

2.1.5 Annualisation of returns

Annualising holding period returns (assuming 360 days per year)
Assuming reinvestment of interests at rate R_τ

$$R_{ann} = (1+R_\tau)^{360/\tau} - 1$$

Euromarket convention, assuming no reinvestment of interests

$$r_{an} = (1+r_\tau)^{360/\tau} - 1$$

where

R_{ann} annualised simple rate of return
R_τ simple return for a time period of τ days

Annualising continuously compounded returns (assuming 360 days per year)

$$r_{an} = \frac{360}{\tau} \times r_\tau$$

where

r_{ann} annualised rate of return
r_τ continuously compounded rate of return earned over a period of τ days

2.1.6 Nominal versus real returns

With **simple returns**

$$R_t^{real} = R_t^{\text{nominal}} - I_t - R_t^{real} \cdot I_t \approx R_t^{\text{nominal}} - I_t$$

With **continuously compounded returns**

$$r_t^{real} = r_t^{\text{nominal}} - i_t$$

where

R_t^{real}	real rate of return on an asset over period t (simple)
R_t^{nominal}	nominal rate of return on an asset over period t (simple)
I_t	rate of inflation over period t (simple)
r_t^{real}	real rate of return on an asset over period t (cont. comp.)
r_t^{nominal}	nominal rate of return on an asset over period t (cont. comp.)
i_t	rate of inflation over period t (cont. comp.)

2.2 Mean, Variance, Covariance, Volatility

Expectation value E(.), **variance** Var(.), **covariance** Cov(.) and **correlation** Corr(.) of two random variables X and Y if the variables take values x_k, y_k in state k with probability p_k

$$\mathrm{E}(X) = \sum_{k=1}^{K} p_{\mathrm{k}} \cdot x_k, \quad \mathrm{E}(Y) = \sum_{k=1}^{K} p_{\mathrm{k}} \cdot y_k$$

$$\mathrm{Var}(X) = \sigma_X^2 = \mathrm{E}\left[(X - \mathrm{E}(X))^2\right] = \mathrm{E}(X^2) - \mathrm{E}(X)^2 = \sum_{k=1}^{K} p_{\mathrm{k}} (x_k - \mathrm{E}(X))^2$$

$$\mathrm{Cov}(X,Y) = \sigma_{XY} = \mathrm{E}\left[(X - \mathrm{E}(X)) \cdot (Y - \mathrm{E}(Y))\right] = \sum_{k=1}^{K} p_{\mathrm{k}} (x_k - \mathrm{E}(X)) \cdot (y_k - \mathrm{E}(Y))$$

$$\mathrm{Corr}(X,Y) = \frac{\sigma_{XY}}{\sigma_X \cdot \sigma_Y}$$

where $\sum_{k=1}^{K} p_{\mathrm{k}} = 1$, and

p_k	probability of state k
x_k	value of X in state k
y_k	value of Y in state k
K	number of possible states

The mean, the variance and the covariance of two random variables X and Y, in a sample of N observations of x_i and y_i, are given by

$$\mathrm{E}(X) = \bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i, \quad \mathrm{Var}(X) = \sigma_X^2 = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})^2$$

$$\mathrm{Cov}(X,Y) = \sigma_{XY} = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x}) \cdot (y_i - \bar{y})$$

where

x_i, y_i	observation i
$\bar{x}, \bar{y}$	mean of X and Y
σ_X, σ_Y	standard deviations
σ_{XY}	covariance of X and Y
N	number of observations

2.2.1 Volatility of returns

$$\sigma = \sqrt{\frac{1}{N-1}\sum_{t=1}^{N}(r_t - \bar{r})^2}, \qquad \bar{r} = \frac{1}{N}\sum_{t=1}^{N} r_t$$

where

σ	standard deviation of the returns (the volatility)
N	number of observed returns
$r_t = \ln \frac{P_t}{P_{t-1}}$	continuously compounded return of asset P over period t

