Standard Approaches to Asset & Liability Risk *

Boualem Djehiche
Department of Mathematics, KTH†

Per Hörfelt
Fraunhofer-Chalmers Research Center Industrial Mathematics‡

December 15, 2004

Abstract

We compare two different models for assets and liabilities for an insurance company that can be considered in the standard approach to solvency assessment and in particular, in determining the required target capital. The first model is suggested by a joint working party by members in CEA, Comité Européen des Assurances, and is based on the duration concept and the second one is based on ideas from Arbitrage Pricing Theory (APT). An application of these approaches to two specific insurance contracts indicates that, among other things, the duration-based approach to solvency assessment suggests larger target capital requirement than the one based on APT.

Acknowledgments: The authors would like to thank Christer Borell and Sture Holm, Chalmers University of Technology, Arne Sandström, Swedish Insurance Federation, and Christer Stolt, Länsförsäkringar, for valuable discussions and support.

*Supported by the Swedish Insurance Federation.
†SE-100 44 Stockholm. e-mail: boualem@math.kth.se
‡SE-412 88 Göteborg. e-mail: perh@fcc.chalmers.se

1 Introduction

The objective of this paper is to describe quantitative methods to assess the asset & liability risk in an insurance company. The International Actuarial Association (IAA) (2004)[20] suggested that a Standard Approach to solvency assessment should be a robust and simple method to determine a minimum level of capital that has to be maintained by an insurance company for one or several lines of business. In particular, the method should include a minimum number of parameters and be analytically tractable. Henceforth, this approach for solvency assessment will be referred to as the *standard approach*. In this report we will compare two different models for assets and liabilities that can be considered in the standard approach. One model is a joint work by members in CEA, Comité Européen des Assurances, and is based on the duration concept. This model will be referred to as Model A. The other model, henceforth referred to as Model B, is based on ideas from the powerful Arbitrage Pricing Theory and stochastic analysis.

We must underline that this paper will only focus on the asset & liability risk. This risk results from the uncertainty in market value of future cash flows from the insurers assets and liabilities. Of course, an insurance company is exposed to many other forms of risk such as mortality risk, surrender and lapses risk, reinsurance risk etc. (see [20] for a discussion on different types of insurance risk). However, it is possible to extend the models so that they include more risk sources than solely asset & liability risk, as we will show at the end of this paper.

The report is structured as follows. In Section 2 we give a mathematical formulation of the concepts solvency requirement and target capital. These concepts are closely related to risk measures. In Section 2 we will also describe three classes of risk measures. In Sections 3 and 4 we describe the processes of a typical asset portfolio for an insurance company, consisting of bonds, equity, and property (real estate), and the liabilities according to Models A and B. In Section 5 we derive analytical formulas that are useful to compute the target capital. In Section 6 we address the issue of calibration of the parameters included in the models. In Section 7 we apply the results to a D(10)-contract from life insurance and a vehicle insurance contract. In the final section, Section 8, we discuss how one may include other risk factors in the models.

2 Solvency Assessment

In this section we give a mathematical formulation of the concepts solvency requirement and target capital. These concepts are closely linked to risk measures that are discussed in the first part of this section.

2.1 Risk Measures

A risk measure is a function that maps random variables to real numbers. The random variable may for instance describe a future loss of a business line. The risk measure should reflect the risk associated with the random variable in the sense that its value increases with higher risk. This section is only meant as an introduction to the theory of risk measures. A comprehensive treatment can be found in Artzner et al. [2] and Delbean [9].

Among the large number of different risk measures that have been suggested in the literature, the Standard Deviation Principle, Value-at-Risk, and Expected Shortfall (also known as the Tail VaR) are the most widely used by practitioners and also discussed by CEA and IAA. In the sequel, we will briefly introduce these measures and mention some of their properties. Let X and Y denote random variables describing a future loss of a business line. The Standard Deviation Principle, SDP_δ, is defined as

$$\mathrm{SDP}_\delta(X) = E\big[X\big] + \delta\sqrt{\mathrm{Var}\big[X\big]}, \tag{2.1}$$

where δ is a positive number. This measure suffers from the fact that it is not monotone increasing in the sense that if $X \leq Y$ then it is not necessarily true that $\mathrm{SDP}_\delta(X)$ is smaller than $\mathrm{SDP}_\delta(Y)$. Moreover, there is no obvious choice of δ (see Delbaen [9] for a further discussion). We will discuss the value of δ in more details below. Another very popular example of risk measure is Value-at-Risk, usually denoted by VaR or VaR_α, and it is defined as

$$\mathrm{VaR}_\alpha(X) = \inf\{x \in \mathbb{R} \,:\, P(X > x) \leq \alpha\}, \tag{2.2}$$

where $0 < \alpha < 1$. It describes the highest value of the loss for more than $100(1-\alpha)$ % of all outcomes. Typically α equals 5, 1, or 0.5 %. VaR only measures the probability for loss and not its magnitude. Moreover, the measure is not in general sub-additive, a risk measure ρ is sub-additive if $\rho(X+Y) \leq \rho(X) + \rho(Y)$. It is natural that a risk measure should be sub-additive since the overall risk measure of a portfolio including several lines of business should be less than the sum of the risk measures of the corresponding individual lines of business simply because the correlation between the business line acts as a diversifier of risk. However, compared to other risk measures, Value-at-Risk is analytically tractable, verifiable, and easy to communicate. The third example of a risk measure that has got a lot of attention in recent years is Expected Shortfall or Tail VaR. This risk measure is defined as

$$\mathrm{ES}_\alpha(X) = E\big[X \,|\, X > \mathrm{VaR}_\alpha(X)\big], \tag{2.3}$$

where $0 < \alpha < 1$ (see Artzner et al. [2] for further details). In the IAA-report [20] it is recommended to use expected shortfall, at least in more advanced models for solvency assessment. However, in many cases it is not possible to compute this measure analytically and therefore it is not useful as a standard approach.

In this report we will focus on two risk measures. The first one is the Standard Deviation Principle suggested in the standard approach as described in Rantala [29][1] and Sandström [34]. The second one is Value-at-Risk suggested in Model B. The value of δ in the definition of Standard Deviation Principle will, unless stated otherwise, be set to

$$\delta = \Phi^{-1}(0.99) \approx 2.33, \tag{2.4}$$

[1]Rantala actually uses Value at Risk. However, after some approximations he obtains a risk measure identical with the Standard Deviation Principle.

where Φ^{-1} is the inverse of standard normal distribution. This is motivated, see Rantala [29], by the fact that if X is normal distributed then

$$\mathrm{SDP}_\delta(X) = \mathrm{VaR}_\alpha(X) \tag{2.5}$$

if and only if $\delta = \Phi^{-1}(1-\alpha)$. If not stated otherwise, in Model B we put $\alpha = 1\%$ in the definition of Value-at-Risk

2.2 Solvency Requirement and Target Capital

Let $A(t)$ denote the value at time t of the assets in an insurance company and suppose $V(t)$ denotes a best estimate at time t of the liability. Think of A and V as stochastic processes. Moreover, suppose $t = 0$ means today and let $T > 0$ denote a fixed future date. Finally, let ρ be a given risk measure.

The so called *solvency requirement* is defined as the requirement that the difference between the liability and the asset at time T should have a negative risk measure:

$$\rho\big(V(T) - A(T)\big) \leq 0. \tag{2.6}$$

The initial value $A(0)$ of the assets that satisfies $\rho\big(V(T) - A(T)\big) = 0$ is referred to as the *target capital* and plays an important role in solvency assessment (see Sandström [34] for a further discussion about the target capital). Henceforth the quantity $\rho\big(V(T) - A(T)\big)$ will be referred to as the asset & liability risk.

The value of T in the definition of the asset & liability risk varies. In [20] it is suggested that T equals one year. Another alternative is to put T equal to a number that better reflects the payment date of the liability, for instance the duration of the liability. In the examples in Section 7 we will see that the value of T may have a great affect on the asset & liability risk.

In the next section we review some models of the assets. The liability modeling will be discussed in Section 4.

3 Assets

A typical portfolio of assets of an insurance company includes the classical asset classes bonds, equity and property. In this section we review models of these assets, as suggested by Model A and Model B respectively. Model A is based on the duration concept. Duration measures the sensitivity of an asset to changes in the interest rates. The concept of duration was first developed by Macaulay [26] and is frequently used in asset & liability management. The valuation suggested in Model B relies on ideas that goes back to the seminal work by Samuelson [33] and Vasicek [35]. An obvious reason of choosing these valuation models among others heavily lies on their simplicity, in the sense that they consist of few parameters and are analytically tractable, verifiable and easy to communicate, which is desirable for a standard model.

In Section 3.1 we discuss interest rates and bond portfolios. Section 3.2 considers equities and Section 3.3 describes valuation methods of property (real estate) as asset class.

3.1 Interest Rate and Bond Portfolios

To begin with, we introduce some notation that will be used throughout the rest of this report. Suppose $B(t)$ denotes the value at time t of a portfolio consisting of default–free bonds, typically government bonds. Without loss of generality, we may assume that the portfolio consists solely of zero-coupon bonds. Next, define the duration $d_B(t)$ at time t of the bond portfolio as

$$d_B(t) = \frac{1}{B(t)} \sum_k (\tau_k - t)\, \pi_k(t) p(t, \tau_k), \tag{3.1}$$

where $p(t, \tau)$, $t \leq \tau$, is the price at time t of a zero-coupon bond with maturity τ and $\pi_k(t)$ is the number of bonds with maturity τ_k held in the portfolio at time t. Thus, duration can either be seen as the negative log-derivative of the value of a bond portfolio with respect to a fix interest rate or as a measure of how long, on average, the holder of the portfolio has to wait before receiving the cash payment.

