Options for Guaranteed Index-linked Life Insurance

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Abstract

The market risk contained in guaranteed equity-linked life insurance products can be managed with options. To prevent premature cancellation, a profit structure is chosen which corresponds to forward cliquet options. The pricing and the delta- and vega-risk of these options is computed for a variety of payoff patterns. When the profit from the index contributes additively to the benefits, then a closed form is possible for prices and delta- resp. gamma-risks. If the profit from the index contributes to the benefit only if it exceeds the guaranteed sum, then these quantities can be computed with Monte Carlo simulations.

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1 Introduction and Summary

Guaranteed equity-linked life insurance products are well established in North America and in some countries in Europe e.g., in United Kingdom, France, Switzerland, Italy, and Belgium. Since traditional life insurance products sold badly in Germany during the last five years (the total sum newly insured dropped from 191,1 billion DM in 1990 to 149,2 billion DM in 1995), also here this concept is discussed nowadays; guaranteed index-linked life products are considered for possible competition with high interest rate products (Equitable Life) or bank products ("Börse Vollkasko" of Commerzbank). See Meisch and Stolz (1996) [11] or Mattar (1996) [10]. Until today (July 1996) guaranteed equity-linked life products are not sold yet by German life insurance companies in Germany, there are some variants which are discussed and considered to fit to German taxation and supervision rules (see Blohm (1996) [3]). A first tranche of guaranteed equity-linked life products, called INDAX, is being offered by Standard Life in Germany from August 1 until September 30 this year. The contracts commence on October 1st, 1996.

Already in one of the first treatments of guaranteed equity-linked life contracts of Brennan and Schwartz ([4]), it is stressed that these products differ essentially from traditional life products. The benefits are linked to an equity such as the S&P 500 stock index or the German DAX, and consequently the risk for the insurer is non-diversifiable across individual policy holders. If an insurance company issues a large portfolio of life products, then by the law of large numbers the mortality risk is diversified away. However, if a large portfolio of equity-linked life contracts is issued, then "a stock market collapse will render the insurance company simultaneously liable under the guarantees of all its expiring policies." (see [4]). While the pricing of the mortality risk is done with the help of the equivalence principle, the pricing of the financial risk must be done with the help of modern finance. The payoff of a guaranteed equity-linked life contract at expiration is - in the most simple case - identical to the payoff from a European call option plus the guaranteed amount. The first component of this payoff can therefore be priced using the Black and Scholes theory. This pricing problem has been addressed in the papers [15], [16], [4], [5], [6], [1] for constant interest rate, and in [2], [13], [12], and [9] for random interest rate. In this paper we con-

sider special equity-linked life products with a certain "lock in" feature in the equity-profit part. We shall derive prices for the corresponding options under the Black and Scholes model with constant interest rate. The delta- and vega-risks are computed as well.

The above form of profit from the index $S(t)$ is connected with the relative total profit in the index from time 0 to time T. If $S(t)$ increases by $x\%$ with $0 < x = S(T)/S(0) * 100$, then the benefit from the index equals $x\,K/100$. Therefore this profit corresponds to a European call option which at maturity pays

$$K\left(S(T)/S(0) - 1\right)^{+}. \tag{1}$$

Here, K is the fixed sum which participates in the index. This definition of the profit has the disadvantage that a policy holder will cancel his contract as soon as the index is far below $S(0)$ or far above $S(0)$: in the first case, the probability of a future gain is very small, so the policy holder will cancel his contract and will invest his money directly into the index; in the second case he will prefer to withdraw the profit since the probability of a future loss is quite high. In order to prevent cancellation of contracts, a lock in variant can be built in, e.g., via a forward cliquet option or a ladder option.

In a ladder option, the profit is locked in as soon as it exceeds a prescribed value. This profit will not be reduced by future losses in the index. The payoff of a ladder option with ladder values $0 < M_1 < M_2 < \ldots$ at expiration equals

$$K\left(\max\left(S(T), M_j\right)/S(0) - 1\right)^{+}$$

where M_j is the largest member in the sequence $M_1 < M_2 < \ldots$ for which $S(t) \geq M_j$ holds for some $0 < t < T$. Also for this profit structure, premature cancellation will occur as soon as $S(t)$ is much smaller than $S(0)$, since then the probability for a future profit is small. Furthermore, this option has a path dependent payoff at expiration, and therefore it will not be considered in this paper.

In a forward cliquet option, the annual profit in the index is locked in, it will not be lost if losses in the index occur in the future. A forward cliquet option written on the underlying index $S(t)$ is a sequence of T forward options on $S(t)$ which are at the money. The value at maturity is

$$K\sum_{t=1}^{T}\left(S(t)/S(t-1) - 1\right)^{+}.$$

The word "cliquet" is the french word for a toothed ratchet wheel. This is a wheel which always continues to spin round in the same direction. Correspondingly, a forward cliquet option accumulates annual profits and neglects all annual losses. The word "forward" indicates that we have a sequence of forward contracts on European call options starting at $t - 1$, expiring at t, and being at the money when starting, for $t = 1, ..., T$. In this forward contract, the exercise price of the one year options for $t > 1$ is not known in advance.

For this kind of profit from the index, the risk of premature cancellation is small since small values of $S(t)$ increase the probability for a large future relative profit. Even a crash of the stock market opens the chance for future profits, provided the stock market recovers after the crash.

