On the Use of Numeraires in Option Pricing

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Significant computational simplification is achieved when option pricing is approached through the change of numeraire technique. By pricing an asset in terms of another traded asset (the numeraire), this technique reduces the number of sources of risk that need to be accounted for. The technique is useful in pricing complicated derivatives.

This article discusses the underlying theory of the numeraire technique, and illustrates it with five pricing problems: pricing savings plans that offer a choice of interest rates; pricing convertible bonds; pricing employee stock ownership plans; pricing options whose strike price is in a currency different from the stock price; and pricing options whose strike price is correlated with the short-term interest rate.

While the numeraire method is well-known in the theoretical literature, it appears to be infrequently used in more applied research, and many practitioners seem unaware of how to use it as well as when it is profitable (or not) to use it. To illustrate the uses (and possible misuses) of the method, we discuss in some detail five concrete applied problems in option pricing:

- Pricing employee stock ownership plans.
- Pricing options whose strike price is in a currency different from the stock price.
- Pricing convertible bonds.
- Pricing savings plans that provide a choice of indexing.
- Pricing options whose strike price is correlated with the short-term interest rate.

The standard Black-Scholes (BS) formula prices a European option on an asset that follows geometric Brownian motion. The asset's uncertainty is the only risk factor in the model. A more general approach developed by Black-Merton-Scholes leads to a partial differential equation. The most general method developed so far for the pricing of contingent claims is the martingale approach to arbitrage theory developed by Harrison and Kreps [1981], Harrison and Pliska [1981], and others.

Whether one uses the PDE or the standard risk-neutral valuation formulas of the martingale method, it is in most cases very hard to obtain analytic pricing formulas. Thus, for many important cases, special formulas (typically modifications of the original BS formula) have been developed. See Haug [1997] for extensive examples.

One of the most typical cases with multiple risk factors occurs when an option involves a choice between two assets with stochastic prices. In this case, it is often of considerable advantage to use a *change of numeraire* in the pricing of the option. We show where the numeraire approach leads to significant simplifications, but also where the numeraire change is trivial, or where an obvious numeraire change really does not simplify the computations.

The History of the Numeraire Approach in Option Pricing

The idea of using a numeraire to simplify option pricing seems to have a history almost as long as the Black-Scholes formula—a formula that might itself be interpreted as using the dollar as a numeraire. In 1973, the year the Black-Scholes paper was published, Merton [1973] used a change of numeraire (though without using the name) to derive the value of a European call (in units of zero coupon bond) with a stochastic yield on the zero. Margrabe's 1978 paper in the *Journal of Finance* on exchange options was the first to give the numeraire idea wide press. Margrabe appears also to have primacy in using the "numeraire" nomenclature. In his paper Margrabe acknowledges a suggestion from Steve Ross, who had suggested that using one of the assets as a numeraire would reduce the problem to the Black-Scholes solution and obviate any further mathematics.

In the same year that Margrabe's paper was published, two other papers, Brenner-Galai [1978] and Fischer [1978], made use of an approach which would now be called the numeraire approach. In the following year Harrison-Kreps [1979, p. 401] used the price of a security with a strictly positive price as numeraire. Hence, their numeraire asset has no market risk and pays an interest rate that equals zero, which is convenient for their analysis. In 1989 papers by Geman [1989] and Jamshidian [1989] formalized the mathematics behind the numeraire approach.

The main message is that in many cases the change of numeraire approach leads to a drastic simplification in the computations. For each of five different option pricing problems, we present the possible choices of numeraire, discuss the pros and cons of the various numeraires, and compute the option prices.

I. CHANGE OF NUMERAIRE APPROACH

The basic idea of the numeraire approach can be described as follows. Suppose that an option's price depends on several (say, n) sources of risk. We may then compute the price of the option according to this scheme:

- Pick a security that embodies one of the sources of risk, and choose this security as the numeraire.
- Express all prices in the market, including that of the option, in terms of the chosen numeraire. In other words, perform all the computations in a relative price system.
- Since the numeraire asset in the new price system is riskless (by definition), we have reduced the number of risk factors by one, from n to $n - 1$. If, for example, we start out with two sources of risk, eliminating one may allow us to apply standard one-risk factor option pricing formulas (such as Black-Scholes).
- We thus derive the option price in terms of the numeraire. A simple translation from the numeraire back to the local currency will then give the price of the option in monetary terms.

The standard numeraire reference in an abstract setting is Geman, El Karoui, and Rochet [1995]. We first consider a Markovian framework that is simpler than theirs, but that is still reasonably general. All details and proofs can be found in Björk [1999].[1]

Assumption 1. Given a priori are:

- An empirically observable $(k + 1)$-dimensional stochastic process:

$$X = (X_1, \ldots, X_{k+1})$$

with the notational convention that the process $k + 1$ is the riskless rate:

$$X_{k+1}(t) = r(t)$$

- We assume that under a fixed risk-neutral martingale measure Q the factor dynamics have the form:

$$dX_i(t) = \mu_i(t, X(t))dt + \delta_i(t, X(t))dW(t)$$
$$i = 1, \ldots, k + 1$$

where $W = (W_1, \ldots, W_d)'$ is a standard d-dimensional Q-Wiener process and $\delta_i = (\delta_{i1}, \delta_{i2}, \ldots, \delta_{id})$ is a row vector. The superscript $'$ denotes transpose.

- A risk-free asset (money account) with the dynamics:

$$dB(t) = r(t)B(t)dt$$

The interpretation is that the components of the vector process X are the underlying factors in the economy. We make no a priori market assumptions, so whether a particular component is the price process of a traded asset in the market will depend on the particular application.

Assumption 2 introduces asset prices, driven by the underlying factors in the economy.

Assumption 2

- We consider a fixed set of price processes $S_0(t)$, …, $S_n(t)$, each assumed to be the arbitrage-free price process for some traded asset without dividends.
- Under the risk-neutral measure Q, the S dynamics have the form

$$dS_i(t) = r(t)S_i(t)dt + S_i(t)\sum_{j=1}^{d}\sigma_{ij}(t, X(t))dW_j(t), \quad (1)$$

for $i = 0, \ldots, n - 1$.

- The n-th asset price is always given by

$$S_n(t) = B(t)$$

and thus (1) also holds for $i = n$ with $\sigma_{nj} = 0$ for $j = 1, \ldots, d$.