3.1.1 Model A

Suppose $\Delta B(t) = B(t) - B(0)$, $t > 0$, is the gain of the bond portfolio in $[0, t]$. In [29] and [30] Rantala suggests that[2]

$$\frac{\Delta B(T)}{B(0)} = I_B - \frac{d_B(0)}{1 + \bar{i}(0)}\, \Delta\bar{i}(T), \tag{3.2}$$

where I_B is the expected income on bonds with the present interest rate levels, $d_B(0)$ is the duration of the bond portfolio, and $\Delta\bar{i}(T) = \bar{i}(T) - \bar{i}(0)$ is the change over the time period $[0, T]$ for an annual compounded 'flat' and risk free interest rate $\bar{i}$. A flat interest rate should be understood as a constant approximation of the yield curve. Rantala ([29],[30]) gives no further details about this approximation. Moreover, he assumes that $\Delta\bar{i}(T)$ is a random variable with mean zero and standard deviation ς_i. This model includes three main approximations; a simple model for the interest rate change, no reinvestment risk, and a first order Taylor approximation in the Macaulay approximation.

3.1.2 Model B

In Model B the valuation of a bond is based on an Arbitrage Pricing argument (see e.g. Brigo et al. [5]). Assume that the risk free instantaneous spot rate $\{i(t)\}_{0 \leq t \leq T}$ is a mean-reverting Gaussian process (Vasicek model) i.e. solves the stochastic differential equation

$$di(t) = \kappa\big(\theta - i(t)\big)dt + \sigma_i dW_i(t) \tag{3.3}$$

where κ, θ, and σ_i are positive constants and $\{W_i(t)\}_{0 \leq t \leq T}$ is a Brownian motion. The solution of Eq. (3.3) equals

$$i(t) = \theta + (i(0) - \theta)e^{-\kappa t} + \sigma_i \int_0^t e^{-\kappa(t-s)} dW_i(s). \tag{3.4}$$

[2] Rantala has a plus sign instead of a minus sign, i.e. $I_B + \frac{d_B(0)}{1+i} \cdots$ instead of $I_B - \frac{d_B(0)}{1+i} \cdots$. However, our formulation is closer to Macauley's formulation.

The Gaussian process $\{i(t)\}_{0\leq t\leq T}$ has mean

$$\alpha_i(t) = \theta + (i(0) - \theta)e^{-\kappa t} \tag{3.5}$$

and covariance

$$\gamma_i(s,t) = \frac{\sigma_i^2}{2\kappa}\big(e^{-\kappa|t-s|} - e^{-\kappa(t+s)}\big). \tag{3.6}$$

The constant θ is the long term average of the short interest rate in the sense that

$$\lim_{t\to\infty} E\big[i(t)\big] = \theta. \tag{3.7}$$

This property can also be gathered from Eq. (3.3), since the drift term $\kappa(\theta - i(t))$ is positive if $i(t) < \theta$ and negative if $i(t) > \theta$. The parameter θ can also be viewed as the interest rate equilibrium. Moreover, the quotient $\sigma_i^2/2\kappa$ can be seen as the lowest upper bound for the variance of the spot rate. The constants σ_i, κ, and θ can be estimated from historical values of the spot rate. Alternatively, the parameters σ_i and κ can be assessed by fitting the theoretical yield curve given by the model with the market yield curve, see Section 6 for further details.

The price at time t of a zero coupon bond $p(t,\tau)$ with maturity date τ is then given by

$$p(t,\tau) = E^Q\big[\exp\big(-\int_t^\tau i(s)ds\big)\big|\,\mathcal{F}_t\big], \quad t\leq\tau, \tag{3.8}$$

where, $(\mathcal{F}_t,\, 0\leq t\leq T)$ denotes the filtration generated by $\{W_i(t)\}_{0\leq t\leq T}$ i.e. the information in inflow is carried out by the driving Brownian motion, and Q is the martingale measure of the market. The Q-dynamics of $\{i(t)\}_{0\leq t\leq T}$ is given by

$$di(t) = \kappa\big(\tilde\theta(t) - i(t)\big)dt + \sigma_i dW_i^Q(t), \tag{3.9}$$

where $\tilde\theta(t)$ is some (random) function related to the choice of the probability measure Q and $\{W_i^Q\}_{0\leq t\leq T}$ is a Q–Brownian motion. Assuming that $\tilde\theta(t)$ is a (non-random) constant with $\tilde\theta(t) = \tilde\theta$ for all t, $\{i(t)\}_{0\leq t\leq T}$ will be a Gaussian process under Q as well. Thus, the distribution of $p(t,\tau)$ is log-normal. To be more specific, the zero coupon bond price is given by

$$p(t,\tau) = a(t,\tau)e^{-b(t,\tau)i(t)} \tag{3.10}$$

where

$$\begin{aligned} b(t,\tau) &= \frac{1}{\kappa}\big(1 - e^{-\kappa(\tau-t)}\big), \\ a(t,\tau) &= \exp\bigg\{\Big(\tilde\theta - \frac{\sigma_i^2}{2\kappa^2}\Big)(b(t,\tau) - \tau + t) - \frac{\sigma_i^2}{4\kappa}b(t,\tau)^2\bigg\}, \end{aligned} \tag{3.11}$$

see e.g. Brigo et al. [5]. In particular, the parameter $\tilde\theta$ may be estimated by fitting the theoretical yield curve with the observed yield curve using for e.g. a least square method, see Section 6 for further details.

The Vasicek model has one obvious drawback, the short rate is not necessarily positive or even bounded from below. However, the probability for the interest rate $i(t), 0\leq t\leq T$, to fall below zero is in practice close to zero, at least if the value of T is small or moderate. Moreover, the interest rate is probably not a diffusion process as in the Vasicek model. Indeed, recent empirical studies indicate that the interest rate has a jump component–see Zhou [38] and the references therein. However, the hypothesis that interest rates are mean reverting is, according to Fama et al. [10], prominent in old and new models of the term structure and has been empirically supported in e.g. Fama et al. [10]. For further details on interest theory in general and the Vasicek model in particular, see Brigo et al. [5].

The value of a bond portfolio depends on the chosen trading strategy. One simple strategy is to perform a perfect matching i.e. to purchase bonds, if available in the market, with a duration equal to the duration of the liabilities (cf. Section 4). If d_V denote the duration of the liabilities then the value at time $T \leq d_V$ of this strategy, from now on denoted $B'(T)$, can be approximated by

$$B'(T) = B(0)\frac{p(T, d_V)}{p(0, d_V)}. \tag{3.12}$$

That is, at time $t = 0$ the amount $B(0)$ is invested in zero coupon bonds with maturity d_V. This hedge of interest rate risk is a simplification of the immunization principle, first introduced in life insurance by Redington, see [31].

Another widely used strategy in the market is to re-balance the portfolio after each (often small) time interval Δt so that the duration in the bond portfolio is kept at a constant level. The value of this strategy at time T, henceforth denoted $B''(T)$, is easily seen to be equal

$$B''(T) = B(0) \prod_{k=0}^{n_0-1} \frac{p((k+1)\Delta t, d_B + k\Delta t)}{p(k\Delta t, d_B + k\Delta t)} \tag{3.13}$$

where d_B is the (constant) duration of the portfolio and n_0 is an integer such that $n_0\Delta t = T$.

The expression of $B''(T)$ can be simplified by letting $\Delta t \to 0$. Note that

$$\begin{aligned} B''(T) =& \frac{B(0)\, p(T, d_B + T - \Delta t)}{p(0, d_B)} \\ & \times \exp\Big(\sum_{k=1}^{n_0-1} \ln p\big(k\Delta t, d_B + (k-1)\Delta t\big) - \ln p\big(k\Delta t, d_B + k\Delta t\big)\Big). \end{aligned} \tag{3.14}$$

If f denotes the instantaneous forward rate with maturity τ, i.e.

$$f(t, \tau) = -\frac{\partial}{\partial \tau} \ln p(t, \tau), \tag{3.15}$$

then, by letting $\Delta t \to 0$ and simultaneously $n_0 \to \infty$, we get the price process of the bond portfolio in Model B:

$$B(T) = B(0) \exp\Big(\int_0^T f(t, d_B + t)dt\Big) \frac{p(T, d_B + T)}{p(0, d_B)}. \tag{3.16}$$

The random variables in Eq.(3.13) or (3.16) are log-normally distributed. The relevant parameters, the mean and the variance, are easily computed. One may interpret Eq.(3.16) as the value of a bank account on the forward rate times a factor $p(T, d_B + T)/p(0, d_B)$ that reflects the reinvestment risk. Note as well that if $d_B = 0$ then

$$B(T) = B(0)e^{\int_0^T i(t)dt}.$$

That is, $\{B(t)\}_{0\le t\le T}$ with $d_B = 0$ describes the value process of a bank account.

We end this section by deriving the dynamics the value of the bond portfolio. Since $a(t, d_B + t)$ and $b(t, d_B + t)$ are indeed independent of t it follows that

$$\frac{dB(t)}{B(t)} = f(t, d_B + t)dt - b(t, d_B + t)di(t)$$

and thus

$$\frac{dB(t)}{B(t)} = \big[f(t, d_B + t) - \kappa b(t, d_B + t)\big(\theta - i(t)\big)\big]dt - \sigma_i b(t, d_B + t)dW_i(t).$$

The forward rate equals

$$f(t, d_B + t) = \big[\tilde{\theta} - \frac{\sigma_i^2}{2\kappa} b(t, d_B + t)\big]\kappa b(t, d_B + t) + e^{-\kappa d_B} i(t)$$

and therefore the drift in the bond process is given by

$$\begin{aligned} &f(t, d_B + t) - \kappa b(t, d_B + t)\big(\theta - i(t)\big) \\ &\qquad = i(t) + (\tilde{\theta} - \theta)\kappa b(t, d_B + t) - \frac{\sigma_i^2}{2}\, b^2(t, d_B + t). \end{aligned}$$

Note that $b(t, d_B + t) = \frac{1}{\kappa}(1 - \exp(-\kappa d_B))$. Thus, if we put

$$\mu_B = (\tilde{\theta} - \theta)(1 - e^{-\kappa d_B}) - \frac{\sigma_i^2}{2\kappa^2}\,(1 - e^{-\kappa d_B})^2$$

then

$$\frac{dB(t)}{B(t)} = (i(t) + \mu_B)dt - \frac{\sigma_i}{\kappa}(1 - e^{-\kappa d_B})dW_i(t). \tag{3.17}$$

3.2 Equity

Equity is probably the most profitable and most risky asset class. This section will consider two different models of the price process of an equity index. The first model, Model A, is proposed by members in Fédération Fraincaise des Sociétés d'Assurances, see [11], and further discussed in Sandström [34]. The other model, in Model B, is the Samuelson or Black-Scholes model.