A guaranteed index-linked life insurance has an expiration time T, a time of payment S, a guaranteed sum G, a participation in the index, a death benefit D, a survival benefit B, a pattern for the profit from the index as well as a definition of the surrender value. We shall use P for the total profit from the index at expiration T. In order to fit to the German taxation rules, an annual premium has to be paid (for at least 5 years, i.e. $S \geq 5$), and T should be 12 or larger. For this purpose we include the case of $T = 12$ and $S = 12$ annual premia, paid at the beginning of each year. To simplify notation we assume that the annual premium is 1. The participation in the single premium case can be defined as a total amount through a percentage α of the single premium. In the case of current premium payments, it seems to be more natural to define it as a percentage of accumulated premia: for the profit in year t the amount participating in the index equals

$$K(t) = \alpha \min(S, t).$$

The yearly profit is a special "policyholder's dividend" which can be accumulated, can yield interest on expiry, or can be included into the amount participating in the index (compounding). In this note we shall consider profit patterns for single premium payment (with constant participation) and current premium payment (with time dependent participation), with profits which are just added, are yielding interest, or are compounded. We deal with the two cases of *additive* ($B = G + P$) and *non-additive* ($B = \max(G, P + K(T))$) survival benefits. In the non-additive case the paid profit equals $(P + K(T) - G)^+$.

2   Payoff Patterns

2.1   Single Premium Payment

Additive Benefits

In our above payoff pattern

$$K \sum_{t=1}^{T} (S(t)/S(t-1) - 1)^{+} \tag{2}$$

the annual profits from the index are just added. We shall next consider the cases in which the annual profits bear interest or are compounded. We assume that the interest rate is nonrandom and constant, say r. The payoff at maturity in the case with interest rate equals

$$K \sum_{t=1}^{T} (S(t)/S(t-1) - 1)^{+} (1+r)^{T-t}. \tag{3}$$

If profit is compounded, i.e. if K is adjusted each year such that all earlier profits are included, then the payoff at expiration equals

$$K \prod_{t=1}^{T} \max \left(\frac{S(t)}{S(t-1)}, 1 \right) - K = K \exp \left(\sum_{t=1}^{T} (\log(S(t)) - \log(S(t-1)))^{+} \right) - K. \tag{4}$$

An alternative method for the definition of a guaranteed sum is via a *guaranteed interest rate* r_0. A payoff yielding a minimal interest rate of r_0 which might be called *compounded with floor* is the following:

$$K \prod_{t=1}^{T} \max \left(\frac{S(t)}{S(t-1)}, 1 + r_0 \right) - K. \tag{5}$$

Non-additive Benefits

Up to now, we have considered only the case in which the profit from participation in the index is additive, i.e. the $B = P + G$. A second possibility is a benefit which is the maximum of the guaranteed amount and the total return from the participation, $B = \max(G, P + K)$. If $G > K$ is the guaranteed amount, then the benefits for the above profit schemes are

$$B = \max \left(K \, S(T)/S(0), G \right) = \frac{K}{S(0)} \max \left(S(T), G \frac{S(0)}{K} \right) \text{ for (1)},$$

$$B = \max \left(K \sum_{t=1}^{T} (S(t)/S(t-1) - 1)^{+} + K, G \right) \text{ for (2)},$$

$$B = \max \left(K \sum_{t=1}^{T} (S(t)/S(t-1) - 1)^{+} (1+r)^{T-t} + K, G \right) \text{ for (3)},$$

$$B = \max \left(K \prod_{t=1}^{T} \max \left(\frac{S(t)}{S(t-1)}, 1 \right), G \right) \text{ for (4), and}$$

$$B = \max \left(K \prod_{t=1}^{T} \max \left(\frac{S(t)}{S(t-1)}, 1 + r_0 \right), G \right) \text{ for (5)}.$$

2.2 Current Premium Payment

In the case of current premium payment, the amount participating in the index at time t, say $K(t)$, depends on t, more precisely on the accumulated sum of premia paid up to time t. Recall that S is the time of premium payment. For current premium payment, the guaranteed endowment value is usually defined through a guaranteed interest rate r_0 paid for accumulated premia:

$$G = \sum_{t=1}^{S} (1 + r_0)^{T-t},$$

and the participation at time t is defined through a percentage α of the accumulated premia paid up to time t (the beginning of year t): $K(t) = \alpha \min(t, S)$. The Standard Life product "INDAX Andante", e.g., has $S = 5$ annual premia, $T = 12$, $\alpha = 0.69$ and $r_0 = 0$. We consider the following four patterns in which the annual profits are just added, are bearing interest, are compounded, or are compounded with floor. In the first two cases the accumulated profits are *not* included in the participating amount. Just adding yields the final profit of

$$\alpha \sum_{t=1}^{T} t \left(\frac{S(t)}{S(t-1)} - 1 \right)^{+}, \tag{6}$$

bearing interest gives a benefit of

$$\alpha \sum_{t=1}^{T} t \left(\frac{S(t)}{S(t-1)} - 1 \right)^{+} (1+r)^{T-t}; \tag{7}$$

the compounded version, i.e. accumulated profits are participating in the index, gives

$$\alpha \sum_{t=1}^{T} \left(\prod_{i=t}^{T} \max \left(\frac{S(i)}{S(i-1)}, 1 \right) - 1 \right), \tag{8}$$

which has the same distribution as

$$\alpha \sum_{t=1}^{T} \left(\prod_{i=1}^{t} \max \left(\frac{S(i)}{S(i-1)}, 1 \right) - 1 \right),$$

while the compounded with floor version yields

$$\alpha \sum_{t=1}^{T} \left(\prod_{i=t}^{T} \max \left(\frac{S(i)}{S(i-1)}, 1 + r_0 \right) - 1 \right), \tag{9}$$

which has the same distribution as

$$\alpha \sum_{t=1}^{T} \left(\prod_{i=1}^{t} \max \left(\frac{S(i)}{S(i-1)}, 1 + r_0 \right) - 1 \right).$$

It is straightforward to write down the corresponding formulae for non-additive benefits $B = \max(P + K(T), G)$.