We now fix an arbitrary asset as the numeraire, and for notational convenience we assume that it is S_0. We may then express all other asset prices in terms of the numeraire S_0, thus obtaining the *normalized* price vector $Z = (Z_0, Z_1, \ldots, Z_n)$, defined by

$$Z_i(t) = \frac{S_i(t)}{S_0(t)}.$$

We now have two formal economies: the S economy where prices are measured in the local currency (such as dollars), and the Z economy, where prices are measured in terms of the numeraire S_0.

The main result is a theorem that shows how to price an arbitrary contingent claim in terms of the chosen numeraire. For brevity, we henceforth refer to a contingent claim with exercise date T as a T-claim.

Main Theorem. Let the numeraire S_0 be the price process for a traded asset with $S_0(t) > 0$ for all t. Then there exists a probability measure, denoted by Q^0, with properties as follows:

- For every T-claim Y, the corresponding arbitrage free price process $\Pi(t;\, Y)$ in the S economy is given by

$$\Pi(t; Y) = S_0(t)\Pi^Z\left(t; \frac{Y}{S_0(T)}\right), \quad (2)$$

where Π^Z denotes the arbitrage-free price in the Z economy.

- For any T-claim $\widetilde{Y}$ ($\widetilde{Y} = Y/S_0(T)$, for example) its arbitrage-free price process Π^Z in the Z economy is given by:

$$\Pi^Z\left(t; \tilde{Y}\right) = E^0_{t,X(t)}\left[\tilde{Y}\right], \quad (3)$$

where E^0 denotes expectations with regard to Q^0. The pricing formula (2) can be written

$$\Pi(t; Y) = S_0(t)E^0_{t,X(t)}\left[\frac{Y}{S_0(T)}\right]. \quad (4)$$

- The Q^0 dynamics of the Z processes are given by

$$dZ_i = Z_i\left[\sigma_i - \sigma_0\right]dW^0, \quad i = 0, \ldots, n. \quad (5)$$

where $\sigma_i = (\sigma_{i1}, \sigma_{i2}, \ldots, \sigma_{id})$, and σ_0 is defined similarly.

- The Q^0 dynamics of the price processes are given by

$$dS_i = S_i\left(r + \sigma_i\sigma_0'\right)dt + S_i\sigma_i dW^0, \quad (6)$$

where W^0 is a Q^0-Wiener process.

- The Q^0 dynamics of the X processes are given by

$$dX_i = \left(\mu_i + \delta_i\sigma_0'\right)dt + \delta_i dW^0. \quad (7)$$

- The measure Q^0 depends upon the choice of numeraire asset S_0, but the same measure is used for all claims, regardless of their exercise dates.

In passing, note that if we use the money account B as the numeraire, the pricing formula above reduces to the well-known standard risk-neutral valuation formula

$$\Pi(t; Y) = B(t)E^0_{t,X(t)}\left[\frac{Y}{B(T)}\right]$$

$$= E^0_{t,X(t)} \left[e^{-\int_t^T r(s)ds} Y \right] \tag{8}$$

In more pedestrian terms, the main points of the Theorem above are as follows:

- Equation (2) shows that the measure Q^0 takes care of the stochasticity related to the numeraire S_0. Note that we do not have to compute the price $S_0(t)$—we simply use the observed market price.

  We also see that if the claim Y is of the form $Y = Y_0 S_0(T)$ (where Y_0 is some T-claim) then the change of numeraire is a huge simplification of the standard risk-neutral formula (8). Instead of computing the joint distribution of $\int_t^T r(s)ds$ and Y (under Q), we have only to compute the distribution of Y_0 (under Q^0).
- Equation (3) shows that in the Z economy, prices are computed as expected values of the claim. Observe that there is no discounting factor in (3). The reason is that in the Z economy, the price process Z_0 has the property that $Z_0(t) = 1$ for all t. Thus, in the Z economy there is a riskless asset with unit price. That is, in the Z economy the short rate equals zero.
- Equation (5) says that the normalized price processes are martingales (i.e., zero drift) under Q^0, and identifies the relevant volatility.
- Equations (6)-(7) show how the dynamics of the asset prices and the underlying factors change when we move from Q to Q^0. Note that the crucial object is the volatility σ_0 of the numeraire asset.

We show examples of the use of the numeraire method that illustrate the considerable conceptual and implementational simplification this method provides.

II. EMPLOYEE STOCK OWNERSHIP PLANS

The first example is useful in valuing ESOPs.

Problem

In employee stock ownership plans (ESOPs), it is common to include an option such as: The holder has the right to buy a stock at the lower of its price in six months and in one year minus a rebate (say, 15%). The option matures in one year.

Mathematical Model

In a more general setting, the ESOP is a contingent claim Y to be paid out at time T_1 of the form

$$Y = S(T_1) - \beta \min[S(T_1),\ S(T_0)] \tag{9}$$

so in the example we would have $\beta = 0.85$, $T_0 = 1/2$, and $T_1 = 1$.

The problem is to price Y at some time $t \leq T_0$, and to this end we assume a standard Black-Scholes model where under the usual risk-neutral measure Q we have the dynamics:

$$dS(t) = rS(t)dt + \sigma S(t)dW(t), \tag{10}$$

$$dB(t) = rB(t)dt, \tag{11}$$

with a deterministic and constant short rate r.

The price $\Pi(t;\ Y)$ of the option can obviously be written

$$\Pi(t; Y) = S(t) - \beta \Pi(t; Y_0)$$

where the T_1 claim Y_0 is defined by

$$Y_0 = \min[S(T_1), S(T_0)].$$

To compute the price of Y_0, we want to:

- Perform a suitable change of numeraire.
- Use a standard version of some well-known option pricing formula.

The problem with this plan is that, at the exercise time T_1, the term $S(T_0)$ does not have a natural interpretation as the spot price of a traded asset. To overcome this difficulty, we therefore introduce a new asset S_0 defined by

$$S_0(t) = \begin{cases} S(t), & 0 \leq t \leq T_0, \\ S(T_0)e^{r(t-T_0)}, & T_0 \leq t \leq T_1. \end{cases}$$

In other words, S_0 can be thought of as the value of a self-financing portfolio where at $t = 0$ you buy one share of the underlying stock and keep it until $t = T_0$. At $t = T_0$, you then sell the share and put all the money into a bank account.

