Pricing of Equity-linked Life Insurance Policies with an Asset Value Guarantee and Periodic Premiums

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Abstract

In the present paper we establish a quasi-explicit formula for the periodic premium under an equity-linked endowment policy with asset value guarantee in an economy with interest rate risk.

Résumé

Le présent article examine l'évaluation d'une prime periodique d'assurance pour le produit unit-linked qui contient complémentairement un rendement garanti. Nous proposons une solution de type quasi-explicite pour ce contract d'assurance vie dans un modèle de courbe des taux stochastiques.

Keywords

Equity-linked life insurance, unit-linked life insurance, option pricing, principle of equivalence.

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1 Introduction

The distinguishing feature of an equity-linked life insurance policy is that the benefit payable at expiration depends upon the market value of some reference portfolio. The *equity-linked endowment policy with asset value guarantee* (henceforth ELEPAVG) is a life insurance product of the unit-linked type where the sellers provide for a minimum benefit or asset value guarantee, payable on death or maturity. As opposed to traditional insurance the benefit is random but the added guarantee relieves the policyholder of a part of the investment risk. This product involves the insurance company both in mortality risk since it is uncertain at what date the guarantee will be effective, and in investment risk since the cost of the guarantee will depend upon the investment performance of the portfolio.

The pricing of equity-linked life insurance policies with asset value guarantee by using modern financial techniques has been discussed in the actuarial literature since the work of Brennan/Schwartz (1976) and Boyle/Schwartz (1977). They recognized that the payoff from an individual equity-linked contract at expiration is identical to the payoff from an European call option plus a certain amount (the guarantee amount) or to the payoff from an European put option plus the value of the reference portfolio. The insurance premium on an ELEPAVG contract was obtained by the application of the theory of contingent claim pricing. The premium was determined in an economy with the equity following a geometric Brownian motion, whereas the interest rate was assumed to be constant. Further investigations with a deterministic interest rate have been discussed in Delbaen (1986) and Aase/Persson (1994). Esspecially Delbaen focused the periodically paid premiums but he obtained no closed form solution and used Monte Carlo simulations in order to get results. Bacinello/Ortu (1993, 1994) and Nielsen/Sandmann (1995) extended these models and allowed for interest rate risk. To obtain a closed form solution for the price of a single premium endowment policy Bacinello/Ortu described the spot rate of interest by an Ornstein-Uhlenbeck process. Nielsen/Sandmann analysed the multi premium case and obtained no closed form solution but discussed different numerical procedures.

The purpose of this paper is to present a model for the multi premium case in the context of a stochastic interest rate process which allows to derive a quasi-explicit closed form solution. This new formula simplifies the calculation of the periodic premium because it avoids time consuming numerical calculations.

The paper is structured as follows. In section 2 we characterize briefly an ELE-PAVG contract and define the applied notation. In section 3 we abstract from the problem of mortality and present a quasi-explicit formula for the call option which is embedded in the life insurance contract. Section 4 allows for mortality risk and shows how the premium must be determined for an individual periodic premium contract, so that the present value of the expected premiums is equal to the present value of the benefits. The concluding remarks in section 5 discuss different extensions of this basic contract type where the periodic premiums can be evaluated by this approach. In the appendix the detailed proof for the option pricing formula is stated.

2 Notation and definition of the contract

The equity-linked endowment insurance policy with asset value guarantee with periodic premiums is a life insurance contract between an insurance company and a policyholder where the buyer is committed to pay regularly a predetermined premium to the company. At maturity or death of the insured person the benefit of the contract then payable consists of the greater of the value of some reference portfolio and some minimum guarantee payment. The reference portfolio is typically a portfolio formed by investing some predetermined component of the policyholders premium in common stocks. We use financial theory to value the benefit and then take mortality in account, assuming that the financial market is independent of the insured's health condition.