3　Pricing of Forward-Cliquet Options

We shall price the forward cliquet options under the different payoff patterns under the Black and Scholes model for the index:

$$dS(t) = \mu S(t) dt + v S(t) dW(t), \ 0 \le t \le T, \ S(0) = s_0 > 0.$$

Here, $W(t)$, $0 \le t \le T$, is a standard Wiener process. In this model, we assume a known constant volatility v. For more realistic models and pricing/hedging formulae see, e.g., [14]. In addition we shall assume that the interest rate $\delta = \log(1 + r)$ is nonrandom and constant. We then have the

ideal situation in modern finance where the market is arbitragefree and complete, i.e. there exists exactly one equivalent martingale measure $\hat{P}$. The pricing of a contingent claim paying an amount C at T which depends on $S(t)$, $0 \le t \le T$, is then done using this equivalent martingale measure $\hat{P}$: the price for C equals

$$\exp(-\delta T)\,\hat{E}C.$$

Under $\hat{P}$ the process $S(t)$ satisfies the following stochastic differential equation:

$$dS(t) = \delta S(t)dt + vS(t)d\widehat{W}(t),\ 0 \le t \le T,\ S(0) = s_0 > 0.$$

Here, $\widehat{W}(t)$, $0 \le t \le T$, is a standard Wiener process under $\hat{P}$.The above equation can be solved explicitly:

$$S(t) = s_0 \exp\left((\delta - \frac{1}{2}v^2)t + v\widehat{W}(t) \right).$$

We now derive the price for all our payoff structures.

3.1 Single Premium Payment

Additive Benefit

For (1) we have the price

$$K\,C(1,T)$$

where $C(s,x,\tau)$ is the classical Black and Scholes price for a European call with exercise price x, when the underlying has value s, and time to maturity is τ, and $C(s,\tau)$ is the value at the money (i.e. $x = s$) :

$$
\begin{aligned}
C(s,x,\tau) &= s\,\Phi\left(\frac{\log(s/x) + (\delta + v^2/2)\tau}{v\sqrt{\tau}} \right) \\
&\quad - \exp(-\delta\tau)\,x\,\Phi\left(\frac{\log(s/x) + (\delta - v^2/2)\tau}{v\sqrt{\tau}} \right) \\
C(s,\tau) &= s\,\Phi\left(\frac{(\delta + v^2/2)\sqrt{\tau}}{v} \right) - \exp(-\delta\tau)\,s\,\Phi\left(\frac{(\delta - v^2/2)\sqrt{\tau}}{v} \right).
\end{aligned}
$$

with $\Phi(x) = 1/\sqrt{2\pi}\int_{-\infty}^{x}\exp(-t^2/2)dt$ the standard normal distribution function. For (2) we obtain the price

$$\exp(-\delta T)\hat{E}\left(K\sum_{t=1}^{T}(S(t)/S(t-1) - 1)^+ \right) =$$

$$K(1+r)^{-T}\sum_{t=1}^{T}\hat{E}\left(S(t)/S(t-1)-1\right)^+ =$$

$$K(1+r)^{-T}\sum_{t=1}^{T}\hat{E}\left(\exp\left(\left(\delta-\frac{1}{2}v^2\right)+v\widehat{W}(1)\right)-1\right)^+ =$$

$$K(1+r)^{-T+1}TC(1,1).$$

Notice that the price for (1) is not always smaller than the price for the forward cliquet option (2). Even more surprisingly, the price of the forward cliquet option is not increasing in time. This is due to the fact that the profit is not included in the number K (as it would be with compound interest). If time increases, then an additional term shows up in the sum, but at the same time the profit is payed later and therefore smaller by discounting.

For (3) we obtain the following price:

$$\exp(-\delta T)\hat{E}K\sum_{t=1}^{T}\left(S(t)/S(t-1)-1\right)^+(1+r)^{T-t} =$$

$$K\sum_{t=1}^{T}(1+r)^{-t}\,\hat{E}\left(S(t)/S(t-1)-1\right)^+ =$$

$$K\sum_{t=1}^{T}(1+r)^{-t}\exp(\delta)\hat{E}\left(\exp\left(-\frac{1}{2}v^2+v\widehat{W}(1)\right)-1\right)^+ =$$

$$K\sum_{t=0}^{T-1}(1+r)^{-t}\,C(1,1)=K\left(1-(1+r)^{-T}\right)\frac{1+r}{r}C(1,1)$$

Also for (4) we obtain a closed form for the price. The exponent in the payoff

$$\sum_{t=1}^{T}\left(\log(S(t))-\log(S(t-1))\right)^+ = \sum_{t=1}^{T}\left(\delta-\frac{1}{2}v^2-v\left(\widehat{W}(t)-\widehat{W}(t-1)\right)\right)^+$$

is the sum of iid random variables, and therefore

$$\exp(-\delta T)\hat{E}K\prod_{t=1}^{T}\max\left(\frac{S(t)}{S(t-1)},1\right) =$$

$$\exp(-\delta T)K\left(\hat{E}\exp\left(\delta-\frac{1}{2}v^2+v\widehat{W}(1)\right)^+\right)^T =$$

$$K\left(\exp(-\delta)\hat{E}\exp\left(\delta-\frac{1}{2}v^2+v\widehat{W}(1)\right)^+\right)^T.$$