We then have $S_0(T_1) = S(T_0)e^{r(T_1 - T_0)}$, so we can now express Y_0 in terms of $S_0(T_1)$ as:

$$Y_0 = \min\left[S(T_1), K \cdot S_0(T_1)\right] \tag{12}$$

where

$$K = e^{-r(T_1 - T_0)} \tag{13}$$

Now $S_0(T_1)$ in (12) can formally be treated as the price at T_1 of a traded asset. In fact, from the definition above we have the trivial Q dynamics for S_0

$$dS_0(t) = rS_0(t)dt + S_0(t)\sigma_0(t)dW(t)$$

where the deterministic volatility is defined by

$$\sigma_0(t) = \begin{cases} \sigma, & 0 \le t \le T_0, \\ 0, & T_0 \le t \le T_1. \end{cases} \tag{14}$$

Now we perform a change of numeraire; we can choose either S or S_0 as the numeraire. From a logical point of view, the choice is irrelevant, but the computations become somewhat easier if we choose S_0. With S_0 as the numeraire we obtain (always with $t < T_0$) a pricing formula from the Main Theorem:

$$\Pi(t; Y_0) = S_0(t)E^0_{t,S_0(t)}\left[\min\{Z(T_1), K\}\right] \tag{15}$$

where

$$Z(t) = \frac{S(t)}{S_0(t)}$$

is the normalized price process.

From (5) we have

$$dZ(t) = Z(t)\left[\sigma - \sigma_0(t)\right]dW^0(t) \tag{16}$$

where W^0 is a Q^0-Wiener process. Using the simple equality

$$\min\{Z(T_1), K\} = Z(T_1) - \max\{Z(T_1) - K, 0\},$$

and noting that for $t \le T_0$ we have $S_0(t) = S(t)$, we obtain from (15):

$$\Pi(t; Y_0) = S(t)E^0_{t,S(t)}\left[Z(T_1)\right] - S(t)E^0_{t,S(t)}\left[\max\{Z(T_1) - K, 0\}\right].$$

Since Z is a Q^0 martingale (zero drift) and $Z(t) = 1$ for $t \le T_0$, we have:

$$S(t)E^0_{t,S(t)}\left[Z(T_1)\right] = S(t)Z(t) = S(t).$$

It now remains only to compute $E^0_{t,S(t)}[\max\{Z(T_1) - K, 0\}]$. This is just the price of a European call with strike price K where the stock price process Z follows GBM as in (16), and zero short rate.

From (16), and the definition of σ_0 in (14), the integrated squared volatility for Z over the time interval $[t, T_1]$ is given by:

$$\int_t^{T_1} \left[\sigma - \sigma_0(u)\right]^2 du = \sigma^2 \cdot (T_1 - T_0).$$

From the Black-Scholes formula with zero short rate and deterministic but time-varying volatility, we now have:

$$E^0_{t,S(t)}\left[\max\{Z(T_1) - K, 0\}\right] = Z(t)N[d_1] - KN[d_2]$$

where

$$\begin{aligned} d_1 &= \frac{\ln(Z(t)/K) + \frac{1}{2}\sigma^2(T_1 - T_0)}{\sigma\sqrt{T_1 - T_0}}, \\ d_2 &= d_1 - \sigma\sqrt{T_1 - T_0}. \end{aligned}$$

Using again the trivial fact that, by definition $Z(t) = 1$ for all $t \le T_0$, and collecting the computations above, we finally obtain the price of the ESOP as

$$\Pi(t; ESOP) = S(t) - \beta S(t)N[d_1] + \beta S(t)KN[d_2], \tag{17}$$

where

$$
\begin{aligned}
d_1 &= \frac{\ln(1/K) + \frac{1}{2}\sigma^2(T_1 - T_0)}{\sigma\sqrt{T_1 - T_0}}, \\
d_2 &= d_1 - \sigma\sqrt{T_1 - T_0},
\end{aligned}
$$

and where K is given by (13).

III. OPTIONS WITH A FOREIGN CURRENCY STRIKE PRICE

The strike prices for some options are linked to a non-domestic currency. Our example is a U.S. dollar strike price on a stock denominated in U.K. pounds. Such options might be part of an executive compensation program designed to motivate managers to maximize the dollar price of their stock. Another example might be an option whose strike price is not in a different currency, but is CPI-indexed.

Problem

We assume that the underlying security is traded in the U.K. in pounds sterling and that the option exercise price is in dollars. The institutional setup is as follows:

- The option is initially (i.e., at $t = 0$) an at-the-money option, when the strike price is expressed in pounds.[2]
- This pound strike price is, at $t = 0$, converted into dollars.
- The dollar strike price thus computed is kept constant during the life of the option.
- At the exercise date $t = T$, the holder can pay the fixed dollar strike price in order to obtain the underlying stock.
- The option is fully dividend protected.

Since the stock is traded in pounds, the fixed dollar strike corresponds to a randomly changing strike price when expressed in pounds; thus we have a non-trivial valuation problem. The numeraire approach can be used to simplify the valuation of such an option.

Mathematical Model

We model the stock price S (in pounds) as a standard geometric Brownian motion under the objective probability measure P, and we assume deterministic short rates r_p and r_d in the U.K. and the U.S. market, respectively. Since we have assumed complete dividend protection, we may as well assume (from a formal point of view) that S is without dividends. We thus have the P dynamics for the stock price:

$$dS(t) = \alpha S(t)dt + S(t)\delta_S dW^S(t)$$

We denote the dollar/pound exchange rate by X, and assume a standard Garman-Kohlhagen [1983] model for X. We thus have P dynamics given by:

$$dX(t) = \alpha_X X(t)dt + X(t)\delta_X dW^X(t)$$

Denoting the pound/dollar exchange rate by Y, where $Y = 1/X$, we immediately have the dynamics:

$$dY(t) = \alpha_Y Y(t)dt + Y(t)\delta_Y dW^Y(t)$$

where α_Y is of no interest for pricing purposes. Here W^S, W^X, and W^Y are scalar Wiener processes, and we have the relations:

$$
\begin{aligned}
\delta_Y &= \delta_X && (18) \\
W^Y &= -W^X, && (19) \\
dW^S(t) \cdot dW^X(t) &= \rho dt, && (20) \\
dW^S(t) \cdot dW^Y(t) &= -\rho dt. && (21)
\end{aligned}
$$