The following notation will be applied:

$[0,T]$	insurance period;
n	number of premiums;
T_i	premium payment date, $T_i \in [0,T], i = 0, \ldots, n-1$;
$S(t)$	price of one unit of the reference fund at time t;
$X(t)$	value of the reference portfolio of an individual contract at time t;
$G(t)$	exogenously given minimum guaranteed benefit at time t;
$b(t)$	benefit payable in t;
D	amount deemed to be invested in the reference portfolio at the premium payment dates;
P	periodic premium payed by the insured if the insured is alive;
$V_s(b(t))$	market value at time $s \leq t$ of the uncertain benefit, $b(t)$,

	payable at time t;
$C_s(X,G,t)$	value of an European call option at time s to purchase the reference portfolio $X(t)$ at time t for the exercise price $G(t)$;
$P_s(X,G,t)$	value of an European put option at time s to sell the reference portfolio $X(t)$ at time t for the exercise price $G(t)$;
$r(t)$	instantaneuos risk free rate of interest at time t;
$B(s,t)$	price of a zero coupon bond at date s with maturity date $t, s \leq t$;
$A(t)$	bank account.

The contractual features of the ELEPAVG require the benefit payable at death or at maturity to be

$$\begin{aligned} b(t) &= \max(X(t), G(t)) \\ &= X(t) + \max(G(t) - X(t), 0) \\ &= G(t) + \max(X(t) - G(t), 0). \end{aligned}$$

The alternative expressions for the benefit decompose it in terms of maturity values of either put or call options on $X(t)$ with strike price $G(t)$. Consider especially the value of the reference portfolio $X(t)$. The value of $X(t)$ is uncertain and it depends on the price of one unit of the fund at time t, the prices of one unit at the past premium payment dates $T_i \leq t$ and it is determined by the amount deemed to be invested in the fund. This leads to the decomposition

$$X(t) = \sum_{i=0}^{n^*(t)-1} \frac{D}{S(T_i)} S(t)$$

with $n^*(t) := \min\{i | T_i > t\}$.

We assume from now on that financial and insurance markets are perfectly competitive, frictionless and free of arbitrage opportunities. Under this assumption the market value at time $t = 0$ of the benefit $b(t)$ is

$$\begin{aligned} V_0(b(t)) &= V_0(\max(X(t), G(t))) \\ &= V_0(X(t)) + P_0(X, G, t) \\ &= B(0,t)G(t) + C_0(X, G, t). \end{aligned} \tag{1}$$

If $C_0(X,G,t)$ can be determined then equation (1) may be used to calculate the equilibrium price the insurance company should charge for providing the uncertain benefit $b(t)$. The next section will consider the determination of this option price.

3 Pricing of the call option in the absence of mortality risk

The previous section has shown that for a known date of policy expiration the ELEPAVG is equivalent either to a plan providing a fixed benefit plus a call option or to an agreement providing a benefit of the value of the reference portfolio plus a put option. To make the model less complicated we will defer the problem of mortality to the next section and investigate the structure of the call option which is embedded in the benefit of the insurance contract. Therefore we assume that the insured person survives the maturity of the contract.

We assume now that one unit of the fund, which creates the reference portfolio, can be described by lognormal distributed stocks satisfying the stochastic differential equation

$$\frac{dS(t)}{S(t)} = \mu_S(t)dt + \sigma_S(t)dW^1(t); \qquad S(0) = S_0.$$

The price process of the zero coupon bonds is assumed to follow the stochastic differential equation

$$\frac{dB(t,t^*)}{B(t,t^*)} = \mu_B(t,t^*)dt + \sigma_B(t,t^*)dW^2(t).$$

where the time dependence is such that $B(t,t) = 1$, and a specific form of the volatility function is choosen: $\sigma(t,t^*) := \sigma(t^* - t)$ with $\sigma(t-t) = 0$.

We restrict ourselves to nonstochastic coefficients in the differential equations for the bonds and the fund.

The development of the bank account can be described by the deterministic differential equation

$$\frac{dA(t)}{A(t)} = r(t)dt.$$

$A(t)$ is an accumulation factor corresponding to the price of a bank account, rolling over at $r(t)$, with the date 0 investment of one unit of account.

The absence of arbitrage in the financial market allows the following characterisation of the price of the call option on the reference portfolio.