With $S = \delta - \frac{1}{2}v^2 + v\widehat{W}(1)$ we obtain

$$
\begin{aligned}
\widehat{E} \exp\left(S^+\right) &= \widehat{E} \exp\left(S\right) 1_{(S>0)} + \widehat{P}(S \le 0) \\
&= \widehat{E}\left(\exp\left(S\right) - 1\right) 1_{(S>0)} + 1 \\
&= 1 + \exp(\delta)C(1,1)
\end{aligned}
$$

This yields the price

$$
K \left(\frac{1}{1+r} + C(1,1)\right)^T - K(1+r)^{-T}
$$

for the corresponding option with payoff pattern (4). Similarly, for (5) we obtain the option price

$$
K \left(\frac{1+r_0}{1+r} + C(1, 1+r_0, 1)\right)^T - K(1+r)^{-T}.
$$

In Figure 3.1 we show the prices for $r = 0.07$, $v = 0.2$, $K = 1$ and $0 \le T \le 30$ for the five payoff pattern (1) -(5).

Non-additive Benefit

If the benefit is not defined as the sum but as the maximum of G and the amount K plus profit from the index, then not all prices of the corresponding contingent claims can be given in closed form. For (1) we still obtain one:

$$
\begin{aligned}
\exp(-\delta T)\widehat{E} \frac{K}{S(0)} \max\left(S(T), \frac{GS(0)}{K}\right) &= \\
(1+r)^{-T}G + \frac{K}{S(0)}C\left(S(0), \frac{GS(0)}{K}, T\right) & \\
= (1+r)^{-T}G + KC(1, \frac{G}{K}, T) \text{ for payoff} \qquad (1).
\end{aligned}
$$

The corresponding option prices are based on the payment $B - G$ at expiration T. So,e.g., for (1) we obtain the option price

$$
KC \left(1, \frac{G}{K}, T\right).
$$

For all other payoff patterns (2) - (5) we must use Monte Carlo methods for the computation of the option prices. However, this is easy since we do not need simulations of a sample path of a solution to the stochastic differential

equation, instead for each Monte Carlo value we only need T independent normal random variables. We use a Monte Carlo sample size of 10,000. In Figure 3.2 we see the simulated option prices for $r = 0.07$, $G = 1.601$, $K = 1$ (or $\alpha = 0.6246$), $T = 12$ and v from 0.1 to 0.4, for the five payoff pattern (2) - (5). In Figure 3.3 we show the corresponding price for payoff pattern (2),one curve is the exact value, the other the result of a Monte Carlo simulation with 10,000 replications. The error increases with the volatility and equals $0.59327913 - 0.576875 = 0.0164$ or 2.765% for $v = 0.4$. The guaranteed amount $G = 1.601$ corresponds to a guaranteed interest rate of 4% :

$$G = 1.04^{12}.$$

3.2 Current Premium Payment

Additive Benefit

For payoff patterns (6) - (9) we obtain explicit option prices:

$$\exp(-\delta T)\widehat{E}\alpha \sum_{t=1}^{T} t \left(\frac{S(t)}{S(t-1)} - 1 \right)^{+} =$$

$$\alpha \left(1+r\right)^{-T+1} \frac{T}{2}(T+1)C(1,1) \text{ for (6)},$$

$$\exp(-\delta T)\widehat{E}\alpha \sum_{t=1}^{T} t \left(\frac{S(t)}{S(t-1)} - 1 \right)^{+} (1+r)^{T-t} =$$

$$\alpha \frac{(1+r)^2}{r^2} \left(1 - (1+r)^{-T-1}(1 + r(T+1))\right) C(1,1) \text{ for (7)},$$

$$\exp(-\delta T)\widehat{E}\alpha \sum_{t=1}^{T} \left(\prod_{i=1}^{t} \max\left(\frac{S(t)}{S(t-1)}, 1 \right) - 1 \right) =$$

$$\frac{\alpha}{(1+r)^T} \sum_{t=1}^{T} \left((1 + (1+r)C(1,1))^t - 1 \right) =$$

$$\frac{\alpha}{(1+r)^T} \left(\frac{(1 + (1+r)C(1,1))^{T+1} - 1}{(1+r)C(1,1)} - T - 1 \right) \text{ for (8), and}$$

$$\exp(-\delta T)\widehat{E}\alpha \sum_{t=1}^{T}\left(\prod_{i=1}^{t}\max\left(\frac{S(t)}{S(t-1)},1+r_0\right)-1\right)\;=$$

$$\frac{\alpha}{(1+r)^T}\sum_{t=1}^{T}\left((1+r_0+(1+r)C(1,1+r_0,1))^t-1\right)\;=$$

$$\frac{\alpha}{(1+r)^T}\left(\frac{(1+r_0+(1+r)C(1,1+r_0,1))^{T+1}-1}{r_0+(1+r)C(1,1+r_0,1)}-T-1\right)\text{ for (9)}$$

In Figure 3.4 we show the prices for $\alpha=1$, $r=0.07$, $v=0.2$, and $0\le T\le 30$ for the four payoff patterns (6) -(9).

Non-additive Benefit

In Figure 3.5 we show the option prices for non-additive benefits for the parameters $r=0.07$, $\alpha=1$, $G=15.6268$, $T=12$ and v from 0.1 to 0.4, for the four payoff patterns (6) - (9). The choice of G corresponds to a guaranteed interest rate of 4%:

$$G=\sum_{t=1}^{12}1.04^t.$$

Also here, we used a Monte Carlo sample size of 10,000.