For computational purposes it is sometimes convenient to express the dynamics in terms of a two-dimensional Wiener process W with independent components instead of using the two correlated processes W^X and W^S. Logically the two approaches are equivalent, and in this approach we then have the P dynamics:

$$
\begin{aligned}
dS(t) &= \alpha S(t)dt + S(t)\sigma_S dW(t), \\
dX(t) &= \alpha_X X(t)dt + X(t)\sigma_X dW(t), \\
dY(t) &= \alpha_Y Y(t)dt + Y(t)\sigma_Y dW(t).
\end{aligned}
$$

The volatilities σ_S, σ_X, and σ_Y are two-dimensional row vectors with the properties that

$$
\begin{aligned}
\sigma_Y &= -\sigma_X \\
\|\sigma_X\|^2 &= \delta_X^2, \\
\|\sigma_Y\|^2 &= \delta_Y^2, \\
\|\sigma_S\|^2 &= \delta_S^2, \\
\sigma_X \sigma_S' &= \rho \delta_X \delta_S \\
\sigma_Y \sigma_S' &= -\rho \delta_Y \delta_S
\end{aligned}
$$

where ´ denotes transpose and $\| \, \|$ denotes the Euclidian norm in R^2.

The initial strike price expressed in pounds is by definition given by

$$K_p(0) = S(0),$$

and the corresponding dollar strike price is thus

$$K_d = K_p(0) \cdot X(0) = S(0)X(0).$$

The dollar strike price is kept constant until the exercise date. Expressed in pounds, however, the strike price evolves dynamically as a result of the varying exchange rate, so the pound strike at maturity is given by:

$$K_p(T) = K_d \cdot X(T)^{-1} = S(0) \cdot X(0) \cdot X(T)^{-1}. \tag{22}$$

There are now two natural ways to value this option, in dollars or in pounds, and initially it is not obvious which way is the easier. We perform the calculations using both alternatives and compare the computational effort. As will be seen below, it turns out to be slightly easier to work in dollars than in pounds.

Pricing the Option in Dollars

In this approach we transfer all data into dollars. The stock price, expressed in dollars, is given by

$$S_d(t) = S(t) \cdot X(t),$$

so in dollar terms the payout Φ_d of the option at maturity is given by the expression:

$$\Phi_d = \max\left[S(T)X(T) - K_d, 0\right]$$

Since the dollar strike K_d is constant, we can use the Black-Scholes formula applied to the dollar price process $S_d(t)$. The Itô formula applied to $S_d(t) = S(t)X(t)$ immediately gives us the P dynamics of $S_d(t)$ as:

$$
\begin{gathered}
dS_d(t) = S_d(t)\left(\alpha + \alpha_X + \sigma_S \sigma_X'\right) dt + \\
S_d(t)\left(\sigma_S + \sigma_X\right) dW(t)
\end{gathered}
$$

We can write this as:

$$dS_d(t) = S_d(t)\left(\alpha + \alpha_X + \sigma_S \sigma_X'\right) dt + S_d(t)\delta_{S,d} dV(t)$$

where V is a scalar Wiener process, and where

$$\delta_{S,d} = \|\sigma_S + \sigma_X\| = \sqrt{\delta_S^2 + \delta_X^2 + 2\rho\delta_S\delta_X}$$

is the dollar volatility of the stock price.

The dollar price (expressed in dollar data) at t of the option is now obtained directly from the Black-Scholes formula as

$$
\begin{aligned}
C_d(t) &= S_d(t)N[d_1] - e^{-r_d(T-t)} K_d N[d_2], \\
d_1 &= \frac{\ln\left(S_d(t)/K_d\right) + \left(r_d + \frac{1}{2}\delta_{S,d}^2\right)(T-t)}{\delta_{S,d}\sqrt{T-t}}, \\
d_2 &= d_1 - \delta_{S,d}\sqrt{T-t}.
\end{aligned} \tag{23}
$$

The corresponding price in pound terms is finally obtained as:

$$C_p(t) = C_d(t) \cdot \frac{1}{X(t)},$$

so the final pricing formula is:

$$
\begin{aligned}
C_p(t) &= S(t)N[d_1] - e^{-r_d(T-t)} \frac{S(0)X(0)}{X(t)} N[d_2], \\
d_1 &= \frac{\ln\left(\frac{S(t)X(t)}{S(0)X(0)}\right) + \left(r_d + \frac{1}{2}\delta_{S,d}^2\right)(T-t)}{\delta_{S,d}\sqrt{T-t}}, \\
d_2 &= d_1 - \delta_{S,d}\sqrt{T-t}, \\
\delta_{S,d} &= \sqrt{\delta_S^2 + \delta_X^2 + 2\rho\delta_S\delta_X}
\end{aligned} \tag{24}
$$

Pricing the Option Directly in Pounds

Although it is not immediately obvious, pricing the option directly in pounds is a bit more complicated than pricing the option in dollars. The pricing problem, expressed in pound terms, is that of pricing the T-claim Φ_p defined by

$$\Phi_p = \max\left[S(T) - K_p(T), 0\right].$$

Using (22) and denoting the pound/dollar exchange rate by Y (where of course $Y = 1/X$) we obtain:

$$\Phi_p = \max\left[S(T) - S(0)\frac{Y(T)}{Y(0)}, 0\right].$$

It is now tempting to use the pound/dollar exchange rate Y as the numeraire, but this is not correct. The reason is that although Y is the price of a traded asset (dollar bills), it is not the price of a traded asset *without dividends*, the obvious reason being that dollars are put into an American (or perhaps Eurodollar) account where they will command the interest rate r_d. Thus the role of Y is rather that of the price of an asset with the continuous dividend yield r_d.

In order to convert the present situation into the standard case covered by the Main Theorem, we therefore do as follows:

- We denote the dollar bank account by B_d with dynamics

$$dB_d(t) = r_d B_d(t)dt.$$

- The value in pounds of the dollar bank account is then given by the process $\hat{Y}_d$, defined by:

$$\hat{Y}_d(t) = B_d(t) \cdot Y(t) = Y(t)e^{r_d t}.$$

- The process $\hat{Y}_d$ can now be interpreted as the price process (denoted in pounds) of a traded asset without dividends.
- We may thus use $\hat{Y}_d$ as a numeraire.