In the sequel we let $S(t)$ denote the price at time t of equity in the asset portfolio. We assume that all dividends paid out by the equities are reinvested in the portfolio. Thus, we will not, as is common in more advanced models (e.g. Koivu et al. [21]) decompose equity in a volatile component and a dividend component.

3.2.1 Model A

In Model A (see [11]), the value of the equity is duration based. More precisely, if $\Delta S(t) = S(t) - S(0)$ denotes the increment of the value of equity in $[0, t]$ then it is suggested in [11] that

$$\frac{\Delta S(T)}{S(0)} = I_S - d_S(0)\Delta\bar{i}(T) + \epsilon_S, \tag{3.18}$$

where I_S is the expected return on equity with the present interest rate level, $d_S(t)$ is the modified duration at time t of the equity portfolio (see below), ϵ_S is a random variable with mean 0 and variance ς_S^2 that represents the change of the equity returns not explained by the interest rate, and $\Delta\bar{i}(T)$ is defined as in Section 3.1.1. The distribution of ϵ_S is not further specified. In the same way as for bonds, we specify the dependence of the expected income and residual on the underlying price. In particular, we assume that the variance of the residual increases linearly with the underlying price, as is common in many models, see e.g. Granger and Morgenstern [15] for a further discussion.

Formally, the modified duration $d_S(0)$ at time 0 is defined as

$$d_S = -\frac{1 + \bar{i}(0)}{S(0)} \frac{\partial S}{\partial \bar{i}}(0), \tag{3.19}$$

see Brown [6][3] or [11]. The duration can thus be seen as the elasticity of the equity portfolio with respect to the interest rate.

There are different ways to estimate the modified equity duration $d_S(0)$. According to [11], the duration as well as the other parameters in Eq. (3.18) can be estimated using regression analysis. This approach will be discussed in more details in Section 6. Alternatively, one may assess the duration using the Gordon-Shapiro model, also known as the dividend discounted model, that stipulates that the price of an equity is the discounted value of all future dividends, i.e.

$$S(t) = \sum_{k=1}^{\infty} \frac{\delta_k}{(1 + \bar{i})^{(t_k - t)}} 1_k(t) \tag{3.20}$$

where δ_k denotes the dividend amount paid out by the equity at time t_k and the function $1_k(t)$ is defined as

$$1_k(t) = \begin{cases} 1, & t < t_k, \\ 0, & \text{otherwise.} \end{cases} \tag{3.21}$$

If $\delta_k = \delta$ for all k where δ is a positive constant, then the duration d_S can be computed simply by differentiating Eq. (3.20) with respect to $\bar{i}$ and subsequently use Eq. (3.19). However, this approach is based on the unrealistic assumption that all the dividends are equal. The model may be improved by assuming that the dividends $\delta_k = \delta \exp(g t_k)$ for some factor g. However, it may be quite difficult to estimate g as well.

[3] Actually, Brown defines the duration based on the instantaneous spot rate instead of the "flat" interest rate

Different methods to assess the duration may give rise to totally different values. Empirical methods based on regression analysis give values on the duration for equity indexes ranging from two to six years whereas the method based on the Gordon-Shapiro model produce values on the duration ranging from 20 to 50 years, see Leibowitz [23] or Leibowitz et al. [24].

The duration based model in Eq.(3.18) have some other drawbacks besides the calibration of the parameters. The model is mainly inspired by ideas from Fixed Income. For bonds, duration and interest rate sensitivity are virtually synonymous. For equity, however, duration is only one of several factors describing risk (cf. Leibowitz et al. [24]). For a more comprehensive discussion about equity duration, see Boquist et al. [3] and Leibowitz et al. [24].

3.2.2 Model B

Model B assumes that the price process of an equity index is described by a geometric Brownian motion:

$$\frac{dS(t)}{S(t)} = (i(t) + \mu_S)dt + \sigma_S dW_S(t), \tag{3.22}$$

where the drift term μ_S and the volatility σ_S are held constant and $\{W_S(t)\}_{0 \leq t \leq T}$ is a Brownian motion. Moreover, assume (W_i, W_S) is a two-dimensional Brownian motion. The instantaneous spot rate i is defined as in Section 3.1.2. This model was introduced in Samuelson [33]. The stochastic differential equation has the explicit solution

$$S(t) = S(0)e^{\int_0^t i(s)ds + (\mu_S - \frac{1}{2}\sigma_S^2)t + \sigma_S W_S(t)}. \tag{3.23}$$

The parameters μ_S and σ_S can be estimated from historical observations (cf. Section 6) or be defined by experts. The parameter μ_S is often referred to as the risk premium and is the subject of extensive research, due to the difficulty to estimate it (see e.g. Merton [27] and Rogers [32]). A natural assumption is that μ_S is positive. For further details on the risk premium, see Arnott et al. [1].

Let γ_{iS} denote the correlation between the driving Brownian motions of $i(t)$ and $S(t)$:

$$\gamma_{iS} = \mathrm{Cor}[W_i(t), W_S(t)].$$

In Munk et al. [28] it is argued that the correlation γ_{iS} is mostly negative. That is, if the short rate i increases, then the equity price has initially a tendency to fall.

The Samuelson model can be improved in several directions such as considering stochastic volatility, fat tailed distributed returns, more sophisticated dependence structures between $i(t)$ and $S(t)$, and time-dependent risk premium function.

3.3 Property

Property, and then typically domestic property is a common asset class in an insurance portfolio. The general opinion about investing in this asset class relies on risk diversification and inflation hedge, see Hoesli et al. ([17]) and Liu et al. ([25]) for a general discussion. According to Koivu et al. ([21]), property resembles stocks in some ways. The return from property consists of large price fluctuations and a fairly stable cash income. However, in contrast to stocks, the cash income component forms the majority of the total return. It is important to remember that there is a distinct difference between the value of a property index and the real market value of the underlying property. The property index is mainly based on appraisals rather than market values. The appraisals make the index smoother than the true market value, see for e.g. Geltner [14] and Fischer et al. ([12]). In the literature there are several methods designed to recover the true market value from appraisal based indexes. These methods are often referred to as unsmoothing techniques. Here is a short review of a simple unsmoothing technique due to Geltner.

Suppose $P^*(t)$ denotes the value of an appraisal based index at time t and $P(t)$ denotes the true market price of properties underlying the index. Geltner ([14]) argues that the relation between P and P^* can be recovered by the relation

$$r^*(t) = ar(t) + (1-a)r^*(t-1), \quad t = 0, 1, \dots, \tag{3.24}$$

where r^* and r are annual returns, i.e.

$$r^*(t) = \frac{P^*(t) - P^*(t-1)}{P^*(t-1)}, \quad r(t) = \frac{P(t) - P(t-1)}{P(t-1)}, \tag{3.25}$$

and $0 \leq a \leq 1$. Moreover, Geltner argues that $a \approx 0.4$. In the literature there are more sophisticated unsmoothing techniques, see e.g. Fischer et al. ([12]) or Cho et al. ([8]).

Next, we will discuss two different models of the price dynamics of P.

3.3.1 Model A

In [11] property is modelled in the same way as equity. That is, the gain $\Delta P(T) = P(T) - P(0)$ due to property is given by (see Footnote 2 above)

$$\frac{\Delta P(T)}{P(0)} = I_P - d_P(0)\Delta\bar{i}(0) + \epsilon_P, \tag{3.26}$$

where I_P is the expected return of the property with the present interest rate level, $d_P(0)$ is the modified duration of the property at time 0 and defined in analogy with $d_S(0)$, and ϵ_P is a random variable with mean 0 and variance ς_P that represents the change in the price of the property not explained by the interest rate. The distribution of ϵ_P is not further specified. The Gordon-Shapiro model can be used to estimate d_P here as well, then with dividends replaced by rents.

3.3.2 Model B

We use the same approach as in Section 3.2.2. That is, we assume that $P(t)$ is given by:

$$\frac{dP(t)}{P(t)} = (i(t) + \mu_P)\, dt + \sigma_P dW_P(t), \tag{3.27}$$

where μ_P and σ_P are constants and $\{W_P(t)\}_{0 \le t \le T}$ is a Brownian motion. We assume that $\big(W_i, W_S, W_P\big)$ is a joint 3-dimensional Brownian motion, The parameters μ_P and σ_P are called risk premium and volatility, respectively, here as well.

A closely related model has been proposed by Buttimer et al. [7]. Similar ideas as in Eq. (3.27), developed using the efficient market hypothesis, can be found in Fischer et al. [12]. However, there are findings showing that the property returns are not normally distributed, see Young et al. [37].

4 Loss

In this section we discuss how the difference between the liabilities and the assets can be modeled in Model A and Model B.