2.2.2 Annualising volatility

Assuming that monthly returns are independent, then

$$\sigma_{ann} = \sqrt{12} \cdot \sigma_m = \frac{\sigma_\tau}{\sqrt{\tau}}$$

where

σ_{ann}	annualised volatility
σ_m	volatility of monthly returns
σ_τ	volatility of returns over periods of length τ
τ	vength of one period in years

2.3 Linear Regression

2.3.1 Simple regression model (OLS)

$$Y_t = a + b \cdot X_t + \varepsilon_t$$
$$b = \frac{\mathrm{Cov}(X,Y)}{\mathrm{Var}(X)}, \quad a = \overline{Y} - b \cdot \overline{X}$$

where

Y_t	dependent variable
X_t	independent variable
$\overline{Y}$	mean of Y
$\overline{X}$	mean of X
ε_t	error term
a	intercept
b	coefficient (slope) of the regression

2.3.2 Multiple regression model (OLS)

$$Y_t = \alpha + \beta_1 \cdot X_{1,t} + ... + \beta_k \cdot X_{k,t} + \varepsilon_t$$

where

Y_t	dependent variable
$X_{i,t}$	independent variable i
ε_t	error term (residual)
α	intercept
β_i	coefficients of the regression

2.3.3 Quality of the linear regression

2.3.3.1 Correlation Coefficient

$$\rho_{XY} = \frac{\mathrm{Cov}(X,Y)}{\sigma_X \cdot \sigma_Y}$$

where

ρ_{XY}	correlation coefficient
$\mathrm{Cov}(\mathrm{X}, \mathrm{Y})$	covariance between X and Y
σ_X	standard deviation of X
σ_Y	standard deviation of Y

2.3.3.2 Coefficient of determination

$$R^2 = 1 - \frac{\sum_{t=1}^{T} \varepsilon_t^2}{\sum_{t=1}^{T} (Y_t - \overline{Y})^2} = \rho^2$$

where

R^2	coefficient of determination
ρ	correlation coefficient
Y_t	dependent variable
$\overline{Y}$	mean of dependent variable
ε_t	error term (residual)

2.3.3.3 Fisher F-statistic

$$F = \frac{R^2}{1 - R^2} \cdot \frac{n - k - 1}{k}$$

where

F	Fisher F-statistic
R^2	coefficient of determination
n	number of observations
k	number of independent variables

2.4 The portfolio concept

2.4.1 Portfolio characteristics

Ex post return on a portfolio P in period t

$$\overline{r}_p = \sum_{i=1}^{N} x_i \overline{r}_i = x_1 \overline{r}_1 + x_2 \overline{r}_2 + \cdots + x_N \overline{r}_N$$

where $\sum x_i = 1$

$R_{P,t}$	return on the portfolio in period t
$R_{i,t}$	return on asset i in period t
x_i	initial (at beginning of period) proportion of the portfolio invested in asset i
N	number of assets in portfolio P

Expectation of the portfolio return

$$E(R_P) = \sum_{t=1}^{N} x_i E(R_i) = x_1 E(R_1) + x_2 E(R_2) + \ldots + x_N E(R_N)$$

where

$E(R_P)$	expected return on the portfolio
$E(R_i)$	expected return on asset *i*
x_i	relative weight of asset *i* in portfolio *P*
N	number of assets in portfolio *P*

Variance of the portfolio return

$$\text{Var}(R_P) = \sigma_P^2 = \sum_{i=1}^{N}\sum_{j=1}^{N} x_i x_j \sigma_{ij} = \sum_{i=1}^{N}\sum_{j=1}^{N} x_i x_j \rho_{ij} \sigma_i \sigma_j$$

where

σ_P^2	variance of the portfolio return
σ_{ij}	covariance between the returns R_i and R_j
ρ_{ij}	correlation coefficient between the returns R_i and R_j
σ_i, σ_j	standard deviations of the returns R_i and R_j
N	number of assets in portfolio *P*
x_i	initial proportion of the portfolio invested in asset i
x_j	initial proportion of the portfolio invested in asset j