4 Vega and Delta-risk of Options

The forward cliquet options mentioned above are OTC products, there is no liquid market for them, and so hedging of their market risk must be done in a (large) portfolio of options (OTC as well as standardized) on the underlying index. The long time to expiry leads to a large vega-risk of our forward cliquet options. The delta- and vega-risk of long positions of cliquet options can be compensated by an appropriate short position in other options. For the computation of the delta- and vega-risk at time $0<t<T$ we need the option price for time t. For additive benefits and payoff patterns (2) - (3) the delta-risk of the whole option is equal to the delta-risk of the part active at time t : If $j\le t<j+1$ and if $C_s(x,s,\tau)$ is the delta-risk of a European call option with time to maturity τ, exercise price x, starting index value s, i.e.

$$
\begin{aligned}
C_s(s,x,\tau) &= \partial C(s,x,\tau)/\partial s\\
&= \Phi\left(\frac{\log(s/x)+(\delta+v^2/2)\tau}{v\sqrt{\tau}}\right)
\end{aligned}
$$

then (again with $\tau = j + 1 - t$) the delta-risk of the option equals

$$K(1+r)^{-T+j+1}C_s(\frac{s}{S(j)}, 1, \tau)/S(j) \text{ for payoff (2), and}$$

$$K\, C_s(\frac{s}{S(j)}, 1, \tau)/S(j) \text{ for payoff (3)}$$

For compounded payoff patterns we have the accumulated profit up to year j participating in the index, and by independence of future gains the delta-risk equals

$$K\prod_{i=1}^{j}\max\left(\frac{S(i)}{S(i-1)}, 1\right)\frac{1}{S(j)}C_s(\frac{s}{S(j)}, 1, \tau)\left(\frac{1}{1+r} + C(1,1)\right)^{T-j-1}$$

for payoff (4) and

$$K\prod_{i=1}^{j}\max\left(\frac{S(i)}{S(i-1)}, 1+r_0\right)$$
$$\times\frac{1}{S(j)}C_s(\frac{s}{S(j)}, 1+r_0, \tau)\left(\frac{1+r_0}{1+r} + C(1,1+r_0,1)\right)^{T-j-1}$$

for payoff (5), respectively. The corresponding values for current premium payment are

$$\alpha(j+1)(1+r)^{-T+j+1}C_s(\frac{s}{S(j)}, 1, \tau)/S(j) \text{ for (6),}$$

$$\alpha(j+1)C_s(\frac{s}{S(j)}, 1, \tau)/S(j) \text{ for (7),}$$

$$\alpha\sum_{i=1}^{T}\prod_{k=1}^{j}\max\left(\frac{S(k)}{S(k-1)}, 1\right)\frac{1}{S(j)}C_s(\frac{s}{S(j)}, 1, \tau)\prod_{k=j+2}^{i}\left(\frac{1}{1+r} + C(1,1)\right)$$

for (8),and finally

$$\alpha\sum_{i=1}^{T}\prod_{k=1}^{j}\max\left(\frac{S(k)}{S(k-1)}, 1+r_0\right)\frac{1}{S(j)}C_s(\frac{s}{S(j)}, 1+r_0, \tau)$$
$$\times\prod_{k=j+2}^{i}\left(\frac{1+r_0}{1+r} + C(1,1+r_0,1)\right) \text{ for (9).}$$

Similar formulae are valid for the vega-risk: The vega-risk $C_\sigma(s, x, \tau)$ for a European call option equals

$$
\begin{aligned}
C_\sigma(s, x, \tau) &= \partial C(s, x, \tau)/\partial \sigma \\
&= s\sqrt{\tau}\varphi\left(\frac{\log(s/x) + (\delta + v^2/2)\tau}{v\sqrt{\tau}}\right)
\end{aligned}
$$

We obtain for additive benefits the following vega-risks (again with $\tau = j + 1 - t$):

$$
KC_\sigma(\frac{s}{S(j)}, 1, \tau)(1+r)^{j+1-T} + \sum_{i=j+2}^{T} C_\sigma(1, 1, 1)(1+r)^{i-T} \text{ for payoff (2)},
$$

$$
KC_\sigma(\frac{s}{S(j)}, 1, \tau) + \sum_{i=j+2}^{T} C_\sigma(1, 1, 1) \text{ for payoff (3)},
$$

$$
K \prod_{i=1}^{j} \max\left(\frac{S(i)}{S(i-1)}, 1\right) \frac{\partial}{\partial \sigma}\left(C(\frac{s}{S(j)}, 1, \tau)\left(\frac{1}{1+r} + C(1, 1, 1)\right)^{T-j-1}\right)
$$

for payoff (4),

$$
K \prod_{i=1}^{j} \max\left(\frac{S(i)}{S(i-1)}, 1 + r_0\right)
$$
$$
\times \frac{\partial}{\partial \sigma}\left(C(\frac{s}{S(j)}, 1 + r_0, \tau)\left(\frac{1+r_0}{1+r} + C(1, 1 + r_0, 1)\right)^{T-j-1}\right)
$$

for payoff (5),

$$
\alpha(j+1)C_\sigma(\frac{s}{S(j)}, 1, \tau)(1+r)^{j+1-T} + \alpha \sum_{i=j+2}^{T} iC_\sigma(1, 1, 1)(1+r)^{i-T}
$$

for payoff (6),

$$
\alpha(j+1)C_\sigma(\frac{s}{S(j)}, 1, \tau) + \alpha \sum_{i=j+2}^{T} iC_\sigma(1, 1, 1)
$$