4.1 Model A

We follow Rantala [29] to derive a best estimate of future liabilities $V(T)$ at some time T. One may decompose the liability as

$$V(T) = V(0) + \Delta_i V(T) + \Delta_g V(T) + \Delta_p V(T), \tag{4.1}$$

where $\Delta_i V(T)$ is the variation in the liabilities between today and time T that are due to changes in the interest rate, $\Delta_g V(T)$ is the variation due to interest rate guarantees and other more or less binding bonus declarations, and $\Delta_p V(T)$ is the change in net premiums (premiums minus claims).

The initial value $V(0)$ can be computed in several different ways. We will discuss this issue in Sections 7 and 8. For the moment we will assume that the quantity $V(0)$ is given.

To describe $\Delta V_i(T)$, let $d_V(t)$ denote the duration of the liabilities at time t that we define in analogy with the bond duration (cf. Eq.(3.1)). Here the zero coupon bonds are replaced by discounted insurance claims. Of course, $d_V(t)$ is a stochastic process since the payoff of an insurance policy depends on the state of the insured like the mortality rate among other things. However, for the sake of simplicity, we assume $d_V(0)$ constant. $d_V(0)$ can be evaluated as best estimate of the duration of future liabilities. The change over the time period $[0, T]$ due to interest rate is described in Model A by

$$\frac{\Delta_i V(T)}{V(0)} = I_V - \frac{d_V(0)}{1 + \bar{i}(0)} \Delta \bar{i}(T), \tag{4.2}$$

where I_V is the expected increase of the liability with the present interest rate level and $\Delta \bar{i}(T)$ is defined as in Section 3.1.1.

According to Rantala [29], in a standard approach one may assume $\Delta_g V(T)$ constant. For instance, in Sweden, the actual gross guaranteed interest rate for life insurance is equal 3% on average.

For life insurance we will thus assume

$$\Delta_g V(T) = V(0)(1.03^T - 1) \tag{4.3}$$

and for non-life insurance we assume that

$$\Delta_g V(T) = 0. \tag{4.4}$$

Finally we consider the assets. Rantala [29] assumes that

$$A(T) = B(T) + S(T) + P(T) + \Delta_p V(T). \tag{4.5}$$

where as above $\Delta_p V(T)$ is the change in net premiums (premiums minus claims) and B, S, and P are defined as in Section 3.1.1, 3.2.1, and 3.3.1, respectively.

One may remark that the model of the assets probably would be more realistic if the model included more advanced portfolio strategies. For a discussion on other portfolio strategies, see the next section or Koivu et al. [21].

To sum up, in Model A, the balance equation of an insurance company in terms of the difference between the value of the liabilities and the assets, at time T, can be described by

$$V(T) - A(T) = V(0) + \Delta_i V(T) + \Delta_g V(T) - B(T) - S(T) - P(T). \tag{4.6}$$

4.2 Model B

Model B will use a model of the liability that can been seen as a simplification of the ideas in Grosen et al. [16]. Assume that i_g is the continuous interest rate guarantee. In our case this means that

$$i_g = \ln(1 + 0.03) \approx 0.03 \tag{4.7}$$

for life insurance and

$$i_g = 0 \tag{4.8}$$

for non-life insurance. One may assume that the continuous policy interest rate at time t equals $\max[i_g, i(t)]$. That is, the insured are given the maximum of the gross guaranteed interest rate and nominal interest rate. Moreover, if $i(t) \geq 0$ we find that $\max[i_g, i(t)]$ is bounded by the sum $i_g + i(t)$. Thus, an upper bound for the liability at time T is given by

$$V(T) = V(0) \exp\Big(\int_0^T (i_g + i(s)) ds \Big). \tag{4.9}$$

We assume here as well that $V(0)$ is a known quantity.

Finally we consider the asset portfolio. One may consider several different portfolio strategies. Here we assume that the insurance company invests a fixed proportion in each asset class. The goal is to derive the dynamics of the asset portfolio. Denote the number of bonds, equities, and properties at time t with $h_B(t)$, $h_S(t)$, and $h_P(t)$, respectively. It is evident that

$$dA(t) = h_B(t)\, dB(t) + h_S(t)\, dS(t) + h_P(t)\, dP(t),$$

or equivalently,

$$\frac{dA(t)}{A(t)} = \frac{h_B(t)B(t)}{A(t)} \frac{dB(t)}{B(t)} + \frac{h_S(t)S(t)}{A(t)} \frac{dS(t)}{S(t)} + \frac{h_P(t)P(t)}{A(t)} \frac{dP(t)}{P(t)}.$$

Let π_B, π_S, and π_P denote the fixed proportions of bonds, equities, and properties, respectively, in the asset portfolio. Then

$$\frac{dA(t)}{A(t)} = \pi_B \frac{dB(t)}{B(t)} + \pi_S \frac{dS(t)}{S(t)} + \pi_P \frac{dP(t)}{P(t)}.$$

Eq. (3.17), (3.22), and (3.27) now yield

$$\begin{aligned}\frac{dA(t)}{A(t)} &= \big[i(t) + \pi_B\mu_B + \pi_S\mu_S + \pi_P\mu_P\big]dt \\ &\quad - \pi_B \frac{\sigma_i}{\kappa}(1 - e^{-\kappa d_B})dW_i(t) + \pi_S\sigma_S dW_S(t) + \pi_P\sigma_P dW_P(t).\end{aligned}$$

Define the parameters ν_1 and ν_2 as

$$\begin{aligned}\nu_1 =& \pi_B\mu_B + \pi_S\mu_S + \pi_P\mu_P, \\ \nu_2^2 =& \pi_B^2 \frac{\sigma_i^2}{\kappa^2}(1 - e^{-\kappa d_B})^2 + \pi_S^2\sigma_S^2 + \pi_P^2\sigma_P^2 - 2\pi_B\pi_S \frac{\rho_{iS}\sigma_i\sigma_S}{\kappa}(1 - e^{-\kappa d_B}) \\ & - 2\pi_B\pi_P \frac{\rho_{iP}\sigma_i\sigma_P}{\kappa}(1 - e^{-\kappa d_B}) + 2\pi_S\pi_P\rho_{SP}\sigma_S\sigma_P.\end{aligned}$$

Using The Martingal Representation Theorem, there exists, on a possibly larger probability space, a Brownian motion W, such that the asset process $A(t)$ satisfies the stochastic differential equation

$$\frac{dA(t)}{A(t)} = \big[i(t) + \nu_1\big]dt + \nu_2 dW(t). \tag{4.10}$$

Now, the balance of an insurance company is given by

$$V(T) - A(T) = V(0)\exp\Big(\int_0^T i_g + i(s)ds\Big) - A(T). \tag{4.11}$$

Note in particular that this model is independent of the liability duration, which is desirable in a standard approach since the duration may be difficult to estimate, see Rantala [30].

5 Computing the Target Capital

In this section we compute the target capital in Model A and Model B. Recall that the target capital is the initial value of assets $A(0)$ for which the asset & liability risk, at time T, i.e. the risk the value of the liabilities are larger than the assets value, is zero:

$$\rho\big(V(T) - A(T)\big) = 0. \tag{5.1}$$

The choice of the risk measure ρ depends on the valuation model chosen by either the supervisory authority or the company. The risk measure suggested in Model A is given by the Standard Deviation Principle, SDP_δ, and the one suggested in Model B is the Value-at-Risk, VaR_α.

5.1 Model A

Consider the processes defined in Sections 3.1.1, 3.2.1, 3.3.1 and 4.1. Proposition 1 below gives an analytical expression of the asset & liability risk, given by the Standard Deviation Principle.

$$\mathrm{SDP}_\delta\big(V(T) - A(T)\big) = E\big[V(T) - A(T)\big] + \delta\sqrt{\mathrm{Var}\big[V(T) - A(T)\big]}. \tag{5.2}$$

First, let π_B, π_S and π_P denote the proportions at time $t = 0$ of the bonds, equities, and property in the asset portfolio. Let $\varrho_{SP} = \mathrm{Cov}[\epsilon_S, \epsilon_P]$, $m_B = d_B(0)/(1 + \bar{i}(0))$, and $m_V = d_V(0)/(1 + \bar{i}(0))$. Moreover, set

$$\alpha_0 = V(0) + I_V + \Delta_g V(T), \quad \alpha_1 = 1 + \pi_B I_B + \pi_S I_S + \pi_P I_P,$$

and

$$\begin{aligned}
\beta_{0,0} &= m_V^2 V(0)^2 \varsigma_i^2,\\
\beta_{0,1} &= -m_V V(0) m_B \pi_B \varsigma_i^2,\\
\beta_{0,2} &= -m_V V(0) d_S(0) \pi_S \varsigma_i^2,\\
\beta_{0,3} &= -m_V V(0) d_P(0) \pi_P \varsigma_i^2,\\
\beta_{1,1} &= m_B^2 \pi_B^2 \varsigma_i^2,\\
\beta_{1,2} &= m_B \pi_B d_S(0) \pi_S \varsigma_i^2,\\
\beta_{1,3} &= m_B \pi_B d_P(0) \pi_P \varsigma_i^2,\\
\beta_{2,2} &= d_S(0) \pi_S^2 \varsigma_i^2 + \varsigma_S^2,\\
\beta_{2,3} &= d_S(0) \pi_S d_P(0) \pi_P \varsigma_i^2 + \varrho_{SP} \pi_S \varsigma_S \pi_P \varsigma_P,\\
\beta_{3,3} &= d_P(0)^2 \pi_P^2 \varsigma_i^2 + \pi_P^2 \varsigma_P^2.
\end{aligned}$$

If $k < l$ then $\beta_{l,k} = \beta_{k,l}$.
If $X_0 = V(0) + \Delta_i V(T) + \Delta_g V(T)$, $X_1 = -B(T)$, $X_2 = -S(T)$, $X_3 = -P(T)$ then it is easily seen that

$$E\Big[\sum_{k=0}^{3} X_k\Big] = \alpha_0 - \alpha_1 A(0),$$

$$\mathrm{Cov}\big[X_k, X_l\big] = \beta_{k,l} A(0)^{\min(k,1)+\min(l,1)}, \quad 0 \le k, l \le 3,$$

and

$$V(T) - A(T) = \sum_{k=0}^{3} X_k.$$

We are now in the position to compute the asset & liability risk.