2.5 The Capital Asset Pricing Model (CAPM)

2.5.1 The Capital Market Line (CML)

$$E(R_P) = R_f + \frac{E(R_M) - R_f}{\sigma_M} \sigma_P$$

where

$E(R_P)$	expected return of portfolio *P*
R_f	risk free rate
$E(R_M)$	expected return of the market portfolio
σ_M	standard deviation of the return on the market portfolio
σ_p	standard deviation of the portfolio return

2.5.2 The Security Market Line (SML)

$$\mathrm{E}(R_i) = R_f + \left[\mathrm{E}(R_M) - R_f\right] \cdot \beta_i$$

$$\beta_i = \frac{\mathrm{Cov}(R_i, R_M)}{\mathrm{Var}(R_M)}$$

where

$\mathrm{E}(R_i)$	expected return of asset i
R_f	risk free rate
$\mathrm{E}(R_M)$	expected return on the market portfolio
β_i	beta of asset i

2.5.3 Beta of a portfolio

$$\beta_p = \sum_{i=1}^{N} x_i \beta_i$$

where

β_P	beta of the portfolio
β_i	beta of asset i
x_i	proportion of the portfolio invested in asset i
N	number of assests in the portfolio

3. Practical Portfolio Management

3.1 Index and market models

3.1.1 The single-index model

Single-index model

$$R_{it} = \alpha_i + \beta_i \cdot R_{\text{index},t} + \varepsilon_{it}$$

Market model

$$R_{it} = \alpha_i + \beta_i \cdot R_{Mt} + \varepsilon_{it}$$

Market model in expectation terms

$$\mathrm{E}(R_{it}) = \alpha_i + \beta_i \cdot \mathrm{E}(R_{Mt})$$

where

R_{it}	return on asset or portfolio i over period t
α_i	intercept for asset or portfolio i
β_i	sensitivity of asset or portfolio i to the index return
$R_{\text{index},t}$	return on the index over period t
R_{Mt}	return on the market portfolio
ε_{it}	random error term ($\mathrm{E}(\varepsilon_{it}) = 0$)

Covariance between two assets in the market model or the CAPM context

$$\sigma_{ij} = \beta_i \cdot \beta_j \cdot \sigma_M^2$$

where

σ_{ij}	covariance between the returns of assets i and j
β_i	beta of portfolio i
β_j	beta of portfolio j
σ_M^2	variance of the return on the market portfolio

3.1.2 Decomposing variance into systematic and diversifiable risk

In the case of a single security

$$\sigma_i^2 = \underbrace{\beta_i^2\sigma_M^2}_{\substack{\text{market}\\ \text{risk}}} + \underbrace{\sigma_{\varepsilon_i}^2}_{\substack{\text{residual}\\ \text{risk}}}$$

where

σ_i^2 :	total variance of the return on asset or portfolio i
$\beta_i^2\sigma_M^2$:	market or systematic risk (explained variance)
$\sigma_{\varepsilon_i}^2$:	idiosyncratic or residual or unsystematic risk (unexplained variance)

Quality of an index model: R^2 and ρ^2

$$R^2 = \frac{\beta_i^2 \cdot \sigma_I^2}{\sigma_i^2} = \frac{\beta_i^2 \cdot \sigma_I^2}{\beta_i^2 \cdot \sigma_I^2 + \sigma_{\varepsilon_i}^2} = 1 - \frac{\sigma_{\varepsilon_i}^2}{\sigma_i^2} = \rho_{iI}^2$$

where

σ_i^2 :	total variance of the returns on asset i
$\beta_i^2\sigma_M^2$:	market or systematic risk (explained variance)
$\sigma_{\varepsilon_i}^2$:	idiosyncratic or residual or unsystematic risk (unexplained variance)
ρ_{iI}	correlation between asset i and the index I
R^2	coefficient of determination in a regression of R_i on R_I