for payoff (7),

$$
\alpha \sum_{i=1}^{T} \prod_{k=1}^{j} \max\left(\frac{S(k)}{S(k-1)}, 1\right)
$$
$$
\times \frac{\partial}{\partial \sigma}\left(C(\frac{s}{S(j)}, 1, \tau) \prod_{k=j+2}^{i}\left(\frac{1}{1+r} + C(1, 1, 1)\right)\right)
$$

for (8) and, finally,

$$\alpha \sum_{i=1}^{T} \prod_{k=1}^{j} \max \left(\frac{S(k)}{S(k-1)}, 1 + r_0 \right)$$
$$\times \frac{\partial}{\partial \sigma} \left(C(\frac{s}{S(j)}, 1 + r_0, \tau) \prod_{k=j+2}^{i} \left(\frac{1 + r_0}{1 + r} + C(1, 1 + r_0, 1) \right) \right)$$

for (9). Numerical experiments show that for the non-additive case the vega- and delta-risks are smaller than in the additive case.

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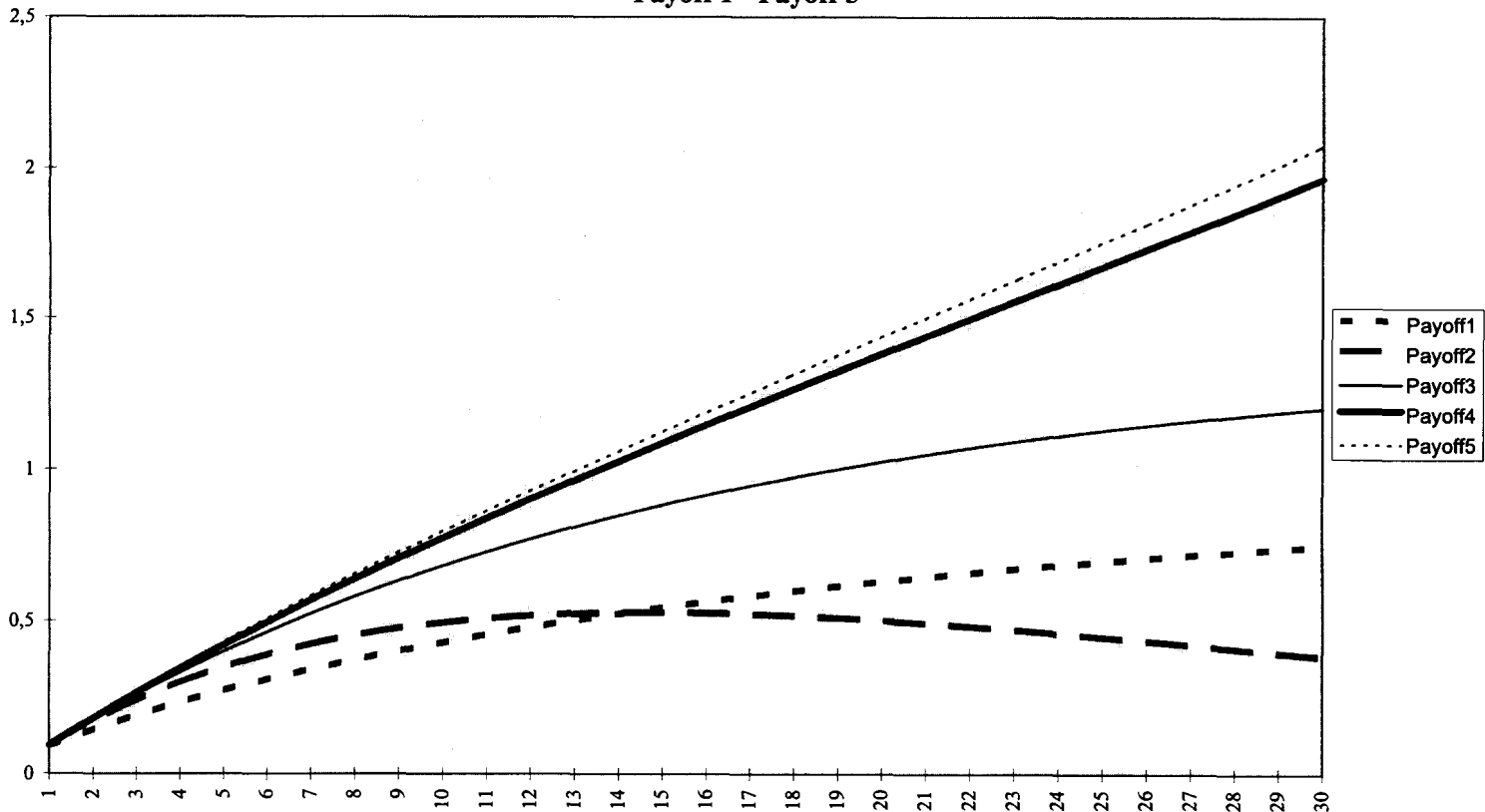**Option prices for additve benefits**
Payoff 1 - Payoff 5