Proposition 1 *The asset & liability risk in Model A is given by the following formula:*

$$SDP_\delta\big(V(T) - A(T)\big) = \alpha_0 - \alpha_1 A(0) + \delta\sqrt{\sum_{k,l=0}^{3} \beta_{k,l} A(0)^{\min(k,1)+\min(l,1)}}. \tag{5.3}$$

In particular, if there exists a target capital then this is given by the positive solution to the equation

$$(\alpha_1^2 - \delta^2 \sum_{k,l=1}^{3} \beta_{k,l}) A(0)^2 - 2(\alpha_0\alpha_1 + \delta^2 \sum_{k=1}^{3} \beta_{k,0}) A(0) + \alpha_0^2 - \delta^2 \beta_{0,0} = 0.$$

Proof The formula for the asset & liability risk follows at once from the definition of Standard Deviation Principle. To prove the statement about the target capital, note that the equation

$$\mathrm{SDP}_\delta\big(V(T) - A(T)\big) = 0$$

may be rewritten as

$$(\alpha_1 A(0) - \alpha_0)^2 = \delta^2 \big(\beta_{0,0} + 2A(0) \sum_{k=1}^{3} \beta_{k,0} + A(0)^2 \sum_{k,l=1}^{3} \beta_{k,l}\big)$$

or, alternatively,

$$(\alpha_1^2 - \delta^2 \sum_{k,l=1}^{3} \beta_{k,l}) A(0)^2 - 2(\alpha_0\alpha_1 + \delta^2 \sum_{k=1}^{3} \beta_{k,0}) A(0) + \alpha_0^2 - \delta^2 \beta_{0,0} = 0.$$

□

We note that using Model A there are cases where there does not exist a target capital. In Section 7 we discuss an example of such a case.

5.2 Model B

Recall the dynamics of $A(t)$ and $V(t)$ from Eq. 4.10 and 4.9 and the parameters ν_1 and ν_2 given by the follwoing expressions:

$$\begin{aligned}
\nu_1 =& \pi_B\mu_B + \pi_S\mu_S + \pi_P\mu_P, \\
\nu_2^2 =& \pi_B^2 \frac{\sigma_i^2}{\kappa^2}(1 - e^{-\kappa d_B})^2 + \pi_S^2\sigma_S^2 + \pi_P^2\sigma_P^2 - 2\pi_B\pi_S \frac{\rho_{iS}\sigma_i\sigma_S}{\kappa}(1 - e^{-\kappa d_B}) \\
& - 2\pi_B\pi_P \frac{\rho_{iP}\sigma_i\sigma_P}{\kappa}(1 - e^{-\kappa d_B}) + 2\pi_S\pi_P\rho_{SP}\sigma_S\sigma_P.
\end{aligned}$$

We are now in the position to compute the asset & liability risk in Model B.

Proposition 2 *The target capital corresponding to the asset & liability risk in Model B is given by*

$$A(0) = V(0) \exp(i_g T) \exp\Big(\nu_2 \sqrt{T}\, \Phi^{-1}(1 - \alpha) - (\nu_1 - \frac{1}{2}\nu_2^2)\, T\Big).$$

Proof Note that if $X(T) = \ln\big(A(T)/V(T)\big)$ then

$$P\big(V(T) - A(T) \geq 0\big) = P\big(X(T) \leq 0\big)$$

Moreover, Using Eq. 4.10 and 4.9 and Itô's Lemma, we get

$$dX(t) = (\nu_1 - i_g - \frac{1}{2}\nu_2^2)dt + \nu_2 dW(t)$$

where W is a standard Brownian motion. In particular,

$$P\big(X(T) \le 0\big) = \Phi\left[\frac{-\ln\frac{A(0)}{V(0)} - (\nu_1 - i_g - \frac{1}{2}\nu_2^2)T}{\nu_2\sqrt{T}}\right]$$

so that if $\mathrm{VaR}_\alpha(V(T) - A(T)) = 0$ then

$$A(0) = V(0)\exp\Big(\nu_2\sqrt{T}\,\Phi^{-1}(1-\alpha) - (\nu_1 - i_g - \frac{1}{2}\nu_2^2)\,T\Big)$$

and the proof is complete. □

6 Model Calibration

In this section we address the calibration issue of the price processes suggested in Models A and B. There are mainly two ways to assess the model parameters; they can be estimated from historical data or be determined by experts. In practice, insurance companies managers often have their own opinion about the parameters, see e.g. Koivu et al. [21]. However, in this section the parameters will be estimated using historical data. We start by considering Model A.

6.1 Model A

Table 1 below collects the parameters in the valuation processes according to Model A.

The duration $d_V(0)$ and expected increase I_V of the liabilities depend of course on the insurance contracts. In the next section we give two examples that show how the liability parameters can be assessed.

First, we estimate the standard deviation of the change in interest rate level ς_i. To this end we suppose that the shift in the yield curve is given by the change in the instantaneous spot rate over time period $[0, T]$ i.e. $\Delta\bar{i}(T) = i(T) - i(0)$. Moreover, assume the interest rate is sampled at the times $t = k\Delta t$, $k = 1, 2, \ldots, N$, and denote the observations by $i_k = i(k\Delta t)$. A common model for estimating the interest rate (see e.g. Fama et al. [10]) is to assume i_k is an AR(1) process, that is

$$i_k = \phi_i + \psi_i i_{k-1} + \epsilon_k,$$

where ϵ_k, $k = 1, 2, \ldots$ are i.i.d. random variables and ϕ_i and $0 < \psi_i < 1$ are constants. The parameters ψ_i and $\mathrm{Var}[\epsilon_1]$ can be estimated with standard regression analysis, see e.g. Larsen et al. [22]. This yields an estimate of ς_i as

$$\varsigma_i^2 = \lim_{k\to\infty} \mathrm{Var}\big[i_k\big] = \frac{\mathrm{Var}[\epsilon_1]}{1 - \psi_i^2}.$$

$d_V(0)$	liability duration
I_V	expected increase of the liabilities
ς_i	standard deviation of the change in interest rate level
I_B	expected income on bonds with the present interest rate levels
ς_S	standard deviation of the change in equity not explained by the interest rate
$d_S(0)$	equity duration
I_S	expected income from equity
ς_P	standard deviation of the change in property not explained by the interest rate
$d_P(0)$	the duration of the property
I_P	expected income from property
ϱ_{SP}	correlation between the residuals of the the equity and the property

Table 1: The parameters in Model A. The parameters are described in greater detail in Section 3.1.1, 3.2.1, and 3.3.1.

Δt	1/250	1/12	1
ς_i	0.018	0.018	0.019

Table 2: Value of the interest rate parameter.

In Table 2 below, we display estimated values of ς_i, given different sampling periods Δt. The historical data for the spot rate are based on the the Swedish interbank rate, STIBOR, during the time period 1994-01-01 - 2003-12-31.

As for the expected income for bonds, if T is smaller than the longest available maturity time in the bond market, the parameter I_B can be extracted from the yield curve. Otherwise one has to determine I_B by extrapolating the yield curve. For a discussion on the estimation of long-term interest rates, see Yong [36], [19], and the references therein.

To estimate the equity parameters, denote the price observations by $S_k = S(k\Delta t)$, $k = 0, 1, 2, \ldots, n$ and assume that the equity returns satisfy

$$\frac{S_k - S_{k-1}}{S_{k-1}} = \phi_S + \psi_S(i_k - i_{k-1}) + \epsilon_k, \tag{6.1}$$

where ϵ_k, $k = 1, 2, \ldots, n$, are i.i.d. random variables and ϕ_S and ψ_S are constants. The constants ϕ_S and ψ_S can be assessed using a standard linear regression. Let $T = n\Delta t$ for some integer n. By Eq. (6.1), a first order approximation of the equity return equals

$$\frac{\Delta S(T)}{S(0)} = \phi_S n + \psi_S[i(T) - i(0)] + \sum_{k=1}^{n} \epsilon_k.$$

It is now immediate that the standard deviation ς_S, the duration $d_S(0)$, and the expected income I_S are given by the relations

$$\varsigma_i^2 = \frac{T}{\Delta t}\mathrm{Var}[\epsilon_k], \quad d_S(0) = -\psi_S, \quad \text{and} \quad I_S = \frac{T}{\Delta t}\phi_S. \tag{6.2}$$

Table 3 below, shows the estimated values of ς_S, $d_S(0)$, and I_S estimated with different sampling periods Δt. The values in the parenthesis describes a 95% confidence interval under the assumption that the residuals ϵ_k are normal distributed. The historical data are based on the SAX-index during the time period 1996-01-01 - 2003-12-31. The dividends are reinvested in the index. The SAX-index is a Swedish equity index. Note the large confidence interval for the duration and the expected income. Moreover, the duration is very sensitive to the sampling frequency. In the examples in Section 7 we have chosen the parameter values corresponding to $\Delta t = 1/12$. This choice gives an estimation of the duration that is closer to the results in Leibowitz [23].