3.1.3 Multi-index models

Multi-index models

$$r_i = \alpha_i + \beta_{i1}I_1 + \beta_{i2}I_2 + \ldots + \beta_{in}I_n + \varepsilon_i$$

where

R_i	return on asset or portfolio i
β_{ij}	beta or sensitivity of the return of asset i to changes in index j
I_j	index j
ε_i	random error term

Portfolio variance under a multi-index model (every index is assumed to be uncorrelated with each other)

$$\sigma_P^2 = \beta_{P,1}^2 \cdot \sigma_1^2 + \ldots + \beta_{P,L}^2 \cdot \sigma_L^2 + \sigma_{\varepsilon P}^2$$

where

σ_i^2 :	variance of the asset or portfolio i
$\beta_{P,j}^2 \sigma_j^2$	systematic risk due to index j
$\sigma_{\varepsilon P}^2$:	residual risk

3.1.4 Tracking error

$$TE^{P,B} = \sqrt{V(R_A^{P,B})}$$

where

$\mathrm{TE}^{\mathrm{P,B}}$	Tracking error
$\mathrm{V}\left(\mathrm{R}_{\mathrm{A}}^{\mathrm{P,B}}\right)$	Volatility of the active return

3.2 Hedging Strategies using Futures

3.2.1 Hedge Ratio and Number of Futures Contracts

$$HR = \frac{\Delta S}{\Delta F} = -\frac{N_F \cdot k}{N_S} \qquad N_F = -HR \cdot \frac{N_S}{k}$$

where

HR	hedge ratio
ΔS	change in spot price per unit
ΔF	change in futures price per unit
N_F	number of futures
N_S	number of spot assets
k	contract size

3.2.2 The Perfect (Naive) Hedge

$$\begin{cases} HR = \pm 1 \\ N_F = \mp \dfrac{N_S}{k} \end{cases}$$

where

HR	hedge ratio
N_F	number of futures
N_S	number of spot assets
k	contract size

3.2.3 Hedged Profit

For a long position in the underlying asset

$$\text{Hedged profit} = (S_T - S_t) - (F_{T,T} - F_{t,T})$$

where

S_T	spot price at the maturity of the futures contract
S_t	spot price at time t
$F_{T,T}$	futures price at its maturity
$F_{t,T}$	futures price at time t with maturity T

3.3 Minimum Variance Hedge Ratio

$$HR = \frac{\text{Cov}(\Delta S, \Delta F)}{\text{Var}(\Delta F)} = \rho_{\Delta S, \Delta F} \cdot \frac{\sigma_{\Delta S}}{\sigma_{\Delta F}}$$

where

HR	hedge ratio
$\text{Cov}(\Delta S, \Delta F)$	covariance between the changes in spot price (ΔS) and the changes in futures price (ΔF)
$\text{Var}(\Delta F)$	variance of changes in futures price (ΔF)
$\rho_{\Delta S, \Delta F}$	coefficient of correlation between ΔS and ΔF
$\sigma_{\Delta S}$	standard deviation of ΔS
$\sigma_{\Delta F}$	standard deviation of ΔF

3.3.1 Hedging when Returns are Normally Distributed (OLS Regression)

$$\frac{\Delta S_t}{S_t} = \alpha + \beta \cdot \frac{\Delta F_t}{F_{t,T}} + \varepsilon_t$$

$$HR = \beta \cdot \frac{S_t}{F_{t,T}}$$

where

ΔS_t	changes in spot price at time t
S_t	spot price at time t
α	intercept of the regression line
β	slope of the regression line
ΔF_t	changes in the futures price at time t
$F_{t,T}$	futures price at time t with maturity T
ε_t	residual term
HR	hedge ratio

3.4 Hedging with Stock Index Futures

3.4.1 Using OLS Regression

$$N_F = -\beta \cdot \frac{N_S \cdot S_t}{k \cdot F_{t,T}}$$

where

β	slope of the regression line
N_F	number of futures
N_S	number of spot assets
S_t	spot price at time t
$F_{t,T}$	futures price at time t with maturity T
k	contract size