Since we have $Y(T) = \hat{Y}_d(T)e^{-r_d T}$ we can write:

$$\Phi_p = \max\left[S(T) - \hat{Y}_d(T)e^{-r_d T}\frac{S(0)}{Y(0)}, 0\right].$$

Using $\hat{Y}_d$ as the numeraire we immediately obtain from the Main Theorem:

$$\Pi(t; \Phi_p) = \hat{Y}_d(t)E_t^{\hat{Q}}\left[\max\left\{Z(T) - K, 0\right\}\right], \tag{25}$$

where $\hat{Q}$ denotes the martingale measure with $\hat{Y}_d$ as the numeraire, where Z is defined by

$$Z(t) = \frac{S(t)}{\hat{Y}_d(t)},$$

and where K is given by

$$K = e^{-r_d T}\frac{S(0)}{Y(0)}. \tag{26}$$

From the Main Theorem we know that Z has zero drift under $\hat{Q}$, and a simple calculation shows that the $\hat{Q}$ dynamics of Z are given by:

$$dZ(t) = Z(t)\left(\sigma_S - \sigma_Y\right)d\hat{W}(t),$$

where $\hat{W}$ is $\hat{Q}$-Wiener. Thus the expectation in (25) is given by the Black-Scholes formula for a call, with strike price K, written on an asset with (scalar) volatility:

$$\delta_Z = \|\sigma_S - \sigma_Y\| = \sqrt{\|\sigma_S\|^2 + \|\sigma_Y\|^2 - 2\sigma_S\sigma_Y^\star} = \sqrt{\delta_S^2 + \delta_Y^2 + 2\rho\delta_S\delta_Y}$$

in a world with zero interest rate. We thus obtain the pricing formula:

$$\begin{aligned}
\Pi(t; \Phi_p) &= \hat{Y}_d(t)\left\{Z(t)N[d_1] - KN[d_2]\right\} \\
d_1 &= \frac{1}{\delta_Z\sqrt{T-t}}\left\{\ln\left(\frac{Z(t)}{K}\right) + \frac{1}{2}\delta_Z^2(T-t)\right\}, \\
d_2 &= d_1 - \delta_Z\sqrt{T-t}
\end{aligned}$$

After simplification, this reduces to a pricing formula that of course coincides with (24):

$$C_p(t) = \Pi(t; \Phi_p) = S(t)N[d_1] - Y(t)e^{-r_d(T-t)}\frac{S(0)}{Y(0)}N[d_2], \quad (27)$$

where

$$\begin{aligned} d_1 &= \frac{1}{\delta_Z\sqrt{T-t}}\left\{\ln\left(\frac{S(t)Y(0)}{Y(t)S(0)}\right) + \left\{r_d + \frac{1}{2}\delta_Z^2\right\}(T-t)\right\} \\ d_2 &= d_1 - \delta_Z\sqrt{T-t}, \\ \delta_Z &= \sqrt{\delta_S^2 + \delta_Y^2 + 2\rho\delta_S\delta_Y}. \end{aligned}$$

We have thus seen that there are two distinct (but logically equivalent) ways of pricing the option. From the computations, it is also clear (ex post) that the easiest way is to use the dollar bank account as the numeraire, rather than the pound value of the same account.

IV. PRICING CONVERTIBLE BONDS

Standard pricing models of convertible bonds concentrate on pricing the bond and its conversion option at date $t = 0$ (see, for example, Brennan and Schwartz [1977] and Bardhan et al. [1993]). A somewhat less-standard problem is the pricing of the bond at some date $0 < t < T$, where T is the maturity date of the bond.

For this problem, again we see that the numeraire approach gives a relatively simple solution. The trick is to use the stock price as the numeraire. This gives a relatively simple pricing formula for the bond.

Problem

A convertible bond involves two underlying objects: a discount bond and a stock. The more precise assumptions are as follows:

- The bond is a zero-coupon bond with face value of 1.
- The bond matures at a fixed date T_1.
- The underlying stock pays no dividends.
- At a fixed date T_0, with $T_0 < T_1$, the bond can be converted to one share of the stock.

The problem is of course to price the convertible bond at time $t < T_0$.

Mathematical Model

We use notation as follows:

$S(t)$ = the price, at time t, of the stock; and
$p(t, T)$ = the price, at time t, of a zero-coupon bond of the same risk class.

We now view the convertible bond as a contingent claim Y with exercise date T_0. The claim Y is thus given by the expression:

$$Y = \max[S(T_0), p(T_0, T_1)].$$

To price this claim, we have two obvious possibilities. We can use either the stock or the zero-coupon bond maturing at T_1 as the numeraire. Assuming that the T_1 bond actually is traded, we immediately obtain the price as:

$$\Pi(t; Y) = p(t, T_1)E_t^1[\max\{Z(T_0), 1\}],$$

where E^1 denotes the expectation under the forward-neutral martingale measure Q^1 with the T_1 bond as numeraire. The process Z is defined by:

$$Z(t) = \frac{S(t)}{p(t, T_1)}.$$

We can now simplify and write

$$\max\{Z(T_0), 1\} = \max\{Z(T_0) - 1, 0\} + 1,$$

giving us

$$\Pi(t; Y) = p(t, T_1)E_t^1[\max\{Z(T_0) - 1, 0\}] + p(t, T_1) \quad (28)$$

In words, this just says that the price of the convertible bond equals the price of a conversion option plus the price of the underlying zero-coupon bond. Since we assumed that the T_1 bond is traded, we do not have to compute the price $p(t, T_1)$ in Equation (28), but instead we simply observe the price in the market. It thus remains only to compute the expectation, which is obviously the price, at time t, of a European call with strike price 1 on the price process Z in a world where the short rate equals zero. Thus the numeraire approach considerably simplifies the computational problem.

To obtain more explicit results, we can make more specific assumptions about the stock and bond price dynamics.