Figure 3.1

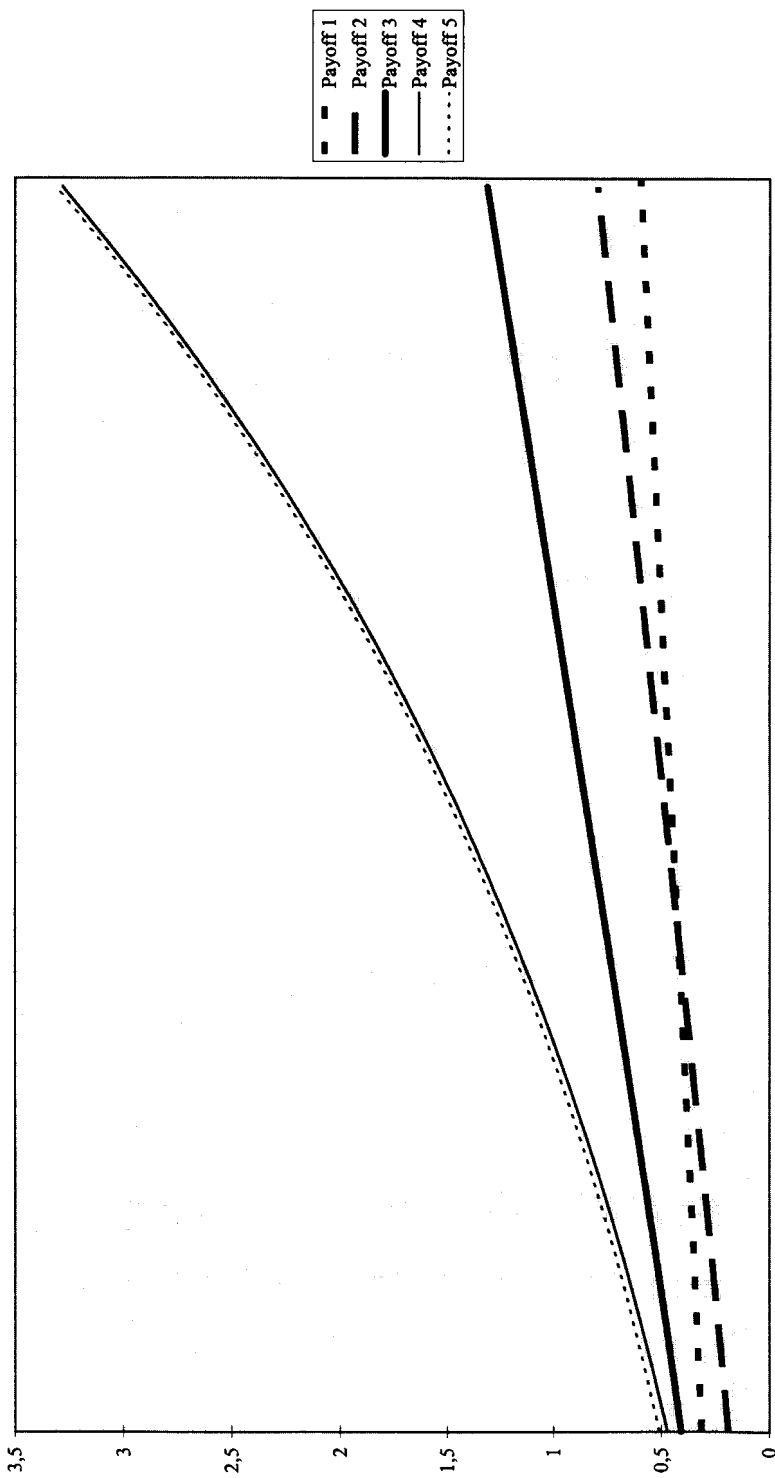Option Prices with non Additive Benefits
for Payoff 1 - Payoff 5
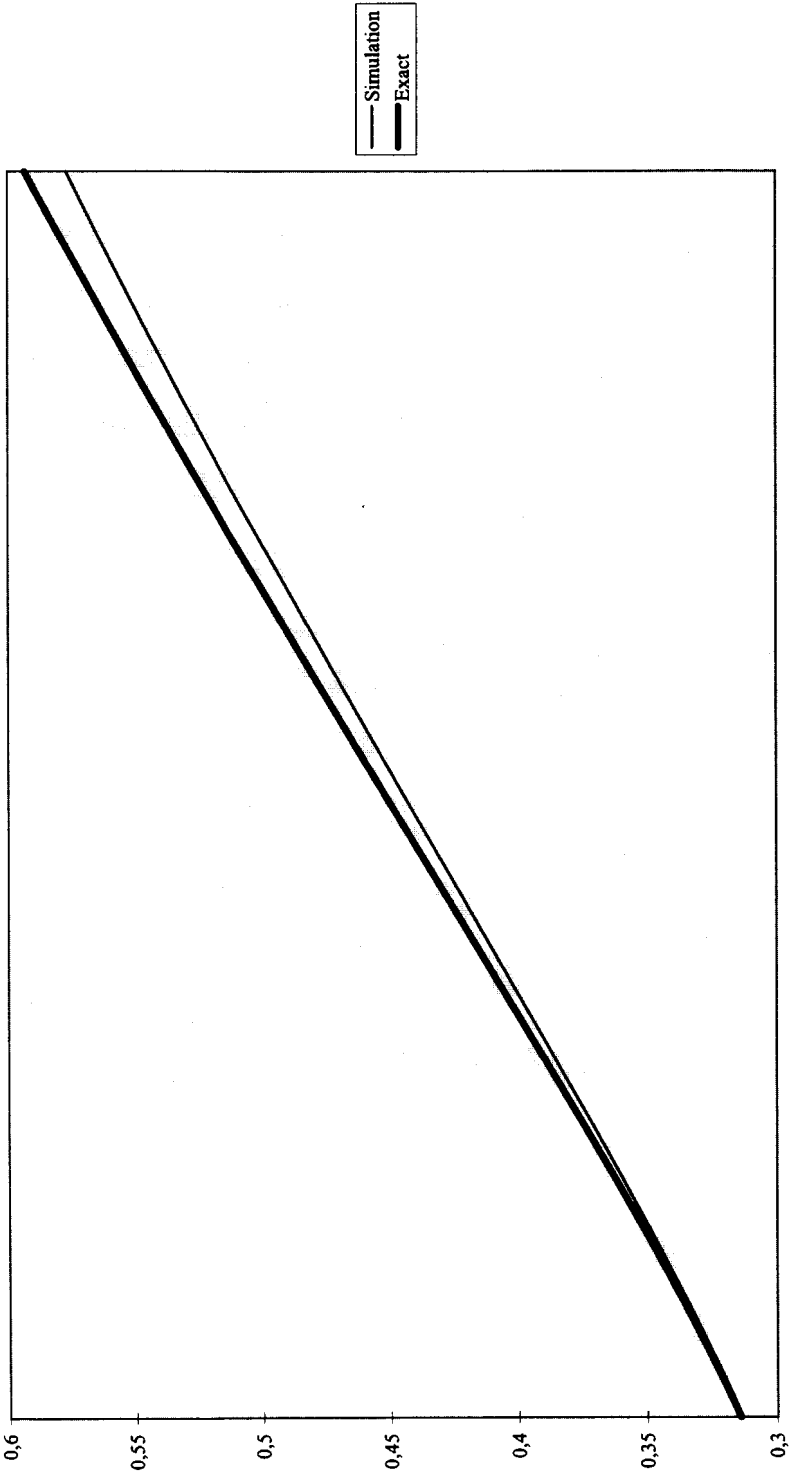
Payoff 1
Payoff 2
Payoff 3
Payoff 4
Payoff 5