3.4.2 Adjusting the Beta of a Stock Portfolio

$$HR_{adj} = (\beta^{\text{actual}} - \beta^{\text{target}}) \cdot \frac{S_t}{F_{t,T}}$$

$$N_F = (\beta^{\text{target}} - \beta^{\text{actual}}) \cdot \frac{N_S \cdot S_t}{k \cdot F_{t,T}}$$

where

HR_{adj}	hedge ratio to adjust the beta to the target beta
β^{actual}	actual beta of the portfolio
β^{target}	target beta of the portfolio
S_t	spot price at time t
$F_{t,T}$	futures price at time t with maturity T
N_F	number of futures contracts
N_S	number of the spot asset to be hedged
k	contract size

3.5 Static Portfolio Insurance

3.5.1 General

3.5.1.1 Floor

$$f = \frac{\Phi}{V_0}$$

where

f	insured fraction of the initial total portfolio value
Φ	floor
V_0	initial value of the portfolio

3.5.1.2 Price Index Return

$$r_{MC} = \frac{r_{PC} + r_{PD} - r_f \cdot (1-\beta) - \beta \cdot r_{MD}}{\beta}$$

where

r_{MC}	price index return
r_{PC}	capital gain of the portfolio
r_{MD}	dividend yield of the index
r_{PD}	dividend yield of the portfolio
r_f	risk-free rate
β	portfolio beta with respect to the index

3.5.2 Paying Insurance on Managed Funds

$$V_0 = N_S \cdot \left(S_0 + \beta \cdot P(S_0, T, K)\right)$$

where

V_0	total initial value of the portfolio
N_S	numbers of units of the risky assets
S_0	risky asset spot price
β	portfolio beta with respect to the underlying S
$P(S_0,T,K)$	put premium for a spot S_0, a strike K and maturity T

3.5.2.1 Strike Price

$$K = f \cdot \left(S_0 + \beta \cdot P(S_0, T, K)\right)$$

where

K	strike price
f	fraction of the initial total portfolio value
S_0	risky asset spot price
β	portfolio beta with respect to the underlying S
$P(S_0,T,K)$	put premium for a spot S_0, a strike K and maturity T

3.5.3 Insurance Paid Externally

$$V_0 = N_S \cdot S_0$$

where

V_0 total initial value of the portfolio (without puts)
N_S numbers of units of the risky assets
S_0 risky asset spot price

3.5.3.1 Strike Price

$$K = I_0 \cdot e^{r^*_{MC}}$$

where

K strike price
I_0 index level at the time the insurance is made
r^*_{MC} return of the price index adjusted for the length of the insurance period

3.5.4 Number of contracts

$$N_P = \beta \cdot \frac{\text{Portfoliovalue}}{\text{Index level} \cdot \text{Option contract size}} = \beta \cdot \frac{N_S}{k}$$

where

N_P number of options
N_S number of units of the risky assets
β portfolio beta with respect to the index
k option contract size

3.6 Dynamic Portfolio Insurance

3.6.1 Price of a Put on an Index Paying a Continuous Dividend Yield y

3.6.1.1 Black&Scholes Model

$$P(S_t, T, K) = K \cdot e^{-r_f \cdot (T-t)} \cdot N(-d_2) - S_t \cdot \left(e^{-y \cdot (T-t)} \cdot N(-d_1)\right)$$

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + (r_f - y) \cdot (T-t)}{\sigma \cdot \sqrt{T-t}} + \frac{1}{2} \cdot \sigma \cdot \sqrt{T-t}$$

$$d_2 = d_1 - \sigma \cdot \sqrt{T-t}$$

where

$P(S_t, T, K)$	put premium for a spot S_t, a strike K and maturity T
S_t	risky asset spot price at time t
K	strike price
r_f	risk-free rate (continuously compounded, p.a.)
y	dividend yield (continuously compounded, p.a.)
σ	volatility of the risky asset (p.a.)
$T-t$	time to maturity (in years)