Assumption. Define, as usual, the forward rates by $f(t, T) = -\partial \ln p(t, T)/\partial T$. We now make the assumptions, all under the risk-neutral martingale measure Q:

- The bond market can be described by a Heath-Jarrow-Morton [1992] model for forward rates of the form:

$$df(t,T) = \left(\sigma_f(t,T) \int_t^T \sigma_f'(t,u)du \right) dt + \sigma_f(t,T)dW(t)$$

  where the volatility structure $\sigma_f(t, T)$ is assumed to be deterministic. W is a (possibly multidimensional) Q-Wiener process.
- The stock price follows geometric Brownian motion:

$$dS(t) = r(t)S(t)dt + S(t)\sigma_S dW(t),$$

  where $r_t = f(t, t)$ is the short rate. The row vector σ_S is assumed to be constant and deterministic.

In essence, we have thus assumed a standard Black-Scholes model for the stock price S_1, and a Gaussian forward rate model. The point is that this will lead to a lognormal distribution for Z, thus allowing us to use a standard Black-Scholes formula.

From the forward rate dynamics above, it now follows that we have bond price dynamics given by (Björk [1999, prop. 15.5]):

$$dp(t,T) = r(t)p(t,T)dt - p(t,T)\Sigma_p(t,T)dW(t),$$

where the bond price volatility is given by:

$$\Sigma_p(t,T) = \int_t^T \sigma_f(t,u)du.$$

We may now attack the expectation in (28), and to this end we compute the Z dynamics under Q^{T_1}. It follows directly from the Itô formula that the Q dynamics of Z are given by:

$$dZ(t) = Z(t)\alpha_Z(t)dt + Z_t \left\{ \sigma_S + \Sigma_p(t,T_1) \right\} dW(t)$$

where for the moment we do not bother about the drift process α_Z.

We know from the general theory that the following hold:

- The Z process is a Q^1 martingale (i.e., zero drift term).
- The volatility does not change when we change measure from Q to Q^1.

The Q^1 dynamics of Z are thus given by:

$$dZ(t) = Z(t)\sigma_Z(t)dW^1(t) \tag{29}$$

where

$$\sigma_Z(t) = \sigma_S + \Sigma_p(t,T_1), \tag{30}$$

and where W^1 is Q^1-Wiener.

Under these assumptions, the volatility σ_Z is deterministic, thus guaranteeing that Z has a lognormal distribution. We can in fact write:

$$dZ(t) = Z(t) \left\| \sigma_Z(t) \right\| dV^1(t),$$

where V^1 is a scalar Q^1 Wiener process. We may thus use a small variation of the Black-Scholes formula to obtain the final pricing result

Proposition. The price at time t of the convertible bond is given by the formula:

$$\Pi(t;Y) = S(t)N[d_1] - p(t,T_1)N[d_2] + p(t,T_1),$$

where

$$
\begin{aligned}
d_1 &= \frac{1}{\sqrt{\sigma^2(t,T_0)}}\left\{\ln\left(\frac{S(t)}{p(t,T_1)}\right)+\frac{1}{2}\sigma^2(t,T_0)\right\} \\
d_2 &= d_1-\sqrt{\sigma^2(t,T_0)}, \\
\sigma^2(t,T_0) &= \int_t^{T_0}\left\|\sigma_Z(u)\right\|^2 du, \\
\sigma_Z(t) &= \sigma_S+\int_t^{T_1}\sigma_f(t,s)ds
\end{aligned}
$$

V. PRICING SAVINGS PLANS WITH CHOICE OF INDEXING

Savings plans that offer the saver some choice about the measure affecting rates are common. Typically they give savers an ex post choice of the interest rate to be paid on their account. With the inception of capital requirements, many financial institutions have to recognize these options and price them.

Problem

We use as an example a bank account in Israel. This account gives savers the ex post choice of indexing their savings to an Israeli shekel interest rate or a U.S. dollar rate.

- The saver deposits NIS 100 (Israeli shekels) today in a shekel/dollar savings account with a maturity of one year.
- In one year, the account pays the maximum of:

  - The sum of NIS 100 + real shekel interest, the whole amount indexed to the inflation rate; or
  - Today's dollar equivalent of NIS 100 + dollar interest, the whole amount indexed to the dollar exchange rate.

The savings plan is thus an option to exchange the Israeli interest rate for the U.S. interest rate, while at the same time taking on exchange rate risk. Since the choice is made ex post, it is clear that both the shekel and the dollar interest rates offered on such an account must be below the respective market rates.

Mathematical Model

We consider two economies, one domestic and one foreign, using notation as follows:

r_d = domestic short rate;
r_f = foreign short rate;
$I(t)$ = domestic inflation process;
$X(t)$ = the exchange rate in terms of domestic currency/foreign currency;
$Y(t)$ = the exchange rate in terms of foreign currency/domestic currency; and
T = the maturity date of the savings plan.

The value of the option is linear in the initial shekel amount invested in the savings plan; without loss of generality, we assume that this amount is one shekel. In the domestic currency, the contingent T-claim Φ_d to be priced is thus given by:

$$\Phi_d=\max\left[e^{r_dT}I(T),\ X(0)^{-1}e^{r_fT}X(T)\right]$$

In the foreign currency, the claim Φ_f is given by:

$$\Phi_f=\max\left[e^{r_dT}I(T)Y(T),\ Y(0)e^{r_fT}\right]$$

It turns out that it is easier to work with Φ_f than with Φ_d, and we have

$$\Phi_f=\max\left[e^{r_dT}I(T)Y(T)-\ Y(0)e^{r_fT},0\right]+Y(0)e^{r_fT}.$$

The price (in the foreign currency) at $t = 0$ of this claim is now given by

$$
\begin{aligned}
\Pi\left(0;\Phi_f\right) &= e^{-r_fT}E^{Q_f}\left[\max\left\{e^{r_dT}I(T)Y(T)-\right.\right. \\
&\qquad Z(t)\left(\sigma_Y+\sigma_I\right)dW(t). \\
&= E^{Q_f}\left[\max\left\{e^{(r_d-r_f)T}I(T)Y(T)-\right.\right. \\
&\qquad \left.\left.Y(0),0\right\}\right]+Y(0),
\end{aligned}
\tag{31}
$$

where Q_f denotes the risk-neutral martingale measure for the foreign market.