Proposition

The price of an European call option with expiration date T on a reference portfolio $X(T)$ and strike price $G(T)$ at date 0 is given by

$$C_0(X,G,T) = \sum_{i=0}^{n-1} DB(0,T_i)N(d_i) - G(T)B(0,T)N(d)$$

where

$$\begin{aligned}
N(x) &:= \int_{-\infty}^{x} \frac{1}{\sqrt{(2\pi)}} \exp(-\frac{1}{2}u^2)du \quad \textit{normal distribution} \\
-U_i &:= -U_i(T) := \int_{T_i}^{T} \sigma_S(t)d\hat{W}^1 \\
v_i^2 &:= v_i^2(T) := var(-U_i) = \int_{T_i}^{T} \sigma_S^2(t)dt \\
d_i &:= d_i(T) := d + v_i \\
d &:= d(T) \quad \textit{is defined implicitely by the following equation}
\end{aligned}$$

$$\sum_{k=0}^{n-1} DB(0,T_k)\exp(-\frac{1}{2}v_k^2 - v_k d(T)) = G(T)B(0,T) \tag{2}$$

The proof of this proposition is given in the appendix.

This call-formula is explicit in the sense that the only unknown parameter $d(T)$ is the unique solution of the simple equation (2). This equation can be solved by an iterative approach. The option price consists of the discounted and weighted investments D and the adjusted guarantee. The formula shows well how the periodically invested amount D changes the value of the call option and has to be considered in the valuation of equity-linked contracts in the multi premium case.

4 Valuation of the periodic premium

Not only the benefit payble unter an ELEPAVG is uncertain, but also the date at which it becomes payable because of the mortality risk. The valuation of the premium P requires the specification of the probabilities that the benefit $b(t)$ is due to be paid at time t, either caused by maturity or premature death. We use time continuous death probabilities so that $\alpha(t)dt$ denotes the probability that the contract terminates in the time intervall $[t, t + dt]$. In addition, we assume that the insurance company behaves risk neutral concerning the mortality risk. Thus, the premium can be calculated by using a revised version of the traditional principle of equivalence where we adjust the basic principle by using the equivalent probability measures. In this context we obtain the premium as the solution of the following equation

$$P\sum_{i=0}^{n-1} B(0,T_i)(1-\int_0^{T_i}\alpha(t)dt) \quad \stackrel{!}{=} \quad \int_0^T \alpha(t)V_0(b(t))dt + (1-\int_0^T \alpha(t)dt)V_0(b(T))$$

$$\Longleftrightarrow$$

$$\begin{aligned} P \quad = \quad & \frac{1}{\ddot{a}^*_{x,\overline{n}|}}\left[\int_0^T \alpha(t)\sum_{i=0}^{n^*(t)-1} DB(0,T_i)N(d_i(t)) + GB(0,t)(1-N(d(t)))dt \right. & (3) \\ & \left. +(1-\int_0^T \alpha(t)dt)\sum_{i=0}^{n-1} DB(0,T_i)N(d_i(T)) + GB(0,t)(1-N(d(T)))\right] \end{aligned}$$

where

$$\ddot{a}^*_{x,\overline{n}|} := \sum_{i=0}^{n-1} B(0,T_i)(1-\int_0^{T_i}\alpha(t)dt).$$

The regular net premium P can be found by an iterative approach caused by the quasi-explicit solution of the call option price. The only unknown parameter besides the premium is the parameter d which can be easily calculated. Hence, equation (3) is the new formula for pricing the multi premium case under an ELEPAVG.

5 Conclusion

In an economy with a stochastic development of the term structure of interest rates a new formula for the periodically paid premium has been presented. The determination of this premium is established in a revised version of the principle of equivalence which is based on a new adjusted measure. This quasi-explicit solution simplifies the valuation of the premium in the multi premium case of an EPELAVG considerably. This formula can also be used to valuate other types of contracts like pure endowment policies or term insurances. It can also be applied for contracts with endogenous guarantees and policies, in which not a predetermined amount but a share of the premium is deemed to be invested in the fund.