Figure 3.2

Figure 3.3

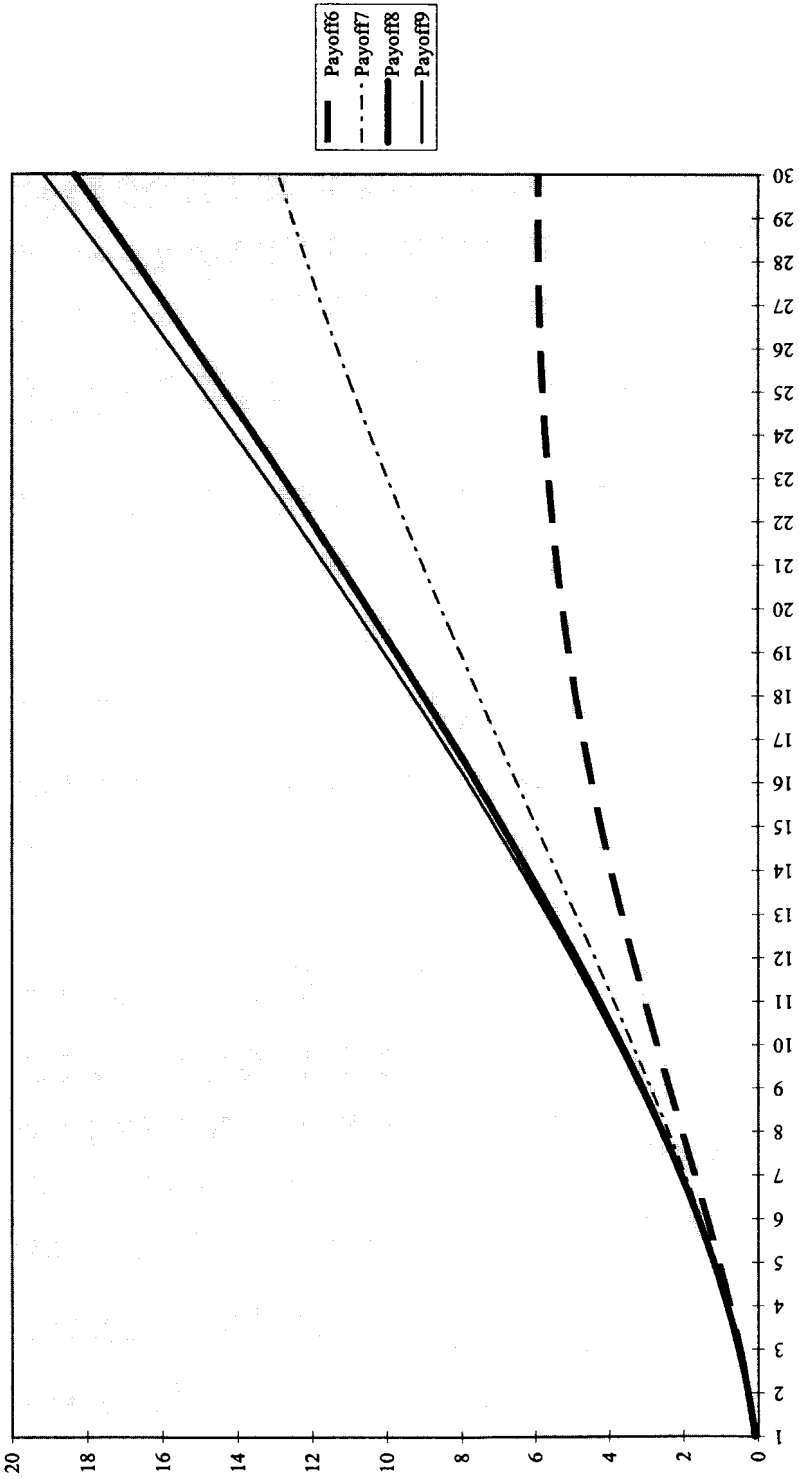Figure 3.4

Option Prices for Non Additive Benefits
Payoff Pattern 6 -9

Payoff6
Payoff7
Payoff8
Payoff9

Figure 3.5