3.6.1.2 Delta of a European Put on an Index Paying a Continuous Dividend Yield y

$$\Delta_P = e^{-y \cdot (T-t)} \cdot [N(d_1) - 1]$$

where

Δ_P	delta of a put on the portfolio

3.6.2 Dynamic Insurance with Futures

$$N_F = -e^{y \cdot (T^* - T)} \cdot e^{-r_f \cdot (T^* - t)} \cdot [1 - N(d_1)] \cdot \beta \cdot \frac{N_S}{k}$$

where

N_F	number of futures
T^*	maturity of the futures contract
T	maturity of the replicated put
β	portfolio beta with respect to the index
N_S	number of units of the risky assets
k	futures contract size

3.6.3 Constant Proportion Portfolio Insurance (CPPI)

3.6.3.1 Cushion

$$c_t = V_t - \Phi$$

where

c_t cushion

Vt value of the portfolio

Φ floor

3.6.3.2 Amount Invested in Risky Assets

$$A_t = N_{S,t} \cdot S_t = m \cdot c$$

where

A_t total amount invested in the risky assets at time t

$N_{S,t}$ number of units of the risky assets

S_t unit price of the risky assets

m multiplier

c cushion

3.6.3.3 Amount Invested in Risk-free Assets

$$B_t = V_t - A_t$$

where

B_t value of the risk-free portfolio at time t

V_t value of the total portfolio at time t

A_t value of the risky portfolio at time t

4. Performance Analysis

4.1 Return

4.1.1 Internal Rate of Return (IRR)

$$CF_0 = -\sum_{t=1}^{N} \frac{CF_t}{(1+IRR)^t}$$

where

CF_0	initial net cash flow
CF_t	net cash flow at the end of period t
IRR	internal rate of return (per period)
N	number of periods

4.1.2 Time Weighted Return (TWR)

Simple return:

$$TWR_{t/t-1} = \frac{MV_{end,t} - MV_{begin,t}}{MV_{begin,t}} = \frac{MV_{end,t}}{MV_{begin,t}} - 1$$

Continuously compounded return:

$$twr_{t/t-1} = \ln \frac{MV_{end,t}}{MV_{begin,t}}$$

where

$TWR_{t/t-1}$	simple time weighted return for sub-period t
$twr_{t/t-1}$	continuously compounded time weighted return for sub-period t
$MV_{begin,t}$	market value at the beginning of sub-period t
$MV_{end,t}$	market value at the end of sub-period t

Total period return:

Simple return:

$$1 + TWR_{tot} = \prod_{t=1}^{N} (1 + TWR_{t/t\text{-}1})$$

Continuously compounded return:

$$twr_{tot} = \sum_{t=1}^{N} twr_{t/t\text{-}1}$$

where

TWR_{tot}	simple time weighted return for the total period
$TWR_{t/t-1}$	simple time weighted return for sub-period t
twr_{tot}	continuously compounded time weighted return for the total period
$twr_{t/t-1}$	continuously compounded time weighted return for sub-period t

4.1.3 Money Weighted Return (MWR)

4.1.3.1 Gain or loss incurred on a portfolio

Gain = (Ending Market Value – Beginning Market Value) – Net Cash Flow

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4.2 Risk Adjusted Performance Measures

Sharpe Ratio or *Reward-to-Variability Ratio*	$RVAR_P = \dfrac{\bar{r}_P - \bar{r}_f}{\sigma_P}$
Treynor Ratio or *Reward-to-Volatility Ratio*	$RVOL_P = \dfrac{\bar{r}_P - \bar{r}_f}{\beta_P}$
Jensen's α	$\alpha_P = (\bar{r}_P - \bar{r}_f) - \beta_P \cdot (\bar{r}_M - \bar{r}_f)$
Treynor-Black Ratio or *Appraisal Ratio*	$AR_P = \dfrac{\alpha_P}{\sigma_\varepsilon}$

where

$\bar{r}_P$	average portfolio return
$\bar{r}_M$	average market return
$\bar{r}_f$	average risk-free rate
α_P	Jensen's alpha
β_P	portfolio beta
σ_P	portfolio volatility
σ_ε	standard deviation of the residuals