At this point, we have to make some probabilistic assumptions. We assume that we have a Garman-Kohlhagen [1983] model for Y. Standard theory then gives us the Q_f dynamics of Y as

$$dY(t) = Y(t)(r_f - r_d)dt + Y(t)\sigma_Y dW(t). \tag{32}$$

For simplicity we assume also that the price level follows a geometric Brownian motion, with Q_f dynamics given by

$$dI(t) = I(t)\alpha_I dt + I(t)\sigma_I dW(t). \tag{33}$$

Note that W is assumed to be two-dimensional, thus allowing for correlation between Y and I. Also note that economic theory does not say anything about the mean inflation rate α_I under Q_f.

When we compute the expectation in Equation (31), we cannot use a standard change of numeraire technique, because none of the processes Y, I, or YI are price processes of traded assets without dividends. Instead we have to attack the expectation directly.

To that end we define the process Z as $Z(t) = Y(t)$ $I(t)$, and obtain the Q_f dynamics:

$$\begin{aligned} dZ(t) = {} & Z(t)\left(r_f - r_d + \alpha_I + \sigma_Y\sigma_I'\right)dt + \\ & Z(t)\left(\sigma_Y + \sigma_I\right)dW(t). \end{aligned}$$

From this it is easy to see that if we define $S(t)$ by

$$S(t) = e^{-(r_f - r_d + \alpha_I + \sigma_Y\sigma_I')t} Z(t),$$

then we will have the Q_f dynamics

$$dS(t) = S(t)\left(\sigma_Y + \sigma_I\right)dW(t),$$

Thus, we can interpret $S(t)$ as a stock price in a Black-Scholes world with zero short rate and Q_f as the risk-neutral measure. With this notation we easily obtain:

$$\Pi\left(0;\Phi_f\right) = e^{cT}E^{Q_f}\left[\max\left[S(T) - e^{-cT}Y(0), 0\right]\right] + Y(0),$$

where

$$c = \alpha_I + \sigma_Y\sigma_I'.$$

The expectation can now be expressed by the Black-Scholes formula for a call option with strike price $e^{-cT}Y(0)$, zero short rate, and volatility given by:

$$\sigma = \sqrt{\|\sigma_Y\|^2 + \|\sigma_I\|^2 + 2\sigma_Y\sigma_I'}$$

The price at $t = 0$ of the claim expressed in the foreign currency is thus given by the formula:

$$\begin{aligned} \Pi\left(0;\Phi_f\right) &= e^{cT}I(0)Y(0)N[d_1] - Y(0)N[d_2] + Y(0), \\ d_1 &= \frac{\ln\left(I(0)\right) + \left(c + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}, \\ d_2 &= d_1 - \sigma\sqrt{T}. \end{aligned} \tag{34}$$

Finally, the price at $t = 0$ in domestic terms is given by

$$\begin{aligned} \Pi(0;\ \Phi_d) &= X(0)\Pi(0;\ \Phi_f) \\ &= e^{cT}I(0)N[d_2] + 1 \end{aligned} \tag{35}$$

For practical purposes it may be more convenient to model Y and I as

$$\begin{aligned} dY(t) &= Y(t)(r_f - r_d)dt + Y(t)\sigma_Y dW^Y(t), \\ dI(t) &= I(t)\alpha_I dt + I(t)\sigma_I dW^I(t), \end{aligned}$$

where now σ_Y and σ_Y are constant scalars, while W^Y and W^I are scalar Wiener processes with local correlation given by $dW^Y(t)dW^I(t) = \rho dt$.

In this model (which of course is logically equivalent to the one above), we have the pricing formulas (34)-(35), but now with the notation

$$\begin{aligned} c &= \alpha_I + \rho\sigma_Y\sigma_I, \\ \sigma &= \sqrt{\sigma_Y^2 + \sigma_I^2 + 2\rho\sigma_Y\sigma_I} \end{aligned}$$

VI. ENDOWMENT WARRANTS

Endowment options, which are primarily traded in Australia and New Zealand, are very long-term call options on equity. These options are discussed by Hoang, Powell, and Shi [1999] (henceforth HPS).

Endowment warrants have two unusual features. Their dividend protection consists of adjustments to the strike price, and the strike price behaves like a money market fund (i.e., increases over time at the short-term interest rate). HPS assume that the dividend adjustment to the strike price is equivalent to the usual dividend adjustment to the stock price; this assumption is now known to be mistaken.[3] Under this assumption they obtain an arbitrage-free warrant price where the short rate is deterministic, and provide an approximation of the option price under stochastic interest rate.

We discuss a *pseudo-endowment option*. This pseudo-endowment option is like the Australian option, except that its dividend protection is the usual adjustment to the stock price (i.e., the stock price is raised by the dividends). The pseudo-endowment option thus depends on two sources of uncertainty: the (dividend-adjusted) stock price and the short-term interest rate.

With a numeraire approach, we can eliminate one of these sources of risk. Choosing an interest rate-related instrument (i.e., a money market account) as a numeraire results in a pricing formula for the pseudo-endowment option that is similar to the standard Black-Scholes formula.[4]

Problem

A pseudo-endowment option is a very long-term call option. Typically the characteristics are as follows:

- At issue, the initial strike price $K(0)$ is set to approximately 50% of the current stock price, so the option is initially deep in the money.
- The endowment options are European.
- The time to exercise is typically ten-plus years.
- The options are interest rate and dividend protected. The protection is achieved by two adjustments:
  - The strike price is not fixed over time. Instead it grows at the short-term interest rate.
  - The stock price is increased by the amount of the dividend each time a dividend is paid.
- The payoff at the exercise date T is that of a standard call option, but with the adjusted (as above) strike price $K(T)$.

Mathematical Model

We model the underlying stock price process $S(t)$ in a standard Black-Scholes setting. In other words, under the objective probability measure P, the price process $S(t)$ follows geometric Brownian motion (between dividends) as:

$$dS(t) = \alpha S(t)dt + S(t)\sigma W^p(t),$$

where α and σ are deterministic constants, and W^p is a P Wiener process. We allow the short rate r to be an arbitrary random process, thus giving the P dynamics of the money market account as:

$$dB(t) = r(t)B(t)dt, \tag{36}$$

$$B(0) = 1. \tag{37}$$

To analyze this option, we have to formalize the protection features of the option as follows.