Δt	1/250	1/12	1
ς_S	1.90 (± 0.07)	1.91 (± 0.28)	2.67 (± 2.00))
$d_S(0)$	0.31 (± 1.18)	3.12 (± 6.71)	6.82 (± 20.20)
I_S	1.04 (± 1.31)	0.86 (± 1.40)	0.72 (± 3.07)

Table 3: Equity parameter values

To estimate the parameters associated with property, define $P_k = P(k\Delta t)$, $k = 1, \ldots, n$, and assume

$$\frac{P_k - P_{k-1}}{P_{k-1}} = \phi_P + \psi_P(i_k - i_{k-1}) + \epsilon_k,$$

where ϵ_k, $k = 1, 2, \ldots$ are i.i.d. random variables and ϕ_P and ψ_P are constants. The parameters ς_P, $d_P(0)$, and I_P can now be computed in a similar way as in Eq. (6.2). Estimated values are shown in Table 4 below, where the values in the parenthesis describe a 95% confidence interval under the assumption that the residuals ϵ_k are normally distributed. The historical data are given by the SFI-index over the time period 1994-01-01 - 2003-12-31. The SFI-index is an annual Swedish property index. The value of Δt is thus 1. The index has been unsmoothed with the technique described in Section 3.3.

ς_P	$d_P(0)$	I_P
1.25 (± 2.07)	-2.33 (± 6.83)	1.42 (± 1.00)

Table 4: Property parameter values.

It remains to assess the correlation between the residuals of the equity and the property. The value of the parameter ϱ_{SP} is estimated to

$$\varrho_{SP} = 0.22$$

with the the same historical data as in Tables 3 and 4. Alternatively, in a conservative standard approach one may put $\varrho_{SP} = 1.0$, see Sandström [34] for further details. Much more can of course be said about the estimation of equity and property duration, see for instance Brown [6] or Leibowitz [23].

6.2 Model B

This section will discuss the calibration of the parameters in the valuation processes according to Model B. The parameters are collected in Table 5 below.

σ_i	volatility of the interest rate
κ	mean reversion coefficient for the interest rate, see Section 3.1.2
θ	long term interest rate or interest rate equilibrium
$\tilde{\theta}$	interest rate equilibrium under the martingale measure
σ_S	volatility of the equities
μ_S	risk premium for the equities
σ_P	volatility of the properties
μ_P	risk premium for the properties
γ_{iS}	correlation between the volatile part of the interest rate and the equities
γ_{iP}	correlation between the volatile parts of the interest rate and the properties
γ_{SP}	correlation between the volatile parts of the equities and the properties

Table 5: The parameters in Model B. The parameters are described in greater detail in Section 3.1.2, 3.2.2, and 3.3.2.

To begin with we consider the interest rate parameters. Assume the spot rate is described as in Section 3.1.2 and its level has been observed at the times $t = k\Delta t$, $k = 1, 2, \ldots, n$ and denote the observations with $i_k = i(k\Delta t)$. It holds

$$i_k = \phi_i + \psi_i i_{k-1} + \epsilon_k$$

where ϵ_k, $k = 1, 2, \ldots, N$, are i.i.d. normal random variables and ϕ_i and ψ_i are constants. One may show, see e.g. Brennan et al. [4], that

$$\kappa = -\frac{\ln \psi_i}{\Delta t}, \quad \theta = \frac{\phi_i}{1 - \psi_i}, \quad \text{and} \quad \sigma_i^2 = \frac{\mathrm{Var}[\epsilon_k] \ln \psi_i}{\Delta t(1 - \psi_i^2)}.$$

Using a linear regression, we get estimates of the parameters ϕ, ψ_i and $\mathrm{Var}[\epsilon_k]$. By these estimates we can assess the parameters σ_i, κ, and θ. The values are shown in Table 6. The historical data are the same as in Table 2.

Δt	1/250	1/12	1
σ_i	0.011	0.008	0.008
κ	0.178	0.092	0.093
θ	0.026	0.005	0.005

Table 6: Interest rate parameter values.

The small value of κ indicates that the interest rate is slowly mean reverting. This property has been observed previously, see Fama et al. [10] and Brennan et al. [4]. Note the surprisingly small value of the equilibrium θ of the interest rate. This value on θ is partly explained by the fact that the interest rate has been decreasing over the last decade. We may add that the estimates of κ are sensitive to the sampling period. Other choices of sampling periods give estimates of κ that may be greater than one as well as negative. The parameter

$\tilde{\theta}$, denoting interest rate equilibrium under the martingale measure, is chosen as the number that minimizes the quadratic sum of the differences between the theoretical yield curve given by the Vasicek model and the yield curve given by the market. To be more specific, $\vartheta = \tilde{\theta}$ minimizes

$$\sum_k \left\{ -\left(\vartheta - \frac{\sigma_i^2}{2\kappa^2}\right) \frac{b(0,\tau_k) - \tau_k}{\tau_k} + \frac{\sigma_i^2}{4\kappa} \frac{b(0,\tau_k)^2}{\tau_k} + \frac{b(0,\tau_k)}{\tau_k} i(0) - R(0,\tau_k) \right\}^2 \quad (6.3)$$

where $R(0,\tau)$ is the market yield of a bond with time to maturity τ. Based on the yield curve at time 2004-01-01 for Swedish government bonds with time to maturity 3/12, 6/12, 9/12, 1, 2, 5, and 10 years one obtain the estimate

$$\tilde{\theta} = 0.052.$$

Next, denote the discounted log-return of the stock over the time period $(k-1)\Delta t \leq t < k\Delta t$ by r_k, that is

$$r_k = \ln \frac{S_D(k\Delta t)}{S_D((k-1)\Delta t)}, \quad k = 1, 2, \ldots, n,$$

where

$$S_D(t) = \frac{S(t)}{\exp(\int_0^t i(s)ds)} = S(0)e^{(\mu_S - \frac{1}{2}\sigma_S^2)T + \sigma_S W_S(T)}.$$

The Samuelson model in Section 3.2.2 stipulates, among other things, that

$$r_k = \phi_S + \epsilon_k, \quad k = 1, \ldots, N,$$

where ϵ_k are i.i.d. normal random variables and ϕ_S is a constant. It is easily seen that

$$\sigma_S^2 = \frac{\mathrm{Var}[\epsilon_k]}{\Delta t} \quad \text{and} \quad \mu_S = \frac{\phi_S}{\Delta t} + \frac{1}{2}\sigma_S^2.$$

Using a linear regression one can estimate the parameters ϕ and $\mathrm{Var}[\epsilon_k]$ and thereby determine σ_S and μ_S.

Table 7 below displays estimated values on σ_S and μ_S for different values of Δt. The values in the parenthesis describes a 95% confidence interval. The historical data are the same as in Table 3. An important aspect of Table 7 is the magnitude of the error in the estimation of the risk premium μ_S. For more details on this subject, see Merton [27], Rogers [32], and the references therein.

Δt	1/250	1/12	1
σ_S	0.237 (± 0.008)	0.241 (± 0.036)	0.303 (± 0.209))
μ_S	0.082 (± 0.165)	0.082 (± 0.170)	0.084 (± 0.253)

Table 7: Equity parameter values

Next we will consider the parameters related to property, that is, σ_P and μ_P. These parameters can be estimated in the same way as the equity parameters. Using historical data as in Table 4, estimated values of σ_P and μ_P are displayed in Table 8 below. The sampling period is $\Delta t = 1$.

σ_P	μ_P
0.119 ($\pm$ 0.082)	0.053 ($\pm$ 0.099)

Table 8: Property parameter values

Alternatively, according to for instance Fischer et al. [12], one can consider a widely used assumption that the volatility of property is the half of the volatility of an equity index, e.g. S & P 500, at least for properties in the U.S.A. Thus a rough estimate of σ_P would be

$$\sigma_P = \frac{1}{2}\sigma_S.$$

This value could be applied in the standard approach.

Finally, we estimate the correlations. Table 9 shows estimated values of the parameters γ_{iS}, γ_{iP}, and γ_{SP}. The values are based on yearly observations during the time period 96-01-01 to 04-01-01. The property index has been unsmoothed according to the techniques described in Section 3.3.

γ_{iS}	γ_{iP}	γ_{SP}
-0.15	0.13	0.24

Table 9: Estimated parameter values for the correlations.

Alternatively, in a conservative standard approach one may put

$$\gamma_{iS} = 0,\ \gamma_{iP} = 1, \quad \text{and} \quad \gamma_{SP} = 1.$$

For a further discussion about the estimation of parameters in a model similar to model B, see Munk et al. [28] and Brennan et al. [4].

7 Applications

The objective of this section is to compute the target capital for two insurance contracts; one from life insurance and the other one from vehicle insurance. We will mainly focus on the effect different choices of asset portfolios may have on the target capital and compare the outcomes from Models A and B.

7.1 The Target Capital for a D(10) Contract

In the first example we consider a life insurance contract called D(10). The holder of a D(10)-contract pays annually a fixed amount to the insurer. In return, after ten years or at the time of death of the insured the insurer pays to the beneficiary a fixed amount. Suppose v_k denotes the payments due to death during year k, $k = 0, 1, \ldots, 9$, and v_{10} denotes the payment to the insured at year 10 provided the insured is alive after ten years. We suppose that v_k, $k = 0, 1, \ldots, 10$ are given constants. The liability at time 0 will in this example be defined as the discounted value of future cash flows. For a D(10) contract we thus have

$$V(0) = \sum_{k=0}^{9} v_k p(0, k+0.5) + v_{10}\, p(0,10), \tag{7.1}$$

where, as before, $p(0,\tau)$ is the value of a zero coupon bond with time to maturity τ. To compensate that the payments due to death, i.e. v_k, $k = 0, 1, \ldots, 9$, may be paid at any time during year k and not necessarily at the beginning of the year, the payments are discounted k plus a half year and not just k year. We may add that this definition of $V(0)$ is just one of many possible definitions. Section 8 discusses other more risk based approaches that may include mortality risk and other insurance risks.

Note that the value of a zero coupon bond depends on the model. In Model A the bond prices have been computed using cubic spline interpolation of the yield curve. In Model B the price of a zero coupon bond is computed by the formula in Eq. (3.10). The values of v_k are based on data given by Skandia Life Insurance Company.