Appendix: The proof of the pricing formula for the embedded call option

In this section we present the proof of propostion.

Proof of the proposition:

The proof of this proposition uses the martingale approach to contingent claim valuation and is stimulated by a pricing formula for an European call option on a coupon bond (see El Karoui/Rochet (1990)).

In the absence of arbitrage the price of the European call option is given by the Feynman-Kac Solution under the equivalent martingale measure Q (see Duffie (1992) pp. 87):

$$
\begin{aligned}
C_0(X,G,T) &= E^Q\left[\exp\left(-\int_0^T r(s)ds\right) C_T(X,G,T)\right] \\
&= E^Q\left[\exp\left(-\int_0^T r(s)ds\right) \max\left(\sum_{i=0}^{n-1}\frac{D}{S(T_i)}S(T)-G(T),0\right)\right] \\
&= E^Q\left[\exp\left(-\int_0^T r(s)ds\right) \mathbf{1}_{\mathcal{E}}\left(\sum_{i=0}^{n-1}\frac{D}{S(T_i)}S(T)-G(T)\right)\right] \\
&= E^Q\left[\exp\left(-\int_0^T r(s)ds\right) \left(\mathbf{1}_{\mathcal{E}}\sum_{i=0}^{n-1}\frac{D}{S(T_i)}S(T)-\mathbf{1}_{\mathcal{E}}G(T)\right)\right] \\
&= E^Q\left[\exp\left(-\int_0^T r(s)ds\right) \mathbf{1}_{\mathcal{E}}\sum_{i=0}^{n-1}\frac{D}{S(T_i)}S(T)-\exp\left(-\int_0^T r(s)ds\right)\mathbf{1}_{\mathcal{E}}G(T)\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=0}^{n-1} DE^Q\left[\exp\left(-\int_0^T r(s)ds\right)\exp\left(\int_{T_i}^T (r(s)-\frac{1}{2}\sigma_S^2(s))ds\right.\right.\\
&\qquad\left.\left.+\int_{T_i}^T \sigma_S(s)d\hat{W}^1\right)\mathbf{1}_{\mathcal{E}}\right] - G(T)E^Q\left[\exp\left(-\int_0^T r(s)ds\right)\mathbf{1}_{\mathcal{E}}\right]\\
&= \sum_{i=0}^{n-1} DE^Q\left[\exp\left(-\int_0^{T_i} r(s)ds\right)\exp\left(-\int_{T_i}^T \frac{1}{2}\sigma_S^2(s)ds+\int_{T_i}^T \sigma_S(s)d\hat{W}^1\right)\mathbf{1}_{\mathcal{E}}\right]\\
&\quad -G(T)E^Q\left[\exp\left(-\int_0^T r(s)ds\right)\mathbf{1}_{\mathcal{E}}\right]\\
&= \sum_{i=0}^{n-1} DE^Q\left[\exp\left(-\int_0^{T_i} r(s)ds\right)\exp\left(-\int_{T_i}^T \frac{1}{2}\sigma_S^2(s)ds+\int_{T_i}^T \sigma_S(s)d\hat{W}^1\right)\mathbf{1}_{\mathcal{E}}\right]\\
&\quad -G(T)E^Q\left[\exp\left(-\int_0^T r(s)ds\right)\mathbf{1}_{\mathcal{E}}\right]
\end{aligned}
$$

where

$$
\mathcal{E} := \{\omega \in \Omega | \left(\sum_{i=0}^{n-1} \frac{D}{S(T_i)} S(T)\right)(\omega) \geq G\}
$$

is the event: "the call is exercised" and $\mathbf{1}_{\mathcal{E}}$ denotes the characteristic function. The processes $\hat{W}^1$ and $\hat{W}^2$ are standard Wiener processes under the equivalent Q-measure.