- We assume that the strike price process $K(t)$ is changed at the continuously compounded instantaneous interest rate. The formal model is thus as follows:

$$dK(t) = r(t)K(t)dt. \tag{38}$$

- For simplicity, we assume that the dividend protection is perfect. More precisely, we assume that the dividend protection is obtained by reinvesting the dividends in the stock itself. Under this assumption, we can view the stock price as the theoretical price of a mutual fund that includes all dividends invested in the stock. Formally this implies that we can treat the stock price process $S(t)$ as the price process of a stock without dividends.

The value of the option at the exercise date T is given by the contingent claim Y, defined by

$$Y = \max\left[S(T) - K(T), 0\right]$$

Clearly there are two sources of risk in endowment options: stock price risk, and the risk of the short-term interest rate. In order to analyze this option, we observe that from (36)-(38) it follows that:

$$K(T) = K(0)B(T).$$

Thus we can express the claim Y as:

$$Y = \max\left[S(T) - K(0)B(T), 0\right]$$

From this expression, we see that the natural numeraire process is now obviously the money account $B(t)$. The martingale measure for this numeraire is the standard risk-neutral martingale measure Q under which we have the stock price dynamics:

$$dS(t) = r(t)S(t)dt + S(t)\sigma dW(t), \tag{39}$$

where W is a Q-Wiener process.

A direct application of the Main Theorem gives us the pricing formula:

$$\Pi(0; Y) = B(0)E^Q\left[\frac{1}{B(T)}\max\left[S(T) - K(0)B(T), 0\right]\right]$$

After a simple algebraic manipulation, and using the fact that $B(0) = 1$, we thus obtain

$$\Pi(0; Y) = E^Q\left[\max\left[Z(T) - K(0), 0\right]\right] \tag{40}$$

where $Z(t) = S(t)/B(t)$ is the normalized stock price process. It follows immediately from (36), (39), and the Itô formula that under Q we have Z dynamics given by

$$dZ(t) = Z(t)\sigma dW(t). \tag{41}$$

and from (40)-(41), we now see that our original pricing problem has been reduced to computing the price of a standard European call, with strike price $K(0)$, on an underlying stock with volatility σ in a world where the short rate is zero.

Thus the Black-Scholes formula gives the endowment warrant price at $t = 0$ directly as:

$$C_{EW} = \Pi(0; Y) = S_0 N(d_1) - K_0 N(d_2) \tag{42}$$

where

$$\begin{aligned} d_1 &= \frac{\ln\left(S(0)/K(0)\right) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}, \\ d_2 &= d_1 - \sigma\sqrt{T}. \end{aligned}$$

Using the numeraire approach, the price of the endowment option in (42) is given by a standard Black-Scholes formula for the case where $r = 0$. The result does not in any way depend upon assumptions made about the stochastic short rate process $r(t)$.

The pricing formula (42) is in fact derived in HPS [1999] but only for the case of a deterministic short rate. The case of a stochastic short rate is not treated in detail. Instead HPS attempt to include the effect of a stochastic interest rate as follows:

- They assume that the short rate r is deterministic and constant.
- The strike price process is assumed to have dynamics of the form

$$dK(t) = rK(t)dt + \gamma dV(t)$$

  where V is a new Wiener process (possibly correlated with W).

- They value the claim $Y = \max[S(T) - K(T), 0]$ by using the Margrabe [1978] result about exchange options.

HPS claim that this setup is an approximation of the case of a stochastic interest rate. Whether it is a good approximation or not is never clarified, and from our analysis above we can see that the entire scheme is in fact unnecessary, since the pricing formula in Equation (42) is invariant under the introduction of a stochastic short rate.

Note that our result relies upon our simplifying assumption about perfect dividend protection. A more realistic modeling of the dividend protection would lead to severe computation problems. To see this, assume that the stock pays a constant dividend yield rate δ. This would change our model in two ways. The Q dynamics of the stock price would be different, and the dynamics of the strike process $K(t)$ would have to be changed.

As for the Q dynamics of the stock price, standard theory immediately gives us:

$$dS(t) = \left(r(t) - \delta\right)S(t)dt + S(t)\sigma dW(t).$$

Furthermore, from the problem description, we see that in real life (rather than in a simplified model), dividend protection is obtained by reducing the strike price process by the dividend amount at every dividend payment. In terms of our model, this means that over an infinitesimal interval $[t, t + dt]$, the strike price should drop by the amount $\delta S(t)dt$.

Thus the K dynamics are given by

$$dK(t) = [r(t)K(t) - \delta S(t)]\,dt.$$

This equation can be solved as:

$$K(T) = e^{\int_0^T r(t)dt} K(0) - \delta \int_0^T e^{\int_t^T r(u)du} S(t)dt$$

The moral is that in the expression of the contingent claim

$$Y = \max\,[S(T) - K(T), 0]$$

we now have the awkward integral expression

$$\int_0^T e^{\int_t^T r(u)du} S(t)dt.$$

Even in the simple case of a deterministic short rate, this integral is quite problematic. It is then basically a sum of lognormally distributed random variables, and thus we have the same difficult computation problems for Asian options.

VII. CONCLUSION

Numeraire methods have been in the derivatives pricing literature since Geman [1989] and Jamshidian [1989]. These methods afford a considerable simplification in the pricing of many complex options, but they appear not to be well known.

We have considered five problems whose computation is vastly aided by the use of numeraire methods. We discuss the pricing of employee stock ownership plans; the pricing of options whose strike price is denominated in a currency different from that of the underlying stock; the pricing of convertible bonds; the pricing of savings plans where the choice of interest paid is ex post chosen by the saver; and finally, the pricing of endowment options.

Numeraire methods are not a cure-all for complex option pricing, although when there are several risk factors that impact an option's price, choosing one of the factors as a numeraire reduces the dimensionality of the computation problem by one. Smart choice of the numeraire can, in addition, produce significant computational simplification in an option's pricing.

ENDNOTES

Benninga thanks the Israel Institute of Business Research at Tel-Aviv University for a grant. Wiener's research was funded by the Krueger Center at the Hebrew University of Jerusalem.

[1]Readers interested only in implementation of the numeraire method can skip over this section.

[2]For tax reasons, most executive stock options are initially at the money.

[3]Brown and Davis [2001].

[4]This formula was derived by HPS as a solution for the deterministic interest rate.

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