The liability duration $d_V(0)$ at time 0 is given by

$$d_V(0) = \frac{1}{V(0)}\Big(\sum_{k=0}^{9}(k+0.5)v_k p(0,k+0.5) + 10 v_{10}\, p(0,10)\Big). \tag{7.2}$$

Table 10 displays the estimated values and the duration of the liabilities. In Model A the value of a zero coupon bond is based on interpolation of the yield curve at time 2004-01-01 for Swedish government bonds with time to maturity 3/12, 6/12, 9/12, 1, 2, 5, and 10 years. The value of a zero coupon bond in Model B is computed using Eq. (3.10)

		Model A	Model B
$V(0)$	(10^3 SEK)	69	71
$d_V(0)$	(years)	9.23	9.28

Table 10: Estimated values of the liability and the liability duration.

Before we go on and compute the target capital we will make some remarks about the model parameters. The values of the model parameters have been estimated in Section 6. Recall that some of the parameters were estimated with several different sampling frequencies. In the examples presented in this section we have, with one exception, chosen the value estimated with sampling interval equal to one month (i.e. $\Delta t = 1/12$). The exception is the value of the parameters associated with property, here we only had data with yearly observations.

Recall that the target capital is defined as the initial value of assets, i.e. $A(0)$, satisfying

$$\rho\big(V(T) - A(T)\big) = 0.$$

with $\rho = \mathrm{SDP}_\delta$ in Model A and $\rho = \mathrm{VaR}_\alpha$ in Model B. In the examples below the value of δ is equal to $\Phi^{-1}(0.99) \approx 2.33$ and $\alpha = 1\%$ with the exception of the example in Table 15, where $\delta = \Phi^{-1}(0.95) \approx 1.64$ and $\alpha = 5\%$.

Proportions			Model A	Model B
Bond	Equity	Property	Target Capital	Target Capital
0.9	0.05	0.05	109	97
0.8	0.1	0.1	133	100
0.6	0.2	0.2	266	110
0.6	0.4	0.0	*	143
0.6	0.0	0.4	196	104
0.5	0.25	0.25	539	117
0.4	0.3	0.3	*	124
0.4	0.6	0.0	*	188
0.2	0.4	0.4	*	142
0.2	0.8	0.0	*	254

Table 11: Target capital for different choices of asset portfolios and $T = d_V(0)$. The target capital is measured in 10^3 SEK. The symbol $*$ means that there is no portfolio with the relevant proportions that satisfy the solvency requirement.

In Table 11, we display the target capital for different asset portfolios. In this example we have put $T = d_V(0)$. This choice of T is motivated by the interpretation of the liability duration as a (economic) mean value of the payment dates. Note that the target capital is quite sensitive to the proportion of equities in the asset portfolio. A large proportion of equity requires a large target capital. Note moreover that in Model A there does not necessarily exist a target capital for all asset portfolios. This is an unrealistic property of the model. Moreover, given an asset portfolio satisfying the solvency requirement, if we increase the amount invested in one asset class then it would be reasonable to assume that we can invest less money in some of the other two asset classes and still the asset portfolio satisfies the solvency requirement. However, this is not the case for Model A, compare for instance the portfolios on row 1 and row 3.

Before we go on and present more examples we will motivate why there are cases where there is no target capital in Model A. Consider the function

$$k \mapsto \mathrm{SDP}_\delta(v - kA(T)),$$

where $A(T)$ is a random variable, $v > 0$ is a constant, and k is a real number. Think of k as units of an asset portfolio with value $A(T)$ and v as a constant liability. The assumption that the liability is constant is not far from our models where the variance of the liabilities are much smaller than the assets, at least if the equity or property component is sufficiently large. Next, by differentiating with respect k we obtain

$$\frac{d}{dk}\mathrm{SDP}_\delta(v - kA(T)) = -E\big[A(T)\big] + \delta\sqrt{\mathrm{Var}\big[A(T)\big]}.$$

Thus, the function $\mathrm{SDP}_\delta(v - kA(T))$ is increasing with respect to k if

$$E[A(T)] \leq \delta\sqrt{\mathrm{Var}[A(T)]}. \tag{7.3}$$

Hence, if Eq. (7.3) is satisfied then there is no k so that $\mathrm{SDP}_\delta(v - kA(T)) \leq 0$. In other words, no matter how much money we invest in the assets, there will always be positive risk if the variance of the assets is sufficiently large.

In the next example, displayed in Table 12, we put $T = 1$. Note that the target capital is now smaller than in the previous example with $T = d_V(0)$, in some cases there is a great difference. In particular, in Model A the target capital is always finite. Thus, we must emphasize that the value of T is of great importance. Note as well that Model A still suggests a larger target capital, thus 'more expensive' for the company than Model B.

Proportions			Model A	Model B
Bond	Equity	Property	Target Capital	Target Capital
0.9	0.05	0.05	91	77
0.8	0.1	0.1	96	78
0.6	0.2	0.2	116	83
0.6	0.4	0.0	171	90
0.6	0.0	0.4	115	80
0.5	0.25	0.25	133	85
0.4	0.3	0.3	155	88
0.4	0.6	0.0	573	100
0.2	0.4	0.4	241	93
0.2	0.8	0.0	*	112

Table 12: Target capital for different choices of asset portfolios. The target capital is measured in 10^3 SEK and $T = 1$.

The next example, displayed in Table 13, shows the target capital for other values of the model parameters than the values estimated in Section 6. The values in Table 14 follow a so called conservative standard approach, see Sandström [34]. This means that in Model A the estimated value of ϱ_{SP} is replaced with $\varrho_{SP} = 1$ and in Model B we put $\gamma_{iS} = 0$, $\gamma_{iP} = 1$, $\gamma_{SP} = 1$ instead of the estimated values.

The final example in this section is presented in Table 15. The table presents the target capital with $\delta = \Phi^{-1}(0.95) \approx 1.64$ and $\alpha = 5\%$ in the definition of the risk measures.

7.2 The Target Capital for a Vehicle Insurance

In this section we consider a vehicle insurance that covers both property damage and bodily injuries. The objective is to compute the target capital for the costs of a certain vehicle insurance issued by Länsförsäkringar Sak AB during year 1997.

Suppose v_k, $k = 0, 1, \ldots$ denote the total payments during year $1997 + k$ due to accidents during year 1997. A best estimate of the liabilities is defined by

$$V(0) = \sum_{k=0}^{6} v_k p(0, k + 0.5). \tag{7.4}$$

In particular, we suppose that the payments after 6 years are negligible. The definition of the liability duration is defined in analogy with the example in the previous section. Table 16 below, collects the value of the liability and its

Proportions				Model A	Model B
Bond	Equity	Property	T	Target Capital	Target Capital
0.8	0.1	0.1	2	102	82
			5	117	90
			10	134	100
0.6	0.2	0.2	2	137	87
			5	203	97
			10	267	110
0.4	0.3	0.3	2	238	94
			5	911	109
			10	*	125
0.4	0.6	0.0	2	*	114
			5	*	147
			10	*	190

Table 13: Target capital for different choices of asset portfolios and values on T. The target capital is measured in 10^3 SEK.

	Proportions			Model A	Model B
T	Bond	Equity	Property	Target Capital	Target Capital
1	0.8	0.1	0.1	101	79
	0.6	0.2	0.2	133	84
	0.4	0.3	0.3	220	91
$d_V(0)$	0.8	0.1	0.1	162	95
	0.6	0.2	0.2	981	118
	0.4	0.3	0.3	*	143

Table 14: Target capital in the conservative standard approach for different choices of asset portfolios and values on T. The target capital is measured in 10^3 SEK. The symbol $*$ means that there is no portfolio with the relevant proportions that satisfy the solvency requirement.

duration in both Models A and B. The data for the payments v_k are given by Länsförsäkringar Sak AB.
Table 17 shows the target capital for different asset portfolios. Model A still gives higher values on the target capital than Model B. The target capital is however always finite in this example.

The final examples displayed in Table 18 and 19, give the target capital in the conservative approach and with $\delta = 1.64$ and $\alpha = 5\%$. Recall that the conservative approach meant that in Model A the estimated value of ϱ_{SP} is replaced with $\varrho_{SP} = 1$ and in Model B we put $\gamma_{iS} = 0$, $\gamma_{iP} = 1$,$\gamma_{SP} = 1$ instead of the estimated values.

8 Including Other Forms of Risk

We conclude this paper with a brief discussion on how one may include other risk factors in the model. This section will only consider Model B. An extension

T	Proportions			Model A	Model B
	Bond	Equity	Property	Target Capital	Target Capital
	0.8	0.1	0.1	90	76
1	0.6	0.2	0.2	99	79
	0.4	0.3	0.3	116	82
	0.8	0.1	0.1	110	93
$d_V(0)$	0.6	0.2	0.2	146	96
	0.4	0.3	0.3	217	102

Table 15: Target capital for different choices of asset portfolios and values on T. The target capital is measured in 10^3 SEK, $\delta = 1.64$, and $\alpha = 5\%$.

		Model A	Model B
$V(0)$	(10^6 SEK)	978	1027
$d_V(0)$	(years)	0.97	1.02

Table 16: Estimated value and duration of the liability. In model A the value of a zero coupon bond is based on interpolation of the yield curve at time 2004-01-01 for Swedish government bonds with time to maturity 3/12, 6/12, 9/12, 1, 2, 5, and 10 years. The value of a zero coupon bonds in Model B is computed using Eq. (3.10).

Proportions			Model A	Model B
Bond	Equity	Property	Target Capital	Target Capital
0.9	0.05	0.05	1144	1099
0.8	0.1	0.1	1242	1122
0.6	0.2	0.2	1555	1183
0.6	0.4	0.0	2416	1294
0.6	0.0	0.4	1477	1151
0.5	0.25	0.25	1790	1218
0.4	0.3	0.3	2103	1254
0.4	0.6	0.0	7658	1435
0.2	0.8	0.0	*	1600

Table 17: The target capital for different choices of asset portfolios. The target capital is measured in 10^6 SEK and $T = d_V(0)$.

Proportions			Model A	Model B
Bond	Equity	Property	Target Capital	Target Capital
0.8	0.1	0.1	1320	1123
0.6	0.2	0.2	1829	1209
0.4	0.3	0.3	3002	1307

Table 18: The target capital in the conservative standard approach for different choices of asset portfolios. The target capital is measured in 10^6 SEK and $T = d_V(0)$.

Proportions			Model A	Model B
Bond	Equity	Property	Target Capital	Target Capital
0.8	0.1	0.1	1164	1095
0.6	0.2	0.2	1340	1134
0.4	0.3	0.3	1575	1178

Table 19: The target capital for different choices of asset portfolios. The target capital is measured in 10^6 SEK, $\delta = 1.64$, $\alpha = 5$, and $T = d_V(0)$.

of Model A can be found in Sandström [34].

We start with a lemma.

Lemma 8.1 *Suppose* $0 < \alpha < 1$ *and* $0 < \lambda < 1$. *Then*

$$VaR_\alpha(X + Y) \leq VaR_{\lambda\alpha}(X) + VaR_{(1-\lambda)\alpha}(Y).$$

Proof Suppose $x = \mathrm{VaR}_{\lambda\alpha}(X)$ and $y = \mathrm{VaR}_{(1-\lambda)\alpha}(Y)$ and introduce the sets $A = \{X > x\}$ and $B = \{Y > y\}$. Firstly, note that

$$\begin{aligned} P(X + Y > x + y) &\leq P(A \cup B) \\ &= P(A) + P(B) - P(A \cap B) \\ &\leq P(X > x) + P(Y > y) \end{aligned}$$

and thus, since $P(X > x) \leq \lambda\alpha$ and $P(X > y) \leq (1 - \lambda)\alpha$,

$$P\big[X + Y > \mathrm{VaR}_{\lambda\alpha}(X) + \mathrm{VaR}_{(1-\lambda)\alpha}(Y)\big] \leq \alpha.$$

The definition of Value-at-Risk now gives the desired result. □

Suppose $\tilde{V}(t)$ describes the value of the liability with mortality, lapse, surrender, reinsurance, and other risk factors included. We want to find a minimum value of $A(0)$ so that

$$\mathrm{VaR}_\alpha\big(\tilde{V}(T) - A(T)\big) \leq 0. \tag{8.1}$$

Let the (simplified) liability $V(T)$ be defined as previous. Lemma 8.1 gives for any $0 < \lambda < 1$ the relation

$$\begin{aligned} &\mathrm{VaR}_\alpha\big(\tilde{V}(T) - A(T)\big) \\ &\qquad \leq \mathrm{VaR}_{\lambda\alpha}\big(\tilde{V}(T) - V(T)\big) + \mathrm{VaR}_{(1-\lambda)\alpha}\big(V(T) - A(T)\big). \end{aligned}$$

Thus, if

$$\mathrm{VaR}_{\lambda\alpha}\big(\tilde{V}(T) - V(T)\big) \leq 0 \quad \text{and} \quad \mathrm{VaR}_{(1-\lambda)\alpha}\big(V(T) - A(T)\big) \leq 0$$

then Eq. (8.1) is satisfied. In other words, first we determine the initial value $V(0)$ so that $\mathrm{VaR}_{\lambda\alpha}\big(\tilde{V}(T) - V(T)\big) \leq 0$ is satisfied. With this value we can determine the target capital $A(0)$ using the formula described by Proposition 2.

References

[1] Arnott, R.D. & Bernstein, P.L. (2002) *What Risk Premium is "Normal"* Financial Analysts Journal, **58**, 64-85

[2] Artzner, P., Delbaen, F., Eber, J.M.,& Heath, D. (1999) *Coherent Measures of Risk* Mathematical Finance **9**, 203-228

[3] Boquist, J.A., Racette, G.A., & Schlarbaum, G.G. (1975) *Duration and Risk Assessment for Bonds and Common Stocks* Journal of Finance **30**, 1360-1365

[4] Brennan, M. & Xia, Y. (2000) *Dynamic Asset Allocation under Inflation* Working paper, The Wharton School, University of Pennsylvania

[5] Brigo, D. & Mercurio, F. (2001) *Interest Rate Models-Theory and Practice* Springer Finance, Springer-Verlag Berlin Heidelberg New York

[6] Brown, G, (2000) *Duration and Risk* Journal of Real Estate Research **20**, 337-356

[7] Buttimer, R.J., Kau, J.B., & Slawson, V.C. (1997) *A Model for Pricing Securities Dependent upon a Real Estate Index* Journal of Housing Economics **7**, 229-264

[8] Cho, H., Kawaguchi, Y., & Shilling, J. (2003) *Unsmoothing Commerical Property Returns: A Revision to Fischer-Geltner-Webb's Unsmoothing Methodology* Journal of Real Estate Economics **27**, 393-405

[9] Delbaen, F. *Risk Measures or Measures that Describe Risk* www.math.ethz.ch/∼ delbaen/

[10] Fama, E. & Bliss, R. (1987) *The Information in Long-Maturity Forward Rates* American Economic Review **77**, 680-692

[11] (2003) *Standard Approach Drafted by the Chairman Jukka Rantala: Extension Proposal for Asset Liability Risk* Working Paper, Members of Féderation Francaise des Sociétés d'Assurances and CEA

[12] Fischer, J., Geltner, D., & Webb, B. (1994) *Value Indices of Commerical Real Estate: A Comparison of Index Construction Methods* Journal of Real Estate Economics **9**, 137-164

[13] Garman, M. & Kohlhagen, S. (1983) *Foreign Currency Option Values* Journal of International Money and Finance **2**, 231-237

[14] Geltner, D. (1993) *Estimating Market Values from Appraised Values without Assuming an Efficient Market* Journal of Real Estate **8**, 325-346

[15] Granger, C., & Morgenstern, O. (1970) *Predictability of Stock Market Prices* D.C. Heath and Company

[16] Grosen, A. & Løchthe Jørgensen, P. (2000) *Fair Valuation of Life Insurance Liabilities: The Impact of Interest Rate Guarantees, Surrender Options, and Bonus Policies* Insurance: Mathematics and Economics **26**, 37-57

[17] Hoesli, M., Lekander, J., & Witkiewicz, W. (2003) *International Evidence on Real Estate as a Portfolio Diversifier* Working paper, Stockholm Institute of Financial Research

[18] Hull, J. (1997) *Options, Futures, and Other Derivatives*, 3rd Ed., Prentice Hall International, London

[19] IAA (2000) *Valuation of Risk Adjusted Cash Flows and the Setting of Discount Rates-Theory and Practice* Comments to the IASC's Insurance Issues Paper

[20] IAA (2004) *A Global Framework for Insurer Solvency Assessment* IAA Insurer Solvency Assessment Working Party

[21] Koivu, M., Pennanen, T., & Ranne, A. (2004) *Modeling Assets and Liabilities of a Finnish Pension Company: A VEqC Approach* Working paper, to appear in Journal of Scandinavian Actuarial Journal

[22] Larsen, R & Marx, M. (1986) *An Introduction to Mathematical Statistics and its Applications* Prentice Hall

[23] Leibowitz, M. (1986) *Total Portfolio Duration: A new Perspective on Asset Allocation* Financial Analysts Journal, **42**, 18-29

[24] Leibowitz, M., Sorensen, E., Arnott, R., & Hanson, N. (1989) *A Total Differential Approach to Equity Duration* Financial Analysts Journal, **45**, 30-37

[25] Liu, C.H., Hartzell, D.J., & Hoesli, M.H. (1997) *International Evidence on Real Estate as an Inflation Hedge* Real Estate Economics **25**, 193-221

[26] Macaulay, F.R. (1938) *Some Theoretical Problems Suggested by the Measurement of Interest Rates, Bond Yields and Stock Prices in the United States Since 1856* National Bureau of Economic Research, Columbia University Press, New York

[27] Merton, R.(1980) *On Estimating the Expected Return on the Market: An Exploratory Investigation.* Journal of Financial Economics **8**

[28] Munk, C., Nygaard Vinther, T., & Sørensen, C (2002) *Dynamic Asset Allocation under Mean-Reverting Returns, Stochastic Interest Rates and Inflation Uncertainty* Working paper, Copenhagen Business School

[29] Rantala, J. (2003) *A Standard Approach for Asset/Liability Risk in Life Insurance* Working Paper, CEA

[30] Rantala, J. (2004) *Some Remarks to the Treatment of the AL-Risk in the SA* Working paper, CEA

[31] Redington, F.M. (1952) *Review of the Principles of Life-Office Valuations* Journal of the Institute of Actuaries **78**, 286-340

[32] Rogers, L.C.G. (2001) *The Relaxed Investor and Parameter Uncertainty* Finance and Stochastics **5**, 131-154

[33] Samuelson, P.A. (1965) *Rational Theory of Warrant Papers* Industrial Management Review **6**, 13-31

[34] Sandström, A. (2004) *Solvency - a Proposal for a Standard Approach* working paper, Swedish Insurance Federation

[35] Vasicek, O. (1977) *An Equilibrium Characterization of the Term Structure* Journal of Financial Economics **5**, 177-188

[36] Yong, Y. (1999) *Term Structure Models: A Perspective from the Long Rate* North American Actuarial Journal, **3**

[37] Young, M.S. & Graff, R.A. (1995) *Real Estate is Not Normal: A Fresh Look at Real Estate Return Distributions* Journal of Real Estate Finance and Economics, **10**, 225-260

[38] Zhou, H. (2001) *Jump-Diffusion Term Structure and Itô Conditional Moment Generator* Working paper, Federal Reserve Board, Washington.