We now introduce some new equivalent probability measures defined by

$$
\begin{aligned}
\frac{dQ^T}{dQ} &:= \frac{\exp(-\int_0^T r(s)ds)}{B(0,T)} \quad \text{and}\\
\frac{dQ^{T_i}}{dQ} &:= \frac{\exp(-\int_0^{T_i} r(s)ds)}{B(0,T_i)} \quad i = 0,\dots,n-1.
\end{aligned}
$$

This leads to the formula

$$
\begin{aligned}
C_0(X,G,T) &= \sum_{i=0}^{n-1} DB(0,T_i)E^{Q^{T_i}}\left[\exp\left(-\int_{T_i}^T \frac{1}{2}\sigma_S^2(s)ds+\int_{T_i}^T \sigma_S(s)d\hat{W}^1\right)\mathbf{1}_{\mathcal{E}}\right]\\
&\quad -G(T)B(0,T)E^{Q^T}[\mathbf{1}_{\mathcal{E}}]
\end{aligned}
$$

Now we define further equivalent probability measures:

$$\frac{dQ^{T_i^*}}{dQ^{T_i}} := \exp\Big(\int_{T_i}^{T} \sigma_S(t)d\hat{W}^1 - \int_{T_i}^{T} \frac{1}{2}\sigma_S^2(t)dt\Big), \quad i = 0, \ldots, n-1.$$

Thus, we obtain

$$C_0(X, G, T) = \sum_{i=0}^{n-1} DB(0, T_i)Q^{T_i^*}(\mathcal{E}) - G(T)B(0, T)Q^T(\mathcal{E}).$$

With one last transformation of the set $\mathcal{E}$ we receive the equation

$$\begin{aligned}\sum_{i=0}^{n-1} DB(0, T_i)Q^{T_i^*}(\mathcal{E}) \quad &- \quad G(T)B(0, T)Q^T(\mathcal{E}) \\ &= \sum_{i=0}^{n-1} DB(0, T_i)Q^T(\mathcal{E}_i) - GB(0, T)Q^T(\mathcal{E}),\end{aligned}$$

where

$$-U_i := -U_i(T) := \int_{T_i}^{T} \sigma_S(t)d\hat{W}^1$$

and

$$\begin{aligned}\mathcal{E}_i := \{\omega \in \Omega | \quad &\sum_{k=0}^{n-1} DB(0, T_k)\exp(-\frac{1}{2}var(U_k) - [U_k - cov(U_i, U_k)])(\omega) \\ &\geq G(T)B(0, T)\}.\end{aligned}$$

The random variables U_i are gaussian with $v_i^2 := var(-U_i) = \int_{T_i}^{T} \sigma_S^2(t)dt$. This variables are also proportional so that we can write

$$U_i = v_i Y, \qquad i = 0, \ldots, n-1,$$

where Y denotes a normalized gaussian random variable.

Now the set $\mathcal{E}$ can be described as:

$$\begin{aligned}\mathcal{E} &= \{\omega \in \Omega | \sum_{k=0}^{n-1} DB(0, T_k)\exp(-\frac{1}{2}var(U_k) - U_k)(\omega) \geq G(T)B(0, T)\} \\ &= \{\omega \in \Omega | \sum_{k=0}^{n-1} DB(0, T_k)\exp(-\frac{1}{2}v_k^2 - v_k Y)(\omega) \geq G(T)B(0, T)\} \\ &= \{\omega \in \Omega | Y \leq d(T)\}.\end{aligned}$$

which implies $Q^T(\mathcal{E}) = N(d(T))$.

This leads to our final formula:

$$C_0(X, G, T) \;=\; \sum_{i=0}^{n-1} DB(0, T_i)N(d_i) - G(T)B(0, T)N(d)$$

where

$$\begin{aligned}
N(x) &:= \int_{-\infty}^{x} \frac{1}{\sqrt{(2\pi)}} \exp(-\frac{1}{2}u^2)du \\
d_i &:= \;:= d_i(T) := d + v_i \\
d &:= d(T) \quad \text{is defined implicitely by the following equation} \\
&\quad \sum_{k=0}^{n-1} DB(0, T_k) \exp(-\frac{1}{2}v_k^2 - v_k d(T)) = G(T)B(0, T).
\end{aligned}